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Nonlinear Eigenvalue Problem for the p-Laplacian

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Abstract

This article is devoted to the study of the nonlinear eigenvalue problem

$$\begin{aligned} -\Delta_p u &= \lambda |u|^{p-2}u \text{ in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &+ \beta |u|^{p-2}u = \lambda |u|^{p-2}u \text{ on } \partial\Omega, \end{aligned}$$

where ν denotes the unit exterior normal, $1 < p < \infty$ and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ denotes the p-laplacian. $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary where $N \geq 2$ and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$. Using Ljusternik-Schnirelman theory, we prove the existence of a nondecreasing sequence of positive eigenvalues and the first eigenvalue is simple and isolated. Moreover, we will prove that the second eigenvalue coincides with the second variational eigenvalue obtained via the Ljusternik-Schnirelman theory.

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1 Introduction

In this work we study the eigenvalue nonlinear problem

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} + \beta |u|^{p-2}u = \lambda |u|^{p-2}u & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

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where ν denotes the unit exterior normal, $1 < p < \infty$ and $\Delta_p u = div(|\nabla u|^{p-2}\nabla u)$ indicates the p-Laplacian. $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary where $N \geq 2$ and $\beta \in L^\infty(\partial\Omega)$ with $\beta^- := \inf_{x\in\partial\Omega}\beta(x) > 0$.
Many authors studies eigenvalue problems for the p-Laplacian under different boundary conditions Dirichlet, Neumann, Robin, no-flux and Steklov (see for instance [1], [6], [5], [9], [14], ...).
In this paper, we extend those results and we study the abstract eigenvalue problem (1.1). Its particularity lies in the fact that the spectral parameter λ is both in the differential equation and on the boundary. It is well known that an eigenvalue problems play a very important role in the studying of linear and nonlinear problems. Therefore, our results in the present paper would be useful to the study of problems of the form

$$\begin{aligned} -\Delta_p u &= f(x,u) \text{ in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} &+ \beta|u|^{p-2}u = g(x,u) \text{ on } \partial\Omega, \end{aligned}$$

where $f(x,u)$ and $g(x,u)$ interact, in some sense, with the spectrum of (1.1).
This paper is motivated by [8], where the author study the Generalized Steklov-Robin spectrum of the following linear problem (with possibly singular (m,n)-weights)

$$\begin{aligned} -\Delta u + c(x)u &= \lambda m(x)u \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} + \sigma(x)u &= \lambda n(x)u \quad \text{on } \partial\Omega \end{aligned}$$

and prove the existence of an unbounded and discrete spectrum. Moreover, the first eigenvalue is simple and its eigenfunction is of constant sign.
The paper is organized as follows. In section 2, we use a version of Ljusternik-Schnirelman theory to prove the existence of nondecreasing sequence of positive eigenvalues $(\lambda_n) \to +\infty$ of problem (1.1). In section 3 we prove some regularity results on eigenfunctions. In section 4, we prove that the first eigenvalue λ_1 characterized by

$$\lambda_1 = \inf_{u\in W^{1,p}(\Omega)}\Big\{\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega}\beta(s)|u|^p\, ds : \int_\Omega |u|^p dx + \int_{\partial\Omega}|u|^p ds = 1\Big\}$$

is simple and isolated. Moreover any associated eigenfunction to a positive eigenvalue $\lambda \neq \lambda_1$ does change sign in $\Omega \cup \partial\Omega$. In section 5 we prove that the eigenvalue λ_2 is actually the second eigenvalue, i.e., $\lambda_2 > \lambda_1$ and

$$\lambda_2 = \inf\Big\{\lambda : \lambda \text{ is an eigenvalue and } \lambda > \lambda_1\Big\}.$$

2 Existence of Ljusternik-Schnirelman Eigenvalue Sequence

Definition 2.1. A pair $(u,\lambda) \in W^{1,p}(\Omega)\times\mathbb{R}$ is a weak solution of (1.1) provided that for all $v \in W^{1,p}(\Omega)$

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx + \int_{\partial\Omega}\beta(s)|u|^{p-2}uv\, ds = \lambda\Big(\int_\Omega |u|^{p-2}uv\, dx + \int_{\partial\Omega}|u|^{p-2}uv\, ds\Big). \tag{2.1}$$

Such a pair (u,λ), with u nontrivial, is called an eigenpair. λ is an eigenvalue and u is called an associated eigenfunction.
By choosing $v = u$ in (2.1), it follows that all eigenvalues λ are nonnegative.
It will be shown that if $\partial\Omega$ is of class $C^{1,\gamma}$ with $0 < \gamma \leq 1$, then eigenfunction of (2.1) belongs to $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Hence, ∇u exists on $\partial\Omega$, and the boundary conditions of the problem (1.1) make sense. The following lemma ensures that if an eigenfunction u is smooth enough, then u solves the corresponding equation.

Lemma 2.2. *Let (u,λ) be an eigenpair, i.e., a weak solution of* (2.1) *such that $u \in W^{2,p}(\Omega)$, then (u,λ) solves* (1.1).

Proof. Let $(u,\lambda) \in W^{2,p}(\Omega)\times\mathbb{R}^+$ be an eigenpair of (2.1). By the first formula of Green, it follows from (2.1), that

$$\int_\Omega(-\Delta_p u)v\,dx+\int_{\partial\Omega}|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}v\,ds+\int_{\partial\Omega}\beta(s)|u|^{p-2}uv\,ds=\lambda\Big(\int_\Omega|u|^{p-2}uv\,dx+\int_{\partial\Omega}|u|^{p-2}uv\,ds\Big),$$

for any $v \in W^{1,p}(\Omega)$. Thus taking any $v \in C_0^\infty(\Omega)$ we obtain

$$\int_\Omega(\Delta_p u+\lambda|u|^{p-2}u)v\,dx=0,$$

which implies $-\Delta_p u = \lambda|u|^{p-2}u$ in Ω. Furthermore, since the range of the trace mapping $W^{1,p}(\Omega) \hookrightarrow L^p(\partial\Omega)$ is continuous and compact, we have

$$\int_{\partial\Omega}|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}v\,d\sigma+\int_{\partial\Omega}\beta|u|^{p-2}uv\,ds=\lambda\int_{\partial\Omega}|u|^{p-2}uv\,ds,\forall v\in L^p(\partial\Omega).$$

Therefore, $|\nabla u|^{p-2}\frac{\partial u}{\partial\nu}+\beta|u|^{p-2}u = \lambda|u|^{p-2}u$, on $\partial\Omega$. □

Let $X := W^{1,p}(\Omega)$ be the Sobolev space equipped with the norm

$$||u||_\beta := \Big(\int_\Omega|\nabla u|^p dx+\int_{\partial\Omega}\beta(s)|u|^p ds\Big)^{1/p}$$

which is equivalent to the usual $W^{1,p}(\Omega)$ norm

$$||u|| := \Big(\int_\Omega|\nabla u|^p dx+\int_\Omega|u|^p dx\Big)^{1/p}.$$

(see [15]).
Now we are going to apply the Ljusternik-Schnirelman principle (see [17, 18, 3]) to establish the existence of a sequence of positive eigenvalues for our eigenvalue problem.
Define on X the functionals

$$F(u)=\int_\Omega a(x)|u(x)|^p dx+\int_{\partial\Omega}b(s)|u(s)|^p ds \tag{2.2}$$

$$G(u)=\int_\Omega(|\nabla u(x)|^p+|u(x)|^p)dx+\int_{\partial\Omega}c(s)|u(s)|^p ds, \tag{2.3}$$

where $a \in L^\infty(\Omega)$ and $b, c \in L^\infty(\partial\Omega)$ such that $a, b, c \geq 0$. Consider the following eigenvalue problem $F'(u) = \mu G'(u)$, $u \in S_G$ and $\mu \in \mathbb{R}$ where S_G is the level

$$S_G = \{u \in W^{1,p}(\Omega) : G(u) = 1\}.$$

For any positive integer n, denote by $\mathbb{A}_n$ the class of all compact, symmetric subsets K of S_G such that $F(u) > 0$ on K and $\gamma(K) \geq n$, where $\gamma(K)$ denotes the genus of K, i.e.,

$$\gamma(K) := \inf\{k \in \mathbb{N} : \exists h : K \to \mathbb{R}^k \backslash \{0\} \text{ such that h is continuous and odd}\}.$$

F and G satisfies assumptions (H1)-(H4) in [6]. Then by [6, Theorem 2.1], we conclude that there exists a nonincreasing sequence of nonnegative eigenvalues $\{\mu_n\}$ obtained from the L-S principle such that $\mu_n \to 0$ as $n \to \infty$, where

$$\mu_n = \sup_{H \in \mathbb{A}_n} \inf_{u \in H} F(u)$$

and each μ_n is an eigenvalue of $F'(u) = \mu G'(u)$. Now, by choosing appropriate functions a, b, c and applying [6, Theorem 2.1] , we have the following :

Theorem 2.3. *Let F, G be the two functionals defined on $W^{1,p}(\Omega)$ in* (2.2), (2.3) *with $a(x) \equiv b(x) \equiv 1$ and $c(x) \equiv 1 + \beta(x)$. Then there exists a nondecreasing sequence of nonnegative eigenvalues $\{\lambda_n\}$ of* (2.1) *obtained from the L-S principle such that $\lambda_n = \frac{1}{\mu_n} - 1$ and $\lambda_n \to +\infty$ as $n \to +\infty$, where each μ_n is an eigenvalue of the corresponding equation $F'(u) = \mu G'(u)$ that satisfies $\mu_1 \geq \mu_2 \geq ... \geq \mu_k \geq ... > 0$ and $\lim_{k \to +\infty} \mu_k = 0$*

Proof. $F(u)$ and $G(u)$ become

$$F(u) = \int_\Omega |u(x)|^p dx + \int_{\partial\Omega} |u(s)|^p ds$$

$$G(u) = \int_\Omega (|\nabla u(x)|^p + |u(x)|^p) dx + \int_{\partial\Omega} (1 + \beta(s))|u(s)|^p ds.$$

Then $F'(u) = \mu G'(u)$ is equivalent to

$$\begin{aligned}\int_\Omega |u|^{p-2} uv\, dx + \int_{\partial\Omega} |u|^{p-2} uv\, ds &= \mu\Big(\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v\, dx + \int_\Omega |u|^{p-2} uv\, dx \\ &+ \int_{\partial\Omega} |u|^{p-2} uv\, ds + \int_{\partial\Omega} \beta(s)|u|^{p-2} uv\, ds\Big)\end{aligned}$$

for any $v \in W^{1,p}(\Omega)$; or

$$\int_\Omega |\nabla u|^{p-2} \nabla u \nabla v\, dx + \int_{\partial\Omega} \beta(s)|u|^{p-2} uv\, ds = \Big(\frac{1}{\mu} - 1\Big)\Big(\int_\Omega |u|^{p-2} uv\, dx + \int_{\partial\Omega} |u|^{p-2} uv\, ds\Big)$$

for any $v \in W^{1,p}(\Omega)$.

The last equation means that u is a weak solution of (1.1) associated to the eigenvalue $\frac{1}{\mu} - 1$. Combining (2.1) and the existence of the L-S sequence principle [6, Theorem 2.1], we obtain $\lambda_n = \frac{1}{\mu_n} - 1 \to +\infty$ as $n \to +\infty$. □

3 Regularity Results on Eigenfunctions

In this section we shall prove boundedness of eigenfunctions and use this fact to obtain $C^{1,\alpha}(\Omega)$ and $C^{1,\alpha}(\overline{\Omega})$ smoothness of weak eigenfunctions of the problem (1.1).
Let Ω be a bounded domain in $\mathbb{R}^N$ with C^1 boundary and $1 < p < +\infty$. First, we will show that eigenfunctions are in $L^\infty(\Omega)$.

Theorem 3.1. *Let* (u,λ) *be an eigensolution of the weak formulation* (2.1)*, then* $u \in L^\infty(\Omega)$*.*

Proof. By Sobolev's embedding theorem it suffices to consider the case $1 < p \leq N$, otherwise we would be done. We will use the Moser iteration technique (see [11]). Let us assume that $u \geq 0$. For $M > 0$ define $v_M(x) = \min\{u(x), M\}$ and $\phi = v_M^{kp+1}$ for $k > 0$, then $\nabla\phi = (kp+1)v_M^{kp}\nabla v_M$. It follows that $\phi \in W^{1,p}(\Omega)\cap L^\infty(\Omega)$ and $v_M|_{\partial\Omega} = \min\{u|_{\partial\Omega}, M\}$. Taking ϕ as a test function we have

$$\begin{aligned}(kp+1)\int_\Omega |\nabla u|^{p-2}\nabla u.\nabla v_M v_M^{kp}\,dx + \int_{\partial\Omega}\beta(s)|u|^{p-2}uv_M^{kp+1}\,ds &= \lambda\int_\Omega |u|^{p-2}uv_M^{kp+1}\,dx \\ &+ \lambda\int_{\partial\Omega}|u|^{p-2}uv_M^{kp+1}\,ds\end{aligned}$$

which implies that

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla v_M^{k+1}|^p\,dx + \int_{\partial\Omega}\beta(s)|u|^{p-2}uv_M^{kp+1}\,ds \leq \lambda\Big(\int_\Omega |u|^{(k+1)p}\,dx + \int_{\partial\Omega}|u|^{(k+1)p}\,ds\Big),$$

Letting $M \to +\infty$, and using Fatou's lemma we obtain

$$\frac{kp+1}{(k+1)^p}\int_\Omega |\nabla u^{k+1}|^p\,dx + \int_{\partial\Omega}\beta(s)|u|^{(k+1)p}\,ds \leq \lambda\Big(\int_\Omega |u|^{(k+1)p}\,dx + \int_{\partial\Omega}|u|^{(k+1)p}\,ds\Big),$$

Since $\frac{kp+1}{(k+1)^p} < 1$ for any $k > 0$, we conclude

$$\frac{kp+1}{(k+1)^p}\Big(\int_\Omega |\nabla u^{k+1}|^p\,dx + \int_{\partial\Omega}\beta(s)|u|^{(k+1)p}\,ds\Big) \leq \lambda\int_\Omega |u|^{(k+1)p}\,dx + \lambda\int_{\partial\Omega}|u|^{(k+1)p}\,ds,$$

thus

$$\frac{kp+1}{(k+1)^p}\|u^{k+1}\|_\beta^p \leq \lambda\|u^{k+1}\|_{L^p(\Omega)}^p + \lambda\|u^{k+1}\|_{L^p(\partial\Omega)}^p \tag{3.1}$$

Now by the multiplicative inequality and the Moser iteration [10, 11, 12] of the form

$$\|u\|_{L^p(\partial\Omega)}^p \leq \varepsilon\|u\|^p + C(\varepsilon)\|u\|_{L^p(\Omega)}^p \leq \varepsilon C'\|u\|_\beta^p + C(\varepsilon)\|u\|_{L^p(\Omega)}^p, \quad \forall\varepsilon > 0, \tag{3.2}$$

we obtain

$$\|u^{k+1}\|_{L^p(\partial\Omega)}^p \leq \varepsilon C'\|u^{k+1}\|_\beta^p + C(\varepsilon)\|u^{k+1}\|_{L^p(\Omega)}^p, \quad \forall\varepsilon > 0, \tag{3.3}$$

for a some positive constant C'.
Combining (3.1) and (3.3) we obtain

$$\Big(\frac{kp+1}{(k+1)^p} - \lambda\varepsilon C'\Big)\|u^{k+1}\|_\beta^p \leq \lambda(1 + C(\varepsilon))\|u^{k+1}\|_{L^p(\Omega)}^p.$$

Since $\varepsilon \to 0$, we may assume that $\frac{kp+1}{(k+1)^p} - \lambda\varepsilon C' > 0$; then

$$\|u\|_\beta \leq \Big(\lambda(1+C(\varepsilon)) \times \frac{1}{\frac{kp+1}{(k+1)^p} - \lambda\varepsilon C'}\Big)^{\frac{1}{(k+1)p}} \|u\|_{L^{(k+1)p}(\Omega)}. \tag{3.4}$$

By Sobolev's embedding theorem, there exists a constant $c_1 > 0$ such that

$$\|u^{k+1}\|_{L^{p^*}(\Omega)} \leq c_1 \|u^{k+1}\|_\beta,$$

and then

$$\|u\|_{L^{(k+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k+1}} \|u\|_\beta, \tag{3.5}$$

here we take $p^* = \frac{Np}{N-p}$ if $p < N$ and $p^* = 2p$, if $p = N$.
For any $k > 0$, we can using calculus find a constant $c_2 > 0$ such that

$$\Big(\lambda(1+C(\varepsilon)) \times \frac{1}{\frac{kp+1}{(k+1)^p} - \lambda\varepsilon C'}\Big)^{\frac{1}{p\sqrt{k+1}}} \leq c_2.$$

Thus

$$\|u\|_{L^{(k+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k+1}} c_2^{\frac{1}{\sqrt{k+1}}} \|u\|_{L^{(k+1)p}(\Omega)}. \tag{3.6}$$

Choosing k_1 such that $(k_1+1)p = p^*$, then taking $k = k_1$ in (3.6), it has

$$\|u\|_{L^{(k_1+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k_1+1}} c_2^{\frac{1}{\sqrt{k_1+1}}} \|u\|_{L^{p^*}(\Omega)}.$$

Next we choose k_2 such that $(k_2+1)p = (k_1+1)p^*$, then taking $k = k_2$ in (3.6), we have

$$\|u\|_{L^{(k_2+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k_2+1}} c_2^{\frac{1}{\sqrt{k_2+1}}} \|u\|_{L^{(k_1+1)p^*}(\Omega)}.$$

By induction we obtain

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \leq c_1^{\frac{1}{k_n+1}} c_2^{\frac{1}{\sqrt{k_n+1}}} \|u\|_{L^{(k_{n-1}+1)p^*}(\Omega)},$$

where the sequence (k_n) is chosen such that $(k_n+1)p = (k_{n-1}+1)p^*$, $k_0 = 0$. It is easy to see that $k_n + 1 = (\frac{p^*}{p})^n$, hence

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \leq c_1^{\Sigma_{i=1}^n \frac{1}{k_i+1}} c_2^{\Sigma_{i=1}^n \frac{1}{\sqrt{k_i+1}}} \|u\|_{L^{p^*}(\Omega)}.$$

As $\frac{p}{p^*} < 1$, there exists $C > 0$ such that for any $n = 1, 2, ...$

$$\|u\|_{L^{(k_n+1)p^*}(\Omega)} \leq C\|u\|_{L^{p^*}(\Omega)},$$

with $r_n = (k_n+1)p^* \to +\infty$ as $n \to +\infty$.
Now we will prove by contradiction that $u \in L^\infty(\Omega)$. Suppose $u \notin L^\infty(\Omega)$, then there exists $\varepsilon_1 > 0$ and a set A of positive measure in Ω such that $|u(x)| > C\|u\|_{L^{p^*}(\Omega)} + \varepsilon_1 = K$, for all $x \in A$. Hence

$$\liminf_{n\to\infty} \|u\|_{L^{r_n}(\Omega)} \geq \liminf_{n\to\infty} \Big(\int_A K^{r_n}\Big)^{1/r_n} = \liminf_{n\to\infty} K|A|^{1/r_n} = K > C\|u\|_{L^{p^*}(\Omega)},$$

which contradicts what has been established above.
If u (as an eigenfunction of (2.1)) changes sign, we consider u^+. It is well known that $u^+ \in W^{1,p}(\Omega)$. We define for each $M > 0$, $v_M(x) = \min\{u^+(x), M\}$. Taking again $\varphi = v_M^{kp+1}$ as a test function in $W^{1,p}(\Omega)$, we obtain

$$\begin{aligned}(kp+1)\int_\Omega |\nabla u^+|^{p-2}\nabla u^+ \nabla v_M v_M^{kp}\,dx + \int_{\partial\Omega}\beta(s)|u^+|^{p-2}u^+ v_M^{kp+1}\,ds &= \lambda\Big(\int_\Omega |u^+|^{p-2}u^+ v_M^{kp+1}\,dx \\ &+ \int_{\partial\Omega}|u^+|^{p-2}u^+ v_M^{kp+1}\,ds\Big).\end{aligned}$$

Proceeding the same way as above, we conclude that $u^+ \in L^\infty(\Omega)$. Similarly we have $u^- \in L^\infty(\Omega)$. Therefore $u = u^+ - u^- \in L^\infty(\Omega)$. □

Let Ω be a bounded domain in $\mathbb{R}^N$, $1 < p < \infty$. Consider the degenerate elliptic equation

$$-\Delta_p u(x) = f(x, u(x)) \text{ in } \Omega, \tag{3.7}$$

where $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function.
A function $u \in W^{1,p}_{loc}(\Omega)$ is called a weak solution of (3.7) if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega f(x,u) v dx \qquad \forall v \in C_0^\infty(\Omega).$$

The following result was established by DiBenedetto [4] and Tolksdorf [13].

Theorem 3.2. *Let u be a weak solution of* (3.7) *and let $g(x) = f(x, u(x))$ a.e. $x \in \Omega$. If $g \in L^q(\Omega)$ with $q > \frac{pN}{p-1}$, then $u \in C^{1,\alpha}(\Omega)$ for some $\alpha > 0$. In particular, the result holds if $g \in L^\infty(\Omega)$*

Combining Theorem 3.1 and Theorem 3.2 with $g(x) = |u(x)|^{p-2}u(x)$ in Ω, we obtain

Theorem 3.3. *If $u \in W^{1,p}(\Omega)$ is an eigenfunction of* (2.1)*, then u is in $C^{1,\alpha}(\Omega)$ for some $\alpha > 0$.*

Having proved that any weak eigenfunction of (1.1) is in $L^\infty(\Omega)$, we now can use boundary regularity results for solutions of degenerate elliptic equations in Liebermann [7] to obtain that u is in $C^{1,\alpha}(\overline{\Omega})$. We state the results as follow

Theorem 3.4. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $C^{1,\gamma}$ boundary with $0 < \gamma \le 1$. Let u be a bounded weak solution of the problem*

$$\begin{cases} -\Delta_p u = g(x) & \text{in } \Omega, \\ |\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = \Phi(x,u) & \text{on } \partial\Omega, \end{cases} \tag{3.8}$$

with $\|u\|_\infty \le M$. If g is in $L^\infty(\Omega)$ with $\|g\|_\infty \le K$ and Φ satisfies the condition

$$|\Phi(x,z) - \Phi(y,w)| \le L(|x-y|^\gamma + |z-w|^\gamma),\ |\Phi(x,z)| \le L,$$

for all (x,z) and (y,w) in $\partial\Omega \times [-M, M]$. Then there exists a positive constant $\alpha = \alpha(\gamma, N, p, M, K)$ such that $u \in C^{1,\alpha}(\overline{\Omega})$ and

$$\|u\|_{C^{1,\alpha}(\overline{\Omega})} \le C(\gamma, N, p, M, K, L, \Omega).$$

We recall that a weak solution u in $W^{1,p}(\Omega)$ of (3.8) satisfies

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla v dx = \int_\Omega g v dx + \int_{\partial\Omega} \Phi(x,u) v ds \ \forall v \in W^{1,p}(\Omega).$$

We observe that if we take $\Phi(x,u) = (\lambda-\beta)|u|^{p-2}u$ then Φ satisfies the hypotheses of Theorem 3.4 for any $0 < \gamma \le \min\{p-1,1\}$. Therefore if $\partial\Omega$ is of class $C^{1,\gamma}$ with $0 < \gamma \le 1$, then eigenfunctions of (1.1) is in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$.

4 Simplicity and Isolation of the First Eigenvalue

In this section we will prove that the first eigenvalue λ_1 is simple and isolated. Moreover, any associated eigenfunction does not change sign in $\overline{\Omega}$. In all that is to follow, we assume that Ω is a bounded domain in $\mathbb{R}^N$ with $C^{1,\gamma}$ boundary, $0 < \gamma \le 1$, and $1 < p < +\infty$. By (2.1), we have

$$\lambda_1 = \inf_{u \in X\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p dx + \int_{\partial\Omega} \beta(s)|u|^p ds}{\int_\Omega |u|^p dx + \int_{\partial\Omega} |u|^p ds}. \tag{4.1}$$

Proposition 4.1. *The eigenfunctions associated to λ_1 are either positive or negative in $\overline{\Omega}$.*

Proof. Let u_1 be an eigenfunction associated to λ_1. We have that $|u_1|$ is also a minimizer. It follows from the Harnack inequality that $|u_1| > 0$ in Ω and $|u_1| > 0$ is in $C^{1,\alpha}(\overline{\Omega})$ for some $\alpha > 0$. Thus if there exists $x_0 \in \partial\Omega$ such that $u_1(x_0) = 0$, then by the Hopf lemma (see [16, Theorem 5]) we obtain $\frac{\partial |u_1|}{\partial \nu}(x_0) < 0$. But the boundary condition $|\nabla u|^{p-2}\frac{\partial u}{\partial \nu} = -\beta|u|^{p-2}u + \lambda|u|^{p-2}u$ impose that $\frac{\partial |u_1|}{\partial \nu}(x_0) = 0$. This contradiction implies that $|u_1| > 0$ in $\overline{\Omega}$ which proves the proposition. □

For the proof of the simplicity of λ_1 we use the following "Picone's identity" proved in [2].

Lemma 4.2. *Let $v > 0, u \ge 0$ be two continuous functions in Ω differentiable a.e. Denote*

$$\begin{aligned} L(u,v) &= |\nabla u|^p + (p-1)\frac{u^p}{v^p}|\nabla v|^p - p\frac{u^{p-1}}{v^{p-1}}|\nabla v|^{p-2}\nabla v \nabla u, \\ R(u,v) &= |\nabla u|^p - |\nabla v|^{p-2}\nabla\Big(\frac{u^p}{v^{p-1}}\Big)\nabla v. \end{aligned}$$

Then **(i)** $L(u,v) = R(u,v)$, **(ii)** $L(u,v) \ge 0$ *a.e. and* **(iii)** $L(u,v) = 0$ *a.e. in Ω if and only if $u = kv$ for some $k \in \mathbb{R}$.*

Theorem 4.3. *The first eigenvalue λ_1 is simple, i.e., if u and v are two eigenfunctions associated with λ_1, then there exists k such that $u = kv$.*

Proof. Let u, v be two eigenfunctions associated to λ_1. We can assume without restriction that u and v are positive in Ω. For any $\varepsilon > 0$ we apply Picone's identity to the pair $u, v+\varepsilon$. We have

$$\begin{aligned} 0 &\le \int_\Omega L(u, v+\varepsilon) dx = \int_\Omega R(u, v+\varepsilon) dx \\ &= -\int_{\partial\Omega} \beta(s) u^p ds + \lambda_1 \int_\Omega u^p dx + \lambda_1 \int_{\partial\Omega} u^p ds - \int_\Omega |\nabla v|^{p-2}\nabla v \nabla\Big(\frac{u^p}{(v+\varepsilon)^{p-1}}\Big) dx \end{aligned} \tag{4.2}$$

Notice that $\frac{u^p}{(v+\varepsilon)^{p-1}} \in W^{1,p}(\Omega)$ is admissible in the weak formulation of v. Then it follows from (4.2) that

$$\begin{aligned} 0 &\le -\int_{\partial\Omega} \beta(s)u^p ds + \lambda_1 \int_\Omega u^p dx + \lambda_1 \int_{\partial\Omega} u^p ds + \int_{\partial\Omega} \beta(s)\Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p ds \\ &- \lambda_1 \int_\Omega \Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p dx - \lambda_1 \int_{\partial\Omega} \Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p ds \end{aligned}$$

By the Dominated Convergence Theorem, which also holds in $L^p(\partial\Omega)$ and letting $\varepsilon \to 0$ it follows that $L(u,v) = 0$. Then by Lemma 4.2, there exists $k \in \mathbb{R}$ such that $u = kv$. □

Proposition 4.4. *Let v be an eigenfunction associated with a positive eigenvalue $\lambda \neq \lambda_1$, then v changes sign in Ω.*

Proof. Suppose that v does not change sign in Ω, then we can assume that $v > 0$ in Ω. Let u be an eigenfunction associated with λ_1. Making similar computations as in the proof of Theorem 4.3, we obtain

$$\begin{aligned} 0 &\le -\int_{\partial\Omega} \beta(s)u^p ds + \lambda_1 \int_\Omega u^p dx + \lambda_1 \int_{\partial\Omega} u^p ds + \int_{\partial\Omega} \beta(s)\Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p ds \\ &- \lambda \int_\Omega \Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p dx - \lambda \int_{\partial\Omega} \Big(\frac{v}{v+\varepsilon}\Big)^{p-1} u^p ds \end{aligned}$$

Letting $\varepsilon \to 0$, we get

$$0 \le (\lambda_1 - \lambda)\Big(\int_\Omega u^p dx + \int_{\partial\Omega} u^p ds\Big)$$

which is impossible since $\lambda > \lambda_1$ and $\int_\Omega u^p dx + \int_{\partial\Omega} u^p ds > 0$. Therefore, u changes sign in Ω. □

Remark 4.5. λ_1 is the unique positive eigenvalue associated to an eigenfunction that does not change sign in Ω.

Proof. Let v an eigenfunction associated to $\lambda > 0$ with a constant sign in Ω, then we can assume that $v > 0$ in Ω. let u be an eigenfunction associated with λ_1. Making similar arguments as in the proof of Proposition 4.4, we obtain

$$0 \le (\lambda_1 - \lambda)\Big(\int_\Omega u^p dx + \int_{\partial\Omega} u^p ds\Big) \le 0$$

which implies that $\lambda = \lambda_1$. □

Theorem 4.6. *Let u be an eigenfunction corresponding to $\lambda > \lambda_1$, then u changes sign on $\Omega \cup \partial\Omega$. Moreover there exists a constant $C > 0$ such that*

$$|\Omega \cap \{u < 0\}|^{1-p/p^*} + |\partial\Omega \cap \{u < 0\}|^{1-p/p^\partial} \ge \frac{1}{\lambda C} \tag{4.3}$$

$$|\Omega \cap \{u > 0\}|^{1-p/p^*} + |\partial\Omega \cap \{u > 0\}|^{1-p/p^\partial} \ge \frac{1}{\lambda C} \tag{4.4}$$

where $p^* = \frac{Np}{N-p}$ and $p^\partial = \frac{(N-1)p}{N-p}$ if $1 < p < N$ and $p^* = 2p = p^\partial$ if $p \ge N$. Here $|A|$ denotes the measure of a subset A.

Proof. Let u be an eigenfunction corresponding to $\lambda > \lambda_1$, then by Proposition 4.4, u changes sign on Ω. Suppose that u does not changes sign on $\partial\Omega$, then we can assume that $u \le 0$ on $\partial\Omega$. Using u^+ as a test function in (2.1), we obtain that u^+ is also an eigenfunction associated to λ. Since $u^+ \not\equiv 0$ in Ω, then using Remark 4.5, we conclude that $\lambda = \lambda_1$ since u^+ has a constant sign. This is a contradiction.
To prove the inequality (4.3), We use u^- as a test function in the weak form of (2.1) satisfied by u. Then we have

$$\|u^-\|_\beta^p = \lambda\Big(\|u^-\|^p_{L^p(\Omega)} + \|u^-\|^p_{L^p(\partial\Omega)}\Big).$$

Now, by the Hölder inequality we have

$$\|u^-\|_\beta^p \le \lambda|\Omega\cap\{u<0\}|^{1-\frac{p}{p^*}}\Big(\int_\Omega |u^-|^{p^*}dx\Big)^{\frac{p}{p^*}} + \lambda|\partial\Omega\cap\{u<0\}|^{1-\frac{p}{p^\partial}}\Big(\int_{\partial\Omega}|u^-|^{p^\partial}ds\Big)^{\frac{p}{p^\partial}}.$$

By the Sobolev embedding $X \hookrightarrow L^{p^*}(\Omega)$ and $X \hookrightarrow L^{p^\partial}(\partial\Omega)$, there exists positive constant C_1, C_2, such that

$$\|u^-\|^p_{L^{p^*}(\Omega)} \le C_1\|u^-\|^p_\beta \quad \text{and} \quad \|u^-\|^p_{L^{p^\partial}(\Omega)} \le C_2\|u^-\|^p_\beta$$

Thus

$$\|u^-\|^p_\beta \le \lambda C_1|\Omega\cap\{u<0\}|^{1-\frac{p}{p^*}}\|u^-\|^p_\beta + \lambda C_2|\Omega\cap\{u<0\}|^{1-\frac{p}{p^\partial}}\|u^-\|^p_\beta,$$

which implies that

$$|\Omega\cap\{u<0\}|^{1-p/p^*} + |\partial\Omega\cap\{u<0\}|^{1-p/p^\partial} \ge \frac{1}{\lambda C}$$

where $C = \max\{C_1, C_2\}$. When $p \ge N$, we choose $p^* = 2p = p^\partial$ and we argue as before using the embedding $X \hookrightarrow L^{2p}(\Omega)$ and $X \hookrightarrow L^{2p}(\partial\Omega)$. A similar argument works for the inequality (4.4). □

Theorem 4.7. *The principal eigenvalue λ_1 is isolated, that is, there exists $\delta > 0$ such that in the interval $(\lambda_1, \lambda_1+\delta)$ there are no other eigenvalues of* (1.1).

Proof. Assume by contradiction that there exists a sequence of eigenvalues λ_n of (1.1) with $0 < \lambda_n \searrow \lambda_1$. Let u_n be an eigenfunction associated to λ_n. Since

$$0 < \int_\Omega |\nabla u_n|^p dx + \beta\int_{\partial\Omega}|u_n|^p ds = \lambda_n\Big(\int_\Omega |u_n|^p dx + \int_{\partial\Omega}|u_n|^p ds\Big),$$

we can define

$$v_n := \frac{u_n}{\big(\int_\Omega |u_n|^p dx + \int_{\partial\Omega}|u_n|^p ds\big)^{1/p}}.$$

v_n is bounded in $W^{1,p}(\Omega)$ so there exist a subsequence (still denoted v_n) and $v \in W^{1,p}(\Omega)$ such that $v_n \rightharpoonup v$ weakly in $W^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$. Moreover

$$\int_\Omega |v_n|^p dx + \int_{\partial\Omega}|v_n|^p ds = 1.$$

On the other hand

$$\int_\Omega |\nabla v|^p dx + \beta \int_{\partial\Omega} |v|^p ds \le \liminf_{n\to\infty} \int_\Omega |\nabla v_n|^p dx + \beta \int_{\partial\Omega} |v_n|^p ds = \lambda_1$$

and then by (4.1) we get

$$\int_\Omega |\nabla v|^p dx + \beta \int_{\partial\Omega} |v|^p ds = \lambda_1$$

and then v is an eigenfunction associated to λ_1. Using proposition 4.1 we obtain $v > 0$ or $v < 0$. In the case $v > 0$ (the other case is analogous) we conclude from the convergence in measure of the sequence v_n towards v that

$$|\Omega_n^-| \to 0 \text{ and } |(\partial\Omega_n)^-| \to 0 \tag{4.5}$$

where

$$\Omega_n^- := \{x \in \Omega : u_n(x) < 0\}, \text{ and } (\partial\Omega_n)^- := \{x \in \partial\Omega : u_n(x) < 0\}.$$

But (4.5) contradicts estimate (4.3). □

5 On the Second Eigenvalue

In this section we will show that the eigenvalue λ_2 obtained via L-S theory is actually the smallest eigenvalue of the spectrum that is greater than the principal eigenvalue λ_1. We need the following lemma.

Lemma 5.1. *[17] Let $(k,q) \in \mathbb{N}^* \times \mathbb{N}$ and let $\lambda \in \mathbb{R}$. If $\lambda_k = \lambda_{k+1} = ... = \lambda_{k+q}$, then $\gamma(K) \ge q+1$ where*

$$K := \{u \in S_G : u \textit{ is an eigenfunction associated to } \lambda_1\}$$

The above lemma is proved by applying a general result from infinite dimensional Ljusternik-Schnirelman theory.

Theorem 5.2.

$$\lambda_2 = \inf\{\lambda : \lambda \textit{ is an eigenvalue and } \lambda > \lambda_1\}$$

Proof. Firstly, note that $\gamma(K_1) = 1$ where K_1 is the set of eigenfunctions associated to λ_1. Thus by Lemma 5.1, $\lambda_1 < \lambda_2$. Now, it suffices to show that there is no eigenvalue α such that $\lambda_1 < \alpha < \lambda_2$. By contradiction, we assume that α is an eigenvalue associated with an eigenfunction u. Since $\alpha \ne \lambda_1$, we deduce that $u^+ \ne 0$ and $u^- \ne 0$. By multiplying respectively by u^+ and u^-, we obtain

$$\begin{aligned} \|u^+\|_\beta^p &= \alpha \int_\Omega |u^+|^p dx + \alpha \int_{\partial\Omega} |u^+|^p ds \\ \|u^-\|_\beta^p &= \alpha \int_\Omega |u^-|^p dx + \alpha \int_{\partial\Omega} |u^-|^p ds. \end{aligned} \tag{5.1}$$

Let $F_2 = span\{u^+, u^-\}$ be the sub vectorial space of X spanned by u^+ and u^- and $K_2 = S_G \cap F_2$ where

$$S_G = \{u \in X : G(u) = 1\}.$$

Let $au^+ + bu^- \in K_2$, we have

$$\begin{aligned} 1 &= G(au^+ + bu^-) \\ &= \int_\Omega (|\nabla(au^+ + bu^-)|^p + |(au^+ + bu^-)|^p)dx + \int_{\partial\Omega} (1+\beta(s))|(au^+ + bu^-)|^p ds \\ &= |a|^p\Big(\int_\Omega |\nabla u^+|^p dx + \int_{\partial\Omega} \beta(s)|u^+|^p ds\Big) + |a|^p\Big(\int_\Omega |u^+|^p dx + \int_{\partial\Omega} |u^+|^p ds\Big) \\ &+ |b|^p\Big(\int_\Omega |\nabla u^-|^p dx + \int_{\partial\Omega} \beta(s)|u^-|^p ds\Big) + |b|^p\Big(\int_\Omega |u^-|^p dx + \int_{\partial\Omega} |u^-|^p ds\Big) \end{aligned}$$

Combining with (5.1), we obtain

$$\begin{aligned} 1 &= |a|^p \||u^+\||_\beta^p + \frac{|a|^p}{\alpha}\||u^+\||_\beta^p + |b|^p\||u^-\||_\beta^p + \frac{|b|^p}{\alpha}\||u^-\||_\beta^p \\ &= \frac{\alpha+1}{\alpha}\Big(|a|^p\||u^+\||_\beta^p + |b|^p\||u^-\||_\beta^p\Big), \end{aligned}$$

and

$$\begin{aligned} \int_\Omega |au^+ + bu^-|^p dx + \int_{\partial\Omega} |au^+ + bu^-|^p ds &= |a|^p \int_\Omega |u^+| dx + |b|^p \int_\Omega |u^-| dx \\ &+ |a|^p \int_{\partial\Omega} |u^+| ds + |b|^p \int_{\partial\Omega} |u^-| ds \\ &= \frac{1}{\alpha}|a|^p\|u^+\|_\beta^p + \frac{1}{\alpha}|b|^p\|u^-\|_\beta^p \\ &= \frac{1}{\alpha+1}. \end{aligned}$$

Thus

$$\begin{aligned} \frac{1}{\alpha+1} &= \min_{v\in K_2}\Big(\int_\Omega |v|^p dx + \int_{\partial\Omega} |v|^p ds\Big) = \min_{v\in K_2} F(v) \\ &\leq \sup_{H\in\mathbb{A}_2} \min_{v\in H} F(v) \\ &= \mu_2 \\ &= \frac{1}{\lambda_2+1}, \end{aligned}$$

which implies that $\alpha \geq \lambda_2$. This is a contradiction. □

Remark 5.3. The proof of Theorem 5.2 shows that λ_2 is actually the smallest eigenvalue of the spectrum that is greater than λ_1. Moreover, it shows the isolation of the principal eigenvalue λ_1 by a direct way without using estimates in Theorem 4.6.

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