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Boundary Value Problems for Degenerate Coupled Systems with Variable Time Delay

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Abstract

The boundary value problems for coupled systems of parabolic and ordinary differential equations, where all equations contain time depended delay and degenerate at initial moment, are considered. Existence and uniqueness of classical solutions of these problems are proved. A priori estimates are obtained.

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1 Introduction

Evolution differential equations with time delay arise in modeling many dynamical real life problems, when response of the system is affected by the current state of the system as well as the past states of the system. The response of the system can be delayed, or depend on the past history of the system in a more complicated way and cause a time lag. Areas, where equations with delay are applied, include the study of materials with memory (viscoelastic materials); dynamics of artificial neural networks which have transmission delays, mathematical demography, and population dynamics. Delay terms can be of different types: constant, time dependent, state dependent etc. Ordinary differential equations with

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time delay are well studied for both constant and variable delay (see, e.g., [2], [5], [4], [6], [14], [17], [19], [20] and others). But in the case of partial differential equations (PDE) with delay, situation with there investigating is not that good. While PDE with constant delay are quite well studied (see, e.g., [3], [21], [22], [25], [26], [34] and others), PDEs with variable delay are still not widely investigated ([10], [11], [16], [33]).

Degenerate nonlinear differential equations used in modeling of different processes, including desalination seawater movement of liquids and gases in porous media. Such equations arise in theory of elasticity, relativity and optimization ([15]). Parabolic degenerate equation and problem for them are investigated in many papers (see., e.g., [12], [13], [15], [27])

Among mathematical models of process, in example, bacterial and cellular growth patterns, tumor growth and tissue development, some can be described by coupled systems. By coupled system we mean systems that contain equations of different type, in particular, systems of parabolic and ordinary differential equations (see, e.g., [1], [8], [11], [18], [24], [29], [35] and references therein). Significant researches of such systems were made by C.V. Pao (see, e.g., [30], [31], [32] and others). In particular, in the paper [30] the time-delayed coupled system was investigated using the method of upper-lower solution and the associated monotone iterations. This method allows obtain existence-comparison theorems. Similar results were obtained also in the papers [7], [18] and others. Note that a lot of results for parabolic equations with constant delay are obtained with the aid of the semi-group theory. Coupled systems with time dependent delay are studied in [11].

In this paper the initial-boundary problems for coupled systems of parabolic and ordinary differential equation, where all equations degenerate at initial time and contain time depended delay, are considered. This work can be considered as continuation of [11]. Uniqueness and existence of the classical solution of the problem are proved, the a priori estimates are obtained. The method similar as in [9] is used.

The paper is organized as follows. In Section 2 main notations and auxiliary facts are given. Statement of the problem and main result are given in Section 3. In Section 4 some auxiliary results are proved. In Section 5 the main results are proved.

2 Notations and Auxiliary Facts

Let $\mathbb{R}^k$, where $k \in \mathbb{N}$, be the standard linear space of ordered collections $z = (z_1, ..., z_k)$ of real numbers with the norm $|z| := (|z_1|^2 + \ldots + |z_k|^2)^{1/2}$. By notation $z^1 \le z^2$ (respectively, $z^1 < z^2$), where $z^1, z^2 \in \mathbb{R}^k$, we mean that $z_i^1 \le z_i^2$ (respectively, $z_i^1 < z_i^2$) for all $i \in \{1, ..., k\}$. By notation $z \le 0$ (respectively, $z \ge 0$), we mean that $z_i \le 0$ (respectively, $z_i \ge 0$) for all $i \in \{1, ..., k\}$.

Denote by $C(H)$, where H is an arbitrary domain in $\mathbb{R}^k$, the linear space of continuous on H functions. If K is a compact set in $\mathbb{R}^k$, then $C(K)$ is a Banach space with the norm $\|u\|_{C(K)} := \max\limits_{z \in K} |u(z)|$. A sequence of functions $\{u_m\}_{m=1}^{\infty}$ converges to u in $C(H)$, where H is arbitrary noncompact set in $\mathbb{R}^k$, if $\|u_m - u\|_{C(K)} \underset{m\to\infty}{\to} 0$ (i.e., $u_m \underset{m\to\infty}{\to} u$ in $C(K)$) for any compact $K \subset H$.

Let $\alpha \in (0,1]$, $n \in \mathbb{N}$, K is a compact set in $\mathbb{R}^{n+1} := \{(x,t) \mid x \in \mathbb{R}^n, t \in \mathbb{R}\}$. Denote by $C^{\alpha,\alpha/2}(K)$ the Banach subspace of space $C(K)$ of the functions $u(x,t)$, $(x,t) \in K$, with the finite norm

$$\|u\|^K_{\alpha,\alpha/2} := \|u\|_{C(K)} + \sup_{(x,t),(x',t)\in K} \frac{|u(x,t)-u(x',t)|}{|x-x'|^\alpha} + \sup_{(x,t),(x,t')\in K} \frac{|u(x,t)-u(x,t')|}{|t-t'|^{\alpha/2}}.$$

The space $C^{\alpha,\alpha/2}(K)$ is the Banach space and it is called the Hölder space (see, e.g. [28]).

By $C^{\alpha,\alpha/2}_{\text{loc}}(H)$, where H is an arbitrary noncompact domain in $\mathbb{R}^{n+1}$, we denote the space of functions u such that $u\in C^{\alpha,\alpha/2}(K)$ for any $K\subset H$.

Denote by $C^{2,1}(G)$ $\big($respectively, $C^{0,1}(G)\big)$, where G is a domain $\mathbb{R}^{n+1}$, a linear space of functions $v(x,t)$, $(x,t)\in G$, which along with their derivatives $v_{x_k}, v_{x_kx_l}$ $(k,l=\overline{1,n})$, v_t (respectively, with their derivative v_t) are define and continuous on G. Define $C^{2+\alpha,1+\alpha/2}(\overline{D})$ (respectively, $C^{\alpha,1+\alpha/2}(\overline{D})$), where D is bounded domain in $\mathbb{R}^{n+1}$, a Banach space of such functions v from $C^{2,1}(\overline{D})$ (respectively, $C^{0,1}(\overline{D})$) with finite norm

$$\|v\|^{\overline{D}}_{2+\alpha,1+\alpha/2} = \|v\|_{C(\overline{D})} + \sum_{k=1}^n \|v_{x_k}\|^{\overline{D}}_{\alpha,\alpha/2} + \sum_{k,l=1}^n \|v_{x_kx_l}\|^{\overline{D}}_{\alpha,\alpha/2} + \|v_t\|^{\overline{D}}_{\alpha,\alpha/2}$$

(respectively, $\|v\|^{\overline{D}}_{\alpha,1+\alpha/2} = \|v\|^{\overline{D}}_{\alpha,\alpha/2} + \|v_t\|^{\overline{D}}_{\alpha,\alpha/2}$). Denote by $C^{2+\alpha,1+\alpha/2}_{\text{loc}}(G)$ (respectively, $C^{\alpha,1+\alpha/2}_{\text{loc}}(G)$), where G is a domain $\mathbb{R}^{n+1}$ or a merge of a domain with part of its boundary, a space of such functions v that $v\in C^{2+\alpha,1+\alpha/2}(\overline{D})$ (respectively, $C^{\alpha,1+\alpha/2}(\overline{D})$) and for any arbitrary bounded domain D such that $\overline{D}\subset G$.

A direct corollary from the Arzela-Ascoli theorem is the following statement.

Proposition 2.1. *Let K be a compact set in $\mathbb{R}^{n+1}$ and $\{u_m\}_{m=1}^\infty$ be a bounded in $C^{\alpha,\alpha/2}(K)$ sequence of functions, i.e., $\|u_m\|^K_{\alpha,\alpha/2}\le C_1, \quad m\in\mathbb{N}$, where $C_1>0$ is a constant independent of m. Then there exist a function $u\in C^{\alpha,\alpha/2}(K)$ and a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of sequence $\{u_m\}_{m=1}^\infty$ such that $u_{m_j}\underset{j\to\infty}{\longrightarrow} u$ in $C(K)$.*

Using the diagonal method and Proposition 2.1 one can easily prove the following statement.

Proposition 2.2. *Let H be an arbitrary noncompact set in $\mathbb{R}^{n+1}$, and $H=\bigcup_{i=1}^\infty K_i$, where $\{K_i\}_{i=1}^\infty$ is a family of compact sets such that $K_i\subset K_{i+1}$ for all $i\in\mathbb{N}$. Suppose that $\{u_m\}_{m=1}^\infty$ is a sequence of functions from $C^{\alpha,\alpha/2}_{loc}(H)$ such that, for any $i\in\mathbb{N}$ the sequence of restrictions of the elements u_m on K_i is bounded in $C^{\alpha,\alpha/2}(K_i)$, i.e., $\|u_m\|^{K_i}_{\alpha,\alpha/2}\le C_2, \quad m\in\mathbb{N}$, where $C_2>0$ is a constant independent of m, but can depend on K_i. Then there exist a subsequence $\{u_{m_j}\}_{j=1}^\infty$ of the sequence $\{u_m\}_{m=1}^\infty$ and a function $u\in C^{\alpha,\alpha/2}_{loc}(H)$ such that $u_{m_j}\underset{j\to\infty}{\longrightarrow} u \quad$ in $C(H)$.*

3 Statement of the Problem and Main Results

Let n, M, L be natural numbers; Ω be a bounded domain in $\mathbb{R}^n\,(n\ge 1)$ with the boundary $\partial\Omega$; $T>0$; $Q:=\Omega\times(0,T]$, $\widetilde{Q}:=\overline{\Omega}\times(0,T]$, $\Sigma:=\partial\Omega\times(0,T]$.

Consider a system

$$P_i w(x,t) := p_i(x,t)\frac{\partial u_i(x,t)}{\partial t} - \sum_{k,l=1}^{n} a_{i,lk}(x,t)\frac{\partial u_i(x,t)}{\partial x_k \partial x_l} + \sum_{k=1}^{n} a_{i,k}(x,t)\frac{\partial u_i(x,t)}{\partial x_k}$$
$$+a_i(x,t)u_i(x,t) - g_i(x,t,w(x,t),w_\tau(x,t)) = f_i(x,t), \quad (x,t)\in Q,\ i=1,...,M, \tag{3.1}$$

$$P_{M+j}w(x,t) := q_j(x,t)\frac{\partial v_j(x,t)}{\partial t} + b_j(x,t)v_j(x,t)$$
$$-g_{M+j}(x,t,w(x,t),w_\tau(x,t)) = f_{M+j}(x,t), \quad (x,t)\in \widetilde{Q}, \quad j=1,...,L, \tag{3.2}$$

where $w_\tau(x,t) := (u_1(x,t-\tau_1(t)),\dots,u_M(x,t-\tau_M(t)); v_1(x,t-\tau_{M+1}(t)),\dots,v_L(x,t-\tau_{M+L}(t)))$, and τ_s $(s=1,...,M+L)$ are continuous nonnegative functions on $(0,T]$ such that $\tau_s(t) < t$ for all $t\in(0,T]$, $s\in\{1,\dots,M+L\}$.

By W we denote the set of vector-functions $w=(u_1,\dots,u_M;v_1,\dots,v_L)$ such that $u_i \in C(\widetilde{Q})\cap C^{2,1}(Q)$ $(i=1,...,M)$, $v_j\in C^{0,1}(\widetilde{Q})$ $(j=1,...,L)$.

Consider the ***problem*** of finding a vector-function $w\in W$ that satisfies system (3.1), (3.2), boundary conditions

$$w_i(x,t) = h_i(x,t), \quad (x,t)\in\Sigma, \quad i=1,...,M, \tag{3.3}$$

and analogy of initial conditions

$$\limsup_{t\to 0+}\big(\max_{x\in\overline{\Omega}}|w_r(x,t)|\big) < \infty, \quad r=1,\dots,M+L \tag{3.4}$$

(note that condition (3.4) is equal to condition $\sup_{(x,t)\in\widetilde{Q}}|w_r(x,t)|<\infty$, $r=1,\dots,M+L$).

We assume that the initial data of problem (3.1)–(3.4) satisfy the following conditions:

$(\mathcal{A}_1)$ $a_{i,kl}=a_{i,lk}$, $a_{i,k}$, a_i $(i=1,...,M;\ k,l=1,...,n)$ are continuous functions on Q, and for each $i\in\{1,\dots,M\}$

$$\sum_{k,l=1}^{n} a_{i,kl}(x,t)\xi_k\xi_l \ge \mu_i(t)\sum_{k=1}^{n}\xi_k^2 \quad \forall (x,t)\in Q,\ \forall \xi\in\mathbb{R}^n, \text{ where } \mu_i(t)\ge 0\ \forall t\in(0,T];$$

b_j $(j=1,...,L)$ are continuous functions on $\widetilde{Q}$;

$(\mathcal{A}_2)$ $p_i : Q\to\mathbb{R}$, $q_j:\widetilde{Q}\to\mathbb{R}$ are continuous positive functions such that $\lim_{t\to 0} p_i(x,t) = 0$, $x\in\Omega$, $\lim_{t\to 0} q_j(x,t) = 0$, $x\in\overline{\Omega}$, and there exists a function $\varphi\in C((0,T])$, which satisfies conditions: $\varphi(t)>0$ when $t\in(0,T]$,

$$\int_0^T \varphi(s)ds = +\infty, \quad \sup_{t\in(0,T]}\int_{t-\tau_k(t)}^{t}\varphi(s)ds < \infty\ (k=1,...,M+L) \tag{3.5}$$

and $\sup_{(x,t)\in Q} p_i\varphi < \infty$, $\sup_{(x,t)\in\widetilde{Q}} q_j\varphi<\infty$ $(i=1,..,M,\ j=1,...,L)$.

($\mathcal{A}_3$) $g_i(x,t,\xi,\eta)$, $(x,t,\xi,\eta)\in\Omega\times(0,T]\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L}$ $(i=1,\dots,M)$, $g_{M+j}(x,t,\xi,\eta)$, $(x,t,\xi,\eta)\in\overline{\Omega}\times(0,T]\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L}$ $(j=1,\dots,L)$, are continuous, and continuously differentiable by the variables ξ and η, functions, and there exist functions $g^1_{r,s}, g^2_{r,s}$ $(r,s=1,\dots,M+L)$ such that

$$0\le\frac{\partial g_r}{\partial\xi_s}(x,t,\xi,\eta)\le g^1_{r,s}(x,t)\quad\forall(x,t)\in Q,\ \forall\xi,\eta\in\mathbb{R}^{M+L},$$

$$0\le\frac{\partial g_r}{\partial\eta_s}(x,t,\xi,\eta)\le g^2_{r,s}(x,t)\quad\forall(x,t)\in Q,\ \forall\xi,\eta\in\mathbb{R}^{M+L},$$

$$\inf_{(x,t)\in Q}\Big[a_i(x,t)-\sum_{s=1}^{M+L}g^1_{i,s}(x,t)\Big]=:a_i^->0,\quad i=1,\dots,M, \tag{3.6}$$

$$\inf_{(x,t)\in\widetilde{Q}}\Big[b_j(x,t)-\sum_{s=1}^{M+L}g^1_{M+j,s}(x,t)\Big]=:b_j^->0,\quad j=1,\dots,L, \tag{3.7}$$

$$\sup_{(x,t)\in Q}\sum_{s=1}^{M+L}g^2_{r,s}(x,t)=:g_r^{2,+}<\infty,\quad r=1,\dots,M+L;$$

moreover, $g_r(x,t,0,0)=0$, $(x,t)\in Q\,(r=1,\dots,M+L)$;

($\mathcal{A}_4$) $f_i\in C(Q)$ $(i=1,\dots,M)$, $f_{M+j}\in C(\widetilde{Q})$ $(j=1,\dots,L)$, $h_i\in C(\Sigma)$ $(i=1,\dots,M)$, moreover, functions f_r,h_i are bounded $(r=1,\dots,M+L,\ i=1,\dots,M)$.

Denote

$$Pw:=(P_1w,\dots,P_Mw,P_{M+1}w,\dots,P_{M+L}w),\quad Rw:=(R_1w,\dots,R_Mw),$$

$$f:=(f_1,\dots,f_M,\dots,f_{M+L}),\quad h:=(h_1,\dots,h_M).$$

Theorem 3.1. *Let conditions* ($\mathcal{A}_1$) – ($\mathcal{A}_3$) *be satisfied and*

$$a_i^--g_i^{2,+}>0,\quad b_j^--g_{M+j}^{2,+}>0\quad(i=1,\dots,M;\ j=1,\dots,L). \tag{3.8}$$

Suppose that w^1,w^2 are solutions of problems that differ from problem (3.1)–(3.4) only in having f^1,h^1 and f^2,h^2 instead of f,h, respectively, with the properties as in ($\mathcal{A}_4$) for f,h, respectively. Then the following inequalities hold

$$\min\Big\{\frac{1}{a_i^--g_i^{2,+}}\inf_{(y,s)\in Q}(f_i^1(y,s)-f_i^2(y,s)),\inf_{(y,s)\in\Sigma}(h_i^1(y,s)-h_i^2(y,s)),0\Big\}$$
$$\le u_i^1(x,t)-u_i^2(x,t)$$
$$\le\max\Big\{\frac{1}{a_i^--g_i^{2,+}}\sup_{(y,s)\in Q}(f_i^1(y,s)-f_i^2(y,s)),\sup_{(y,s)\in\Sigma}(h_i^1(y,s)-h_i^2(y,s)),0\Big\},$$
$$(x,t)\in Q,\ i\in\{1,\dots,M\}, \tag{3.9}$$

$$\min\Big\{\frac{1}{b_j^--g_{M+j}^{2,+}}\inf_{(y,s)\in\widetilde{Q}}(f^1_{M+j}(y,s)-f^2_{M+j}(y,s)),0\Big\}\le v_j^1(x,t)-v_j^2(x,t)$$
$$\le\max\Big\{\frac{1}{b_j^--g_{M+j}^{2,+}}\sup_{(y,s)\in\widetilde{Q}}(f^1_{M+j}(y,s)-f^2_{M+j}(y,s)),0\Big\},\quad(x,t)\in\widetilde{Q},\ j\in\{1,\dots,L\}. \tag{3.10}$$

Corollary 3.2. *Let conditions of Theorem 3.1 hold and, besides,*

$$f_r^1(x,t) \le f_r^2(x,t) \quad \forall (x,t) \in Q \quad (r = 1,\dots,M+L),$$

$$h_i^1(x,t) \le h_i^2(x,t) \quad \forall (x,t) \in \Sigma \quad (i = 1,\dots,M).$$

Then inequalities $w_r^1(x,t) \le w_r^2(x,t)\ \forall (x,t) \in Q\ (r = 1,\dots,M+L)$ *hold.*

Corollary 3.3. *Let conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ *and (3.8) hold. Then a solution* $w = (u_1,...,u_M; v_1,...,v_L)$ *of problem (3.1)–(3.4) satisfies the following estimate*

$$\forall i \in \{1,\dots,M\}: \quad \min\{\frac{1}{a_i^- - g_i^{2,+}} \inf_{(y,s)\in Q} f_i(y,s), \inf_{(y,s)\in\Sigma} h_i(y,s), 0\}$$

$$\le u_i(x,t) \le \max\{\frac{1}{a_i^- - g_i^{2,+}} \sup_{(y,s)\in Q} f_i(y,s), \sup_{(y,s)\in\Sigma} h_i(y,s), 0\}, \quad (x,t) \in Q, \tag{3.11}$$

$$\forall j \in \{1,\dots,L\}: \quad \min\{\frac{1}{b_j^- - g_{M+j}^{2,+}} \inf_{(y,s)\in\widetilde{Q}} f_{M+j}(y,s), 0\}$$

$$\le v_j(x,t) \le \max\{\frac{1}{b_j^- - g_{M+j}^{2,+}} \sup_{(y,s)\in\widetilde{Q}} f_{M+j}(y,s), 0\}, \quad (x,t) \in \widetilde{Q}. \tag{3.12}$$

Corollary 3.4. *Let conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ *and (3.8) hold. Then the solution of problem (3.1)–(3.4) is unique.*

Denote by $C^{\alpha,\alpha/2}_{\mathrm{loc}}(Q)$ a space of functions $v \in C(Q)$ such that for any strictly internal subdomain Ω' of domain Ω (e.i., $\overline{\Omega'} \subset \Omega$) and any number $\delta \in (0,T)$ a restriction of v on $\overline{\Omega'} \times [\delta,T]$ belongs to $C^{2+\alpha,1+\alpha/2}(\overline{\Omega'} \times [\delta,T])$, and by $C^{\alpha,\alpha/2}_{\mathrm{loc}}(\widetilde{Q})$ (respectively, $C^{\alpha,\alpha/2}_{\mathrm{loc}}(\Sigma)$) – a space of functions $v \in C(\widetilde{Q})$ (respectively, $C(\Sigma)$) such that for any number $\delta \in (0,T)$ a restriction of v on $\overline{\Omega} \times [\delta,T]$ (respectively, $\partial\Omega \times [\delta,T]$) belongs to $C^{\alpha,\alpha/2}(\overline{\Omega} \times [\delta,T])$ (respectively, $C^{\alpha,\alpha/2}(\partial\Omega \times [\delta,T])$).

Denote by $C^{2+\alpha,1+\alpha/2}_{\mathrm{loc}}(Q)$ (respectively, $C^{\alpha,1+\alpha/2}_{\mathrm{loc}}(\widetilde{Q})$) a space of functions $v \in C^{2,1}(Q)$ (respectively, $v \in C^{0,1}(\widetilde{Q})$) such that their derivatives v_{x_k}, $v_{x_k x_l}$ $(k,l = \overline{1,n})$, v_t (respectively, derivative v_t) belong to $C^{\alpha,\alpha/2}_{\mathrm{loc}}(Q)$ (respectively, $C^{\alpha,\alpha/2}_{\mathrm{loc}}(\widetilde{Q})$).

Denote by $C^{\alpha,\alpha/2,1,1}_{\mathrm{loc}}(\overline{\Omega}\times(0,T]\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L})$ a space of continuous functions $\widetilde{g}(x,t,\xi,\eta)$, $(x,t,\xi,\eta) \in \overline{\Omega}\times(0,T]\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L}$, that are continuously differentiable by the variables ξ,η and these derivatives are bounded, and for any $\delta \in (0,T)$, for some constant $L \ge 0$ and for any$(x,t),(x',t') \in \overline{\Omega}\times[\delta,T]$, $(\xi,\eta) \in \mathbb{R}^{M+L}\times\mathbb{R}^{M+L}$ satisfy the inequality

$$\left|\widetilde{g}(x,t,\xi,\eta) - \widetilde{g}(x',t',\xi,\eta)\right| \le L(|x-y|^{\alpha} + |t-s|^{\alpha/2}).$$

Denote by $\mathrm{Lip}_{\mathrm{loc}}((0,T])$ a space of functions, which satisfy Lipschitz condition on each closed interval on $(0,T]$.

Theorem 3.5. *Let conditions* $(\mathcal{A}_1)$–$(\mathcal{A}_4)$ *and* (3.8) *hold. Suppose that for some* $\alpha \in (0,1]$
$(\mathcal{B}_1)$ $\partial\Omega \in C^{2+\alpha}$,

($\mathcal{B}_2$) *p_i, $a_{i,kl}$, $a_{i,k}$, a_i, q_j, $b_j \in C^{\alpha,\alpha/2}_{loc}(\widetilde{Q})$, $g_r \in C^{\alpha,\alpha/2,1,1}_{loc}(\overline{\Omega}\times(0,T]\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L})$, $f_r \in C^{\alpha,\alpha/2}_{loc}(\widetilde{Q})$, $h_i \in C^{2+\alpha,1+\alpha/2}_{loc}(\Sigma)$ $(r=1,\dots,M+L;\ i=1,\dots,M;\ j=1,\dots,L)$.*

Moreover,

($\mathcal{B}_3$) *$\partial a_{i,kl}/\partial x_r \in C(Q)$ $(k,l,r=1,\dots,n;\ i=1,\dots,M)$,*

($\mathcal{B}_4$) *$\tau_s \in Lip_{loc}((0,T])\,(s=1,\dots,M+L)$, $\inf\limits_{t\in(0,T]}\mu_i(t)>0$ $(i=1,\dots,M)$.*

Then there exists the unique solution $w=(u_1,\dots,u_M;v_1,\dots,v_L)$ of problem (3.1)–(3.4)*, and $u_i \in C^{\alpha,\alpha/2}_{loc}(\widetilde{Q})\cap C^{2+\alpha,1+\alpha/2}_{loc}(Q)$, $v_j \in C^{\alpha,\alpha/2}(\widetilde{Q})\cap C^{\alpha,1+\alpha/2}_{loc}(\widetilde{Q})$ $(i=1,\dots,M;\ j=1,\dots,L)$ and also estimates (3.11), (3.12) hold.*

4 Auxiliary Results

Consider problem of finding a vector-function $w=(u_1,\dots,u_M,\ v_1,\dots,v_L)\in W$, which satisfies the system

$$p_i(x,t)\frac{\partial u_i(x,t)}{\partial t}-\sum_{k,l=1}^{n}a_{i,lk}(x,t)\frac{\partial u_i(x,t)}{\partial x_k\partial x_l}+\sum_{k=1}^{n}a_{i,k}(x,t)\frac{\partial u_i(x,t)}{\partial x_k}+a_i(x,t)u_i(x,t)$$
$$-\sum_{s=1}^{M+L}\widetilde{g}^1_{i,s}(x,t)w_s(x,t)-\sum_{s=1}^{M+L}\widetilde{g}^2_{i,s}(x,t)w_{s,\tau_s}(x,t)=f_i(x,t),\quad (x,t)\in Q,\quad i=1,\dots,M,\tag{4.1}$$

$$q_j(x,t)\frac{\partial v_j(x,t)}{\partial t}+b_j(x,t)v_j(x,t)-\sum_{s=1}^{M+L}\widetilde{g}^1_{M+j,s}(x,t)w_s(x,t)$$
$$-\sum_{s=1}^{M+L}\widetilde{g}^2_{M+j,s}(x,t)w_{s,\tau_s}(x,t)=f_{M+j}(x,t),\quad (x,t)\in\widetilde{Q},\quad j=1,\dots,L,\tag{4.2}$$

and conditions (3.3), (3.4).

Suppose that functions $p_i, a_{i,kl}, a_{i,k}, a_i, q_j, b_j, f_r, \tau_r, h_i$ $(r=1,\dots,M+L;\ i=1,\dots,M;\ j=1,\dots,L;\ k,l=\overline{1,n})$ satisfy conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}_4$) and functions $\widetilde{g}^1_{r,s}$, $\widetilde{g}^2_{r,s}$ $(r,s=1,\dots,M+L)$ satisfy condition

($\mathcal{A}^*_3$) $\widetilde{g}^1_{i,s}, \widetilde{g}^2_{i,s}\in C(Q), \widetilde{g}^1_{M+j,s}, \widetilde{g}^2_{M+j,s}\in C(\widetilde{Q})$,

$\widetilde{g}^1_{i,s}\ge 0$, $\widetilde{g}^2_{i,s}\ge 0$ on Q, $\widetilde{g}^1_{M+j,s}\ge 0$, $\widetilde{g}^2_{M+j,s}\ge 0$ on $\widetilde{Q}$ $(i=1,\dots,M;\ j=1,\dots,L;\ s=1,\dots,M+L;)$,

$\inf\limits_{(x,t)\in Q}\big(a_i(x,t)-\sum\limits_{s=1}^{M+L}\widetilde{g}^1_{i,s}(x,t)\big)=:\widetilde{a}^-_i>-\infty\quad(i=1,\dots,M)$,

$\inf\limits_{(x,t)\in\widetilde{Q}}\big(b_j(x,t)-\sum\limits_{s=1}^{M+L}\widetilde{g}^1_{M+j,s}(x,t)\big)=:\widetilde{b}^-_j>-\infty\quad(j=1,\dots,L)$,

$\sup\limits_{(x,t)\in Q}\sum\limits_{s=1}^{M+L}\widetilde{g}^2_{r,s}(x,t)=:\widetilde{g}^{2,+}_r<+\infty\quad(r=1,\dots,M+L)$.

Proposition 4.1. *Let conditions ($\mathcal{A}_1$), ($\mathcal{A}_2$), ($\mathcal{A}^*_3$), ($\mathcal{A}_4$) and*

$$\widetilde{a}^-_i-\widetilde{g}^{2,+}_i>0,\quad \widetilde{b}^-_j-\widetilde{g}^{2,+}_{M+j}>0\quad(i=1,\dots,M;\ j=1,\dots,L).\tag{4.3}$$

Then a solution w of problem (4.1), (4.2), (3.3), (3.4) satisfies the following estimate

$$\forall i \in \{1,\dots,M\}: \quad \min\Big\{\frac{1}{\widetilde{a}_i^- - \widetilde{g}_i^{2,+}} \inf_{(y,s)\in Q} f_i(y,s), \inf_{(y,s)\in\Sigma} h_i(y,s), 0\Big\}$$

$$\le u_i(x,t) \le \max\Big\{\frac{1}{\widetilde{a}_i^- - \widetilde{g}_i^{2,+}} \sup_{(y,s)\in Q} f_i(y,s), \sup_{(y,s)\in\Sigma} h_i(y,s), 0\Big\}, \quad (x,t)\in Q, \tag{4.4}$$

$$\forall j \in \{1,\dots,L\}: \min\Big\{\frac{1}{\widetilde{b}_j^- - \widetilde{g}_{M+j}^{2,+}} \inf_{(y,s)\in\widetilde{Q}} f_{M+j}(y,s), 0\Big\}$$

$$\le v_j(x,t) \le \max\Big\{\frac{1}{\widetilde{b}_j^- - \widetilde{g}_{M+j}^{2,+}} \sup_{(y,s)\in\widetilde{Q}} f_{M+j}(y,s), 0\Big\}, \quad (x,t)\in\widetilde{Q}. \tag{4.5}$$

Proof. We set the following notations:

$$\theta(t) := \int_T^t \varphi(\rho)d\rho, \quad \varkappa_s(t) := \int_{t-\tau_s(t)}^t \varphi(\rho)d\rho, \quad t\in(0,T], \quad s=1,\dots,M+L. \tag{4.6}$$

It is obvious that $\theta(t)\le 0$ for all $t\in(0,T]$, and θ is monotonously growing on $(0,T]$, $\theta(T)=0$, $\theta(t)\to-\infty$ when $t\to-\infty$; $\varkappa_s(t)\ge 0$ for all $t\in(0,T]$, and $\varkappa_s$ are bounded ($s=1,\dots,M+L$).

Let w be a solution of problem (4.1), (4.2), (3.3), (3.4). Denote by $M\ge 0$ a constant such that

$$|w(x,t)| \le M, \;\; (x,t)\in\widetilde{Q}, \tag{4.7}$$

and by w^μ we denote a function such that

$$w^\mu(x,t) = w(x,t)e^{\mu\theta(t)}, \quad (x,t)\in\widetilde{Q}, \tag{4.8}$$

that is, $w(x,t) = w^\mu(x,t)e^{-\mu\theta(t)}$, $(x,t)\in\widetilde{Q}$, where $\mu>0$ is, for now, an arbitrary number.

From equalities (4.1), (4.2), taking into account equalities

$$w_{r,t}(x,t) = w^\mu_{r,t}(x,t)e^{-\mu\theta(t)} - \mu\varphi(t)w^\mu_r(x,t)e^{-\mu\theta(t)}, \quad r=1,\dots,M+L,$$

$$u_{i,x_k}(x,t) = u^\mu_{i,x_k}(x,t)e^{-\mu\theta(t)}, \quad u_{i,x_kx_l}(x,t) = u^\mu_{i,x_kx_l}(x,t)e^{-\mu\theta(t)}, \quad k=1,\dots,n,\; i=1,\dots,M,$$

$$w_{s,\tau_s}(x,t) = w^\mu_{s,\tau_s}(x,t)e^{-\mu\int_T^{t-\tau_s(t)}\varphi(\rho)d\rho} \equiv e^{\mu\varkappa_s(t)}w^\mu_{s,\tau_s}(x,t)e^{-\mu\theta(t)}, \quad s=1,\dots,M+L,$$

we obtain

$$p_i(x,t)\frac{\partial u_i^\mu(x,t)}{\partial t} - \sum_{k,l=1}^n a_{i,lk}(x,t)\frac{\partial u_i^\mu(x,t)}{\partial x_k\partial x_l} + \sum_{k=1}^n a_{i,k}(x,t)\frac{\partial u_i^\mu(x,t)}{\partial x_k} + a_i^\mu(x,t)u_i^\mu(x,t)$$

$$-\sum_{s=1}^{M+L}\widetilde{g}^1_{i,s}(x,t)w^\mu_s(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^{2,\mu}_{i,s}(x,t)w^\mu_{s,\tau_s}(x,t) = f_i^\mu(x,t), \quad (x,t)\in Q, \quad i=1,\dots,M,$$

$$q_j(x,t)\frac{\partial v_j^\mu(x,t)}{\partial t} + b_j^\mu(x,t)v_j^\mu(x,t) - \sum_{s=1}^{M}\widetilde{g}^1_{M+j,s}(x,t)w^\mu_s(x,t)$$

$$-\sum_{s=1}^{L}\widetilde{g}^{2,\mu}_{M+j,s}(x,t)w^\mu_{s,\tau_s}(x,t) = f^\mu_{M+j}(x,t), \quad (x,t)\in\widetilde{Q}, \quad j=1,\dots,L, \tag{4.9}$$

where

$$a_i^\mu(x,t) := a_i(x,t) - \mu p_i(x,t)\varphi(t) \quad (i = 1,\dots,M),$$
$$b_j^\mu(x,t) := b_j(x,t) - \mu q_j(x,t)\varphi(t) \quad (j = 1,\dots,L),$$
$$\widetilde{g}_{r,s}^{2,\mu}(x,t) := \widetilde{g}_{r,s}^{2}(x,t)e^{\mu\varkappa_s(t)}, \quad f_r^\mu(x,t) := f_r(x,t)e^{\mu\theta(t)} \quad (r,s = 1,\dots,M+L). \tag{4.10}$$

From condition (3.3) and correlation (4.8) we obtain

$$u_i^\mu(x,t) = h_i^\mu(x,t), \quad (x,t) \in \Sigma, \tag{4.11}$$

where $h_i^\mu(x,t) := h_i(x,t)e^{\mu\theta(t)}$, $(x,t) \in \Sigma$ $(i = 1,\dots,M)$.

Let $\varepsilon \in (0,T)$ be an arbitrary number. Denote by $E_{s,\varepsilon}$ $(s = 1,\dots,M+L)$ a set, which contains of numbers $t-\tau_s(t)$ such that $t-\tau_s(t) < \varepsilon$ when $t \ge \varepsilon$, and also number ε. Denote

$$Q_\varepsilon := \Omega\times(\varepsilon,T], \quad \overline{Q_\varepsilon} := \overline{\Omega}\times[\varepsilon,T], \quad \Sigma_\varepsilon := \partial\Omega\times(\varepsilon,T].$$

Consider a problem of finding a function $w^{\mu,\varepsilon} = (u_1^{\mu,\varepsilon},\dots,u_M^{\mu,\varepsilon}; v_1^{\mu,\varepsilon},\dots,v_L^{\mu,\varepsilon})$ such that $u_i^{\mu,\varepsilon} \in C(\overline{\Omega}\times(E_{i,\varepsilon}\cup(\varepsilon,T]))\cap C^{2,1}(Q_\varepsilon)$ $(i = 1,...,M)$, $v_j^{\mu,\varepsilon} \in C(\overline{\Omega}\times(E_{M+j,\varepsilon}\cup(\varepsilon,T]))\cap C^{0,1}(\overline{Q_\varepsilon})$ $(j = 1,...,L)$, and which satisfies system

$$p_i(x,t)\frac{\partial u_i^{\mu,\varepsilon}(x,t)}{\partial t} - \sum_{k,l=1}^{n} a_{i,lk}(x,t)\frac{\partial u_i^{\mu,\varepsilon}(x,t)}{\partial x_k\partial x_l} + \sum_{k=1}^{n} a_{i,k}(x,t)\frac{\partial u_i^{\mu,\varepsilon}(x,t)}{\partial x_k} + a_i^\mu(x,t)u_i^{\mu,\varepsilon}(x,t)$$
$$- \sum_{s=1}^{M+L}\widetilde{g}_{i,s}^{1}(x,t)w_s^{\mu,\varepsilon}(x,t) - \sum_{s=1}^{M+L}\widetilde{g}_{i,s}^{2,\mu}(x,t)w_{s,\tau_s}^{\mu,\varepsilon}(x,t) = f_i^\mu(x,t), \quad (x,t)\in Q_\varepsilon, \quad i = 1,\dots,M, \tag{4.12}$$

$$q_j(x,t)\frac{\partial v_j^{\mu,\varepsilon}(x,t)}{\partial t} + b_j^\mu(x,t)v_j^{\mu,\varepsilon}(x,t) - \sum_{s=1}^{M+L}\widetilde{g}_{M+j,k}^{1}(x,t)w_s^{\mu,\varepsilon}(x,t)$$
$$- \sum_{s=1}^{M+L}\widetilde{g}_{M+j,k}^{2,\mu}(x,t)w_{s,\tau_s}^{\mu,\varepsilon}(x,t) = f_{M+j}^\mu(x,t), \quad (x,t)\in\overline{Q}_\varepsilon, \quad j = 1,\dots,L, \tag{4.13}$$

boundary condition

$$u_i^{\mu,\varepsilon}(x,t) = h_i^\mu(x,t), \quad (x,t)\in\Sigma_\varepsilon,\ i = 1,\dots,M, \tag{4.14}$$

and initial condition

$$w_r^{\mu,\varepsilon}(x,t) = w_r^\mu(x,t), \quad (x,t)\in\overline{\Omega}\times E_{r,\varepsilon},\ r = 1,\dots,M+L. \tag{4.15}$$

In this problem the equations are not degenerated, therefore we can use results from [11]. Ensure that for problem (4.12)–(4.15), with small enough values of μ, conditions of Corollary 2 [11] are valid.

It is obvious that from conditions $\widetilde{g}_{r,s}^{2} \ge 0$ follow conditions $\widetilde{g}_{r,s}^{2,\mu} \ge 0$ $(r,s = 1,\dots,M+L)$ for all $\mu > 0$. We shall show that there exists such $\mu_* > 0$ that $\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu+} > 0$ $(i = 1,...,M)$ and $\widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+} > 0$ $(j = 1,...,L)$ for any $\mu \in (0,\mu_*]$, where $\widetilde{a}_i^{\mu,-} := \inf\limits_{(x,t)\in Q}(a_i^\mu(x,t) - \sum\limits_{s=1}^{M+L}\widetilde{g}_{i,s}^{1}(x,t))$,

$\widetilde{b}_j^{\mu,-} := \inf_{(x,t)\in Q}(b_j^\mu(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^1_{M+j,s}(x,t))$, $\widetilde{g}_r^{2,\mu,+} := \sup_{(x,t)\in Q}\sum_{s=1}^{M+L}\widetilde{g}_{r,s}^{2,\mu}(x,t)$. For this purpose, we put

$$(p_i\varphi)^+ := \sup_{(x,t)\in Q}(p_i(x,t)\varphi(t)), \quad (q_j\varphi)^+ := \sup_{(x,t)\in Q}(q_j(x,t)\varphi(t)), \quad \varkappa^+ := \max_{s\in\{1,\dots,M+L\}}\sup_{t\in(0,T]}\varkappa_s(t).$$

It is obvious that for each $\mu > 0$

$$\widetilde{a}_i^{\mu,-} = \inf_{(x,t)\in Q}(a_i(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^1_{i,s}(x,t) - \mu p_i(x,t)\varphi(t)) \ge \widetilde{a}_i^- - \mu(p_i\varphi)^+, \tag{4.16}$$

$$\widetilde{b}_j^{\mu,-} = \inf_{(x,t)\in Q}(b_j(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^1_{M+j,s}(x,t) - \mu q_j(x,t)\varphi(t)) \ge \widetilde{b}_j^- - \mu(q_j\varphi)^+, \tag{4.17}$$

and

$$\widetilde{g}_r^{2,\mu,+} = \sup_{(x,t)\in Q}(\sum_{s=1}^{M+L}\widetilde{g}^2_{r,s}(x,t)e^{\mu\varkappa_s(t)}) \le \widetilde{g}_r^{2+}e^{\mu\varkappa^+}, \quad \mu > 0. \tag{4.18}$$

From (4.16) – (4.18) for all $\mu > 0$ we have

$$\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+} \ge \widetilde{a}_i^- - \widetilde{g}_i^{2+}e^{\mu\varkappa^+} - \mu(p_i\varphi)^+, \qquad \widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+} \ge \widetilde{b}_j^- - \widetilde{g}_{M+j}^{2,+}e^{\mu\varkappa^+} - \mu(q_j\varphi)^+.$$

For each $i \in \{1,\dots,M\}$ consider a function $l_i(\mu) := \widetilde{a}_i^- - \widetilde{g}_i^{2+}e^{\mu\varkappa^+} - \mu(p_i\varphi)^+$, $\mu \in [0,+\infty)$. It is obvious that it is continuous and $l_i(0) = \widetilde{a}_i^- - \widetilde{g}_i^{2+} > 0$. From here it follows existence of such $\mu_i > 0$ that $l_i(\mu) > 0$ for $\mu \in [0,\mu_i]$. For each $j \in \{1,\dots,L\}$ consider a function $l_{M+j}(\mu) := \widetilde{b}_j^- - \widetilde{g}_{M+j}^{2+}e^{\mu\varkappa^+} - \mu(q_j\varphi)^+$, $\mu \in [0,+\infty)$. It is obvious that it is continuous and $l_{M+j}(0) = \widetilde{b}_j^- - \widetilde{g}_{M+j}^{2+} > 0$. From here it follows existence of such $\mu_{M+L} > 0$ that $l_{M+j}(\mu) > 0$ for $\mu \in [0,\mu_{M+j}]$. Let us take $\mu_* = \min\{\mu_1,\dots,\mu_{M+L}\}$. From above it follows

$$\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+} \ge l_i(\mu) > 0, \qquad \widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+} \ge l_{M+j}(\mu) > 0 \quad \text{for} \quad \mu \in [0,\mu_*]. \tag{4.19}$$

Hence, for $\mu \in [0,\mu_*]$ conditions of Corollary 2 [11] for problem (4.12)–(4.15) are valid.

From (4.9) and (4.11) it follows that restriction w_r^μ on $\overline{\Omega}\times(E_{r,\varepsilon}\cup(\varepsilon,T])$ $(r = 1,\dots,M+L)$ is a solution of problem (4.12)–(4.15). Therefore, according to Corollary 2 [11] for $\mu \in [0,\mu_*]$ we have an estimate

$$\min\Big\{\frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+}}\inf_{(y,s)\in Q_\varepsilon} f_i^\mu(y,s), \inf_{(y,s)\in\Sigma_\varepsilon} h_i^\mu(y,s), \inf_{(y,s)\in\overline{\Omega}\times E_{i,\varepsilon}} w^\mu(y,s), 0\Big\} \le u_i^\mu(x,t)$$
$$\le \max\Big\{\frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+}}\sup_{(y,s)\in Q_\varepsilon} f_i^\mu(y,s), \sup_{(y,s)\in\Sigma_\varepsilon} h_i^\mu(y,s), \sup_{(x,s)\in\overline{\Omega}\times E_{i,\varepsilon}} w^\mu(y,s), 0\Big\}, \quad (x,t) \in Q\varepsilon,\tag{4.20}$$

$$\min\Big\{\frac{1}{\widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+}}\inf_{(y,s)\in\overline{Q}_\varepsilon} f_{M+j}^\mu(y,s), \inf_{(y,s)\in\overline{\Omega}\times E_{M+j,\varepsilon}} w^\mu(y,s), 0\Big\} \le v_j^\mu(x,t)$$
$$\le \max\Big\{\frac{1}{\widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+}}\sup_{(y,s)\in\overline{Q}_\varepsilon} f_{M+j}^\mu(y,s), \sup_{(x,s)\in\overline{\Omega}\times E_{M+j,\varepsilon}} w^\mu(y,s), 0\Big\}, \quad (x,t) \in \overline{Q_\varepsilon}. \tag{4.21}$$

It is obvious that for any $\varepsilon \in (0,T)$ we have

$$\inf_{(y,s)\in Q_\varepsilon} f_r^\mu(y,s) \geq \inf_{(y,s)\in Q} f_r^\mu(y,s), \quad \inf_{(y,s)\in \Sigma_\varepsilon} h_i^\mu(y,s) \geq \inf_{(y,s)\in \Sigma} h_i^\mu(y,s), \tag{4.22}$$

$$\sup_{(y,s)\in Q_\varepsilon} f_r^\mu(y,s) \leq \sup_{(y,s)\in Q} f_r^\mu(y,s), \quad \sup_{(y,s)\in \Sigma_\varepsilon} h_i^\mu(y,s) \leq \sup_{(y,s)\in \Sigma} h_i^\mu(y,s). \tag{4.23}$$

It easy to show, using estimate (4.7) and monotonicity of θ, that

$$\sup_{(y,s)\in\overline{\Omega}\times E_\varepsilon} |w_r^\mu(y,s)| \leq \sup_{(y,s)\in\overline{\Omega}\times(0,\varepsilon]} |w_r(y,s)e^{\mu\theta(s)}| \leq Me^{\mu\theta(\varepsilon)} \underset{\varepsilon\to+0}{\longrightarrow} 0. \tag{4.24}$$

According to (4.22) – (4.24), taking $\varepsilon \to 0$ in (4.20), (4.21) we obtain

$$\min\Big\{\frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+}} \inf_{(y,s)\in Q} f_i^\mu(y,s), \inf_{(y,s)\in\Sigma} h_i^\mu(y,s), 0\Big\} \leq u_i^\mu(x,t)$$

$$\leq \max\Big\{\frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{2,\mu,+}} \sup_{(y,s)\in Q} f_i^\mu(y,s), \sup_{(y,s)\in\Sigma} h_i^\mu(y,s), 0\Big\}, \quad (x,t)\in Q, \tag{4.25}$$

$$\min\Big\{\frac{1}{\widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+}} \inf_{(y,s)\in \widetilde{Q}} f_{M+j}^\mu(y,s), 0\Big\} \leq v_j^\mu(x,t)$$

$$\leq \max\Big\{\frac{1}{\widetilde{b}_j^{\mu,-} - \widetilde{g}_{M+j}^{2,\mu,+}} \sup_{(y,s)\in \widetilde{Q}} f_{M+j}^\mu(y,s), 0\Big\}, \quad (x,t)\in \widetilde{Q}. \tag{4.26}$$

Let $Q_{r,-} := \{(x,t)\in Q \,|\, f_r^\mu(x,t) < 0\}$, $Q_{r,+} := \{(x,t)\in Q \,|\, f_r^\mu(x,t) > 0\}$, $\Sigma_{i,-} := \{(x,t)\in Q \,|\, h_i^\mu(x,t) < 0\}$, $\Sigma_{i,+} := \{(x,t)\in Q \,|\, h_i^\mu(x,t) > 0\}$.

In case $Q_{r,-} \neq \varnothing$, implying inequality $0 < e^{\mu\theta(\rho)} \leq 1$, $\rho \in (0,T]$, we obtain

$$\inf_{(y,s)\in Q} f_r^\mu(y,s) = \inf_{(y,s)\in Q_{r,-}} f_r e^{\mu\theta(\rho)} \geq \inf_{(y,s)\in Q_{r,-}} f_r(y,s) = \inf_{(y,s)\in Q} f_r(y,s),$$

and so (see (4.19)) we have

$$\frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{\mu,+}} \inf_{(y,s)\in Q} f_i^\mu(y,s) \geq \frac{1}{\widetilde{a}_i^{\mu,-} - \widetilde{g}_i^{\mu,+}} \inf_{(y,s)\in Q} f_i(y,s) \geq \frac{1}{l_i(\mu)} \inf_{(y,s)\in Q} f_i(y,s). \tag{4.27}$$

Hence, in this case in the left part of inequality (4.20) first term can be replace with $\frac{1}{l_i(\mu)} \inf\limits_{(y,s)\in Q} f_i(y,s)$. It is obvious that same replacement can be done in case $Q_{i,-} = \varnothing$, because in this case first term of inequality (4.20) is nonnegative, and therefore, does not determine the value of the left side of inequality (4.20).

After similar transformations regarding the rest of the terms of inequalities (4.25), (4.26) we obtain

$$\min\Big\{\frac{1}{l_i(\mu)} \inf_{(y,s)\in Q} f_i(y,s), \inf_{(y,s)\in\Sigma} h_i(y,s), 0\Big\} \leq u_i(x,t)e^{\mu\theta(t)}$$

$$\leq \max\Big\{\frac{1}{l_i(\mu)} \sup_{(y,s)\in Q} f_i(y,s), \sup_{(y,s)\in\Sigma} h_i(y,s), 0\Big\}, \quad (x,t)\in Q, \quad \mu\in(0,\mu_*], \tag{4.28}$$

$$\min\Big\{\frac{1}{l_{M+j}(\mu)}\inf_{(y,s)\in Q} f_{M+j}(y,s),0\Big\} \le v_j(x,t)e^{\mu\theta(t)} \le \max\Big\{\frac{1}{l_{M+j}(\mu)}\sup_{(y,s)\in Q} f_{M+j}(y,s),0\Big\}, \quad (x,t)\in\widetilde{Q},\quad \mu\in(0,\mu_*]. \tag{4.29}$$

Fixing arbitrary taken point $(x,t)\in Q$, and taking a limit in (4.28), (4.29) with $\mu\to+0$. As a result, taking into account $l_i(\mu)\underset{\mu\to 0}{\longrightarrow}\widetilde{a}_i^- - \widetilde{g}_i^{2,+}$, $l_{M+j}(\mu)\underset{\mu\to 0}{\longrightarrow}\widetilde{b}_j^- - \widetilde{g}_{M+j}^{2,+}$, we get estimates (4.4), (4.5). □

Lemma 4.2. *For any $(x,t)\in Q$, $\xi^1,\xi^2,\eta^1,\eta^2\in\mathbb{R}^{M+L}$ following equality is valid*

$$g_r(x,t,\xi^1,\eta^1)-g_r(x,t,\xi^2,\eta^2)$$

$$=\sum_{s=1}^{M+L}(\xi_s^1-\xi_s^2)G_{r,s}^1(x,t,\xi^1,\xi^2,\eta^1,\eta^2)+\sum_{s=1}^{M+L}(\eta_s^1-\eta_s^2)G_{r,s}^2(x,t,\xi^1,\xi^2,\eta^1,\eta^2),$$

where

$$G_{r,s}^1(x,t,\xi^1,\xi^2,\eta^1,\eta^2):=\int_0^1\frac{\partial g_r}{\partial\xi_s}\Big(x,t,z(\xi^1-\xi^2)+\xi^2,z(\eta^1-\eta^2)+\eta^2\Big)dz, \tag{4.30}$$

$$G_{r,s}^2(x,t,\xi^1,\xi^2,\eta^1,\eta^2):=\int_0^1\frac{\partial g_r}{\partial\eta_s}\Big(x,t,z(\xi^1-\xi^2)+\xi^2,z(\eta^1-\eta^2)+\eta^2\Big)dz, \tag{4.31}$$

moreover

$$0\le G_{r,s}^i(x,t,\xi_1,\xi_2,\eta_1,\eta_2)\le g_{r,s}^i(x,t)\quad (i=1,2). \tag{4.32}$$

Proof. Equalities (4.30), (4.31) follows directly from the Hadamard Lemma, and (4.32) follows from condition $(\mathcal{A}_3)$. □

5 Proof of the Main Results

Proof of Theorem 3.1. Consider problems for $w^1=(u^1;v^1)$ and $w^2=(u^2;v^2)$. Denote by $\widehat{w}=(\widehat{u};\widehat{v})$ a vector-function, which components are $\widehat{w}_i(x,t)=\widehat{u}_i:=u_i^1(x,t)-u_i^2(x,t)$, $(x,t)\in\widetilde{Q}$, for $i=1,...,M$, and $\widehat{w}_{M+j}(x,t)=\widehat{v}_j:=v_j^1(x,t)-v_j^2(x,t)$, $(x,t)\in\widetilde{Q}$, for $j=1,...,L$. Considering a difference between Pw^1 and Pw^2, and using Lemma 4.2, we obtain equalities

$$P_i\widehat{w}(x,t):=p_i(x,t)\frac{\partial\widehat{u}_i(x,t)}{\partial t}-\sum_{k,l=1}^n a_{i,lk}(x,t)\frac{\partial\widehat{u}_i(x,t)}{\partial x_k\partial x_l}+\sum_{k=1}^n a_{i,k}(x,t)\frac{\partial\widehat{u}_i(x,t)}{\partial x_k}+a_i(x,t)\widehat{u}_i(x,t)$$
$$-\sum_{s=1}^{M+L}\widetilde{g}_{i,s}^1(x,t)\widehat{w}_s(x,t)-\sum_{s=1}^{M+L}\widetilde{g}_{i,s}^2(x,t)\widehat{w}_{s,\tau_s}(x,t)=\widehat{f}_i(x,t),\quad (x,t)\in Q,\quad i=1,\dots,M, \tag{5.1}$$

$$P_{M+j}\widehat{w}(x,t) := q_j(x,t)\frac{\partial \widehat{v}_j(x,t)}{\partial t} + b_j(x,t)\widehat{v}_j(x,t) - \sum_{s=1}^{M+L} \widetilde{g}^1_{M+j,s}(x,t)\widehat{w}_s(x,t)$$

$$- \sum_{s=1}^{M+L} \widetilde{g}^2_{M+j,s}(x,t)\widehat{w}_{s,\tau_s}(x,t) = \widehat{f}_{M+j}(x,t), \quad (x,t)\in \widetilde{Q}, \quad j=1,\dots,L, \tag{5.2}$$

$$R_i\widehat{w}(x,t) := \widehat{u}_i(x,t) = \widehat{h}_i(x,t), \quad (x,t)\in\Sigma, \quad i=1,\dots,M, \tag{5.3}$$

$$\limsup_{t\to 0+} \max_{x\in\overline{\Omega}} |\widehat{w}_r(x,t)| < \infty, \quad r=1,\dots,M+L, \tag{5.4}$$

where

$$\widetilde{g}^1_{r,s}(x,t) = G^1_{r,s}\Big(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t)\Big),$$

$$\widetilde{g}^2_{r,s}(x,t) = G^2_{r,s}\Big(x,t,w^1(x,t),w^2(x,t),w^1_\tau(x,t),w^2_\tau(x,t)\Big),$$

$$\widehat{f}(x,t) := f^1(x,t) - f^2(x,t), \qquad \widehat{h}(x,t) := w^1(x,t) - w^2(x,t).$$

Let us verify that conditions of Proposition 4.1 hold, that is, ensure, that $\widetilde{g}^1_{i,s}\ge 0$, $\widetilde{g}^2_{i,s}\ge 0$ on Q, $\widetilde{g}^1_{M+j,s}\ge 0$, $\widetilde{g}^2_{M+j,s}\ge 0$ on $\widetilde{Q}$ ($i=1,\dots,M$; $j=1,\dots,L$; $s=1,\dots,M+L$) and $\widetilde{a}_i^- - \widetilde{g}_i^{2,+} > 0$, $\widetilde{b}_j^- - \widetilde{g}^{2,+}_{M+j} > 0$ ($i=1,\dots,M$; $j=1,\dots,L$). From Lemma 4.2 (see (4.32)) it follows that $\widetilde{g}^1_{i,s}(x,t)\ge 0$, $\widetilde{g}^2_{i,s}(x,t)\ge 0$ for any $(x,t)\in Q$, and $\widetilde{g}^1_{M+j,s}(x,t)\ge 0$, $\widetilde{g}^2_{M+j,s}(x,t)\ge 0$ for any $(x,t)\in\widetilde{Q}$ ($i=1,\dots,M$; $j=1,\dots,L$; $s=1,\dots,M+L$) for any $(x,t)\in Q$. Using condition $(\mathcal{A}_3)$ and Lemma 4.2 (see (4.32)), we obtain

$$\widetilde{a}_i^- := \inf_{(x,t)\in Q}\Big[a_i(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^1_{i,s}(x,t)\Big] \ge \inf_{(x,t)\in Q}\Big[a_i(x,t) - \sum_{s=1}^{M+L} g^1_{i,s}(x,t)\Big] = a_i^-, \quad i=1,\dots,M,$$

$$\widetilde{b}_j^- := \inf_{(x,t)\in Q}\Big[b_j(x,t) - \sum_{s=1}^{M+L}\widetilde{g}^1_{M+j,s}(x,t)\Big] \ge \inf_{(x,t)\in Q}\Big[b_j(x,t) - \sum_{s=1}^{M+L} g^1_{M+j,s}(x,t)\Big] = b_j^-, \quad j=1,\dots,L,$$

$$\widetilde{g}_r^{2,+} := \sup_{(x,t)\in Q}\sum_{s=1}^{M+L}\widetilde{g}^2_{r,s}(x,t) \le \sup_{(x,t)\in Q}\sum_{s=1}^{M+L} g^2_{r,s}(x,t) = g_r^{2,+}, \quad r=1,\dots,M+L.$$

Therefore, $\widetilde{a}_i^- - \widetilde{g}_i^{2,+} \ge a_i^- - g_i^{2,+} > 0$ $(i=1,...,M)$, $\widetilde{b}_j^- - \widetilde{g}^{2,+}_{M+j} \ge b_j^- - g^{2,+}_{M+j} > 0$ $(j=1,...,L)$. Hence, conditions of Proposition 4.1 hold, this means that for function $\widehat{w}$ estimates (4.4), (4.5), with replacement f,h,u_i ($i=1,\dots,M$), v_j ($j=1,\dots,L$) on $\widehat{f},\widehat{h},\widehat{u}_i$ ($i=1,\dots,M$), $\widehat{v}_j$ ($j=1,\dots,L$), hold. From here it follows (3.9), (3.10). □

Proof of Corollary 3.2. From conditions of the statement we have $f^1(x,t)-f^2(x,t)\le 0$ $\forall (x,t)\in Q$, $h^1(x,t)-h^2(x,t)\le 0$ $\forall(x,t)\in\Sigma$. From (3.9), (3.10) we have $w^1(x,t)-w^2(x,t)\le 0$, i.e., $w^1(x,t)\le w^2(x,t)$ $\forall (x,t)\in\overline{Q}$. □

Proof of Corollary 3.3. This follows directly from Theorem 3.1, by substituting $w^1 = w$ and $w^2 = 0$. □

Proof of Corollary 3.4. Suppose the opposite and let w^1, w^2 be two different solutions of problem (3.1)–(3.4). Then, according to Theorem 3.1, we have $0 \le w^1(x,t) - w^2(x,t) \le 0$, $(x,t) \in \widetilde{Q}$, that is, $w^1 = w^2$ on $\widetilde{Q}$, a contradiction. Thus, the corollary is proved. □

Proof of Theorem 3.5. Let ε be an arbitrary number from interval $(0, T/3)$, and notations Q_ε, Σ_ε, E_ε same as in proof of Proposition 4.1.

Let us take a function $\theta_\varepsilon \in C^\infty((0,T])$, which satisfies conditions: $0 \le \theta_\varepsilon(t) \le 1$ for $t \in (0,T]$, $\theta_\varepsilon(t) = 0$ for $t \in (0, 2\varepsilon]$ and $\theta_\varepsilon(t) = 1$ for $t \in (3\varepsilon, T]$. Put

$$h_i^\varepsilon(x,t) := \theta_\varepsilon(t) h_i(x,t),\ (x,t) \in \Sigma, \quad f_r^\varepsilon(x,t) := \theta_\varepsilon(t) f_r(x,t),\ (x,t) \in \widetilde{Q}, \quad i = 1,...,M,\ r = 1,...,M+L.$$

Note, that

$$|h_i^\varepsilon(x,t)| \le |h_i(x,t)|\ \forall (x,t) \in \Sigma,\ |f_i^\varepsilon(x,t)| \le |f_i(x,t)|\ \forall (x,t) \in \widetilde{Q},\ i = 1,...,M,\ r = 1,...,M+L. \tag{5.5}$$

Consider a problem of finding a vector-function $w^\varepsilon = (u_1^\varepsilon, \dots, u_M^\varepsilon; v_1^\varepsilon, \dots, v_L^\varepsilon)$ such that $u_i^\varepsilon \in C(\overline{\Omega} \times (E_{i,\varepsilon} \cup (0,T])) \cap C^{2,1}(Q_\varepsilon)$ $(i = 1,...,M)$, $v_j^\varepsilon \in C(\overline{\Omega} \times (E_{M+j,\varepsilon} \cup (0,T])) \cap C^{0,1}(\overline{Q_\varepsilon})$ $(j = 1,...,L)$, and which satisfies system

$$P_i w^\varepsilon(x,t) = f_i^\varepsilon(x,t), \quad (x,t) \in Q_\varepsilon,\ i = 1,...,M, \tag{5.6}$$

$$P_{M+j} w^\varepsilon(x,t) = f_{M+j}^\varepsilon(x,t), \quad (x,t) \in \overline{Q_\varepsilon}, \quad j = 1,...,L, \tag{5.7}$$

and conditions

$$u_i^\varepsilon(x,t) = h_i^\varepsilon(x,t), \quad (x,t) \in \Sigma_\varepsilon, \quad i = 1,\dots,M, \tag{5.8}$$

$$w_r^\varepsilon(x,t) = 0, \quad (x,t) \in \overline{\Omega} \times E_{r,\varepsilon},\ r = 1,\dots,M+L, \tag{5.9}$$

where P_r $(r = 1,\dots,M+L)$ are differential operators defined in (3.1), (3.2).

From Theorem 2 [11] it follows existence of unique solution $w^\varepsilon = (u_1^\varepsilon, \dots, u_M^\varepsilon; v_1^\varepsilon, \dots, v_L^\varepsilon)$ of problem (5.6)–(5.9), such that $u_i \in C^{\alpha,\alpha/2}(\overline{\Omega} \times (E_{i,\varepsilon} \cup (\varepsilon,T])) \cap C_{\text{loc}}^{2+\alpha,1+\alpha/2}(Q_\varepsilon)$, $v_j \in C^{\alpha,\alpha/2}(\overline{\Omega} \times (E_{M+j,\varepsilon} \cup (\varepsilon,T])) \cap C^{\alpha,1+\alpha/2}(\overline{Q_\varepsilon})$ $(i = 1,\dots,M;\ j = 1,\dots,L)$. According to Corollary 2 [11] for restriction u_i^ε on $\overline{\Omega} \times (E_{i,\varepsilon} \cup (\varepsilon, 2\varepsilon])$ we have an estimate

$$|u_i^\varepsilon(x,t)| \le \max\{\frac{1}{a_i^- - g_i^{2,+}} \sup_{(y,s)\in Q_\varepsilon/Q_{2\varepsilon}} |f_i^\varepsilon(y,s)|, \sup_{(y,s)\in\Sigma_\varepsilon/\Sigma_{2\varepsilon}} |h_i^\varepsilon(y,s)|\}, \quad (x,t) \in \overline{Q_\varepsilon}/\overline{Q_{2\varepsilon}}, \tag{5.10}$$

and for restriction v_j^ε on $\overline{\Omega} \times (E_{M+j,\varepsilon} \cup (\varepsilon, 2\varepsilon])$ we have following estimate

$$|v_j^\varepsilon(x,t)| \le \max\{\frac{1}{b_j^- - g_{M+j}^{2,+}} \sup_{(y,s)\in\overline{Q_\varepsilon}/\overline{Q_{2\varepsilon}}} |f_{M+j}^\varepsilon(y,s)|\}, \quad (x,t) \in \overline{Q_\varepsilon}/\overline{Q_{2\varepsilon}}. \tag{5.11}$$

From definitions of f^ε and h^ε it follows that right parts of (5.10), (5.11) is equal to zero, and, therefore, $w_r^\varepsilon(x,t) = 0$ for each $(x,t) \in \overline{\Omega} \times (E_{r,\varepsilon} \cup (\varepsilon, 2\varepsilon])$ $(r = 1,\dots,M+L)$. Extend w^ε by zero on whole $\widetilde{Q}$ and leave for this extension notation w^ε. It is easy to show that w^ε

is a solution of a problem, which differ from problem (3.1)–(3.4) only by having f^ε and h^ε instead of f and h, respectively. From here and according to Corollary 3.4 and (5.5) it follows that

$$|u_i^\varepsilon(x,t)| \le \max\{\frac{1}{a_i^- - g_i^{2,+}} \sup_{(y,s)\in Q} |f_i(y,s)|, \sup_{(y,s)\in\Sigma} |h_i(y,s)|\}, \quad (x,t)\in \widetilde{Q}, \tag{5.12}$$

$$|v_j^\varepsilon(x,t)| \le \max\{\frac{1}{b_j^- - g_{M+j}^{2,+}} \sup_{(y,s)\in Q} |f_{M+j}(y,s)|, 0\}, \quad (x,t)\in \widetilde{Q}. \tag{5.13}$$

Let $\{\varepsilon_m\}_{m=1}^\infty$ be a sequence of numbers from interval $(0,T/2)$, such that $\varepsilon_m \downarrow 0$ when $m\to\infty$. Denote $w^m := w^{\varepsilon_m}$, $f_r^m := f_r^{\varepsilon_m}$, $h_i^m := h_i^{\varepsilon_m}$ $(i=1,\dots,M;\ r=1,\dots,M+L)$ for each $m\in\mathbb{N}$. From (5.12), (5.13) it follows that

$$\sup_{(x,t)\in\widetilde{Q}} |w_r^m(x,t)| \le C_3, \qquad r=1,\dots,M+L,\ m\in\mathbb{N}, \tag{5.14}$$

where $C_3>0$ is a constant independent of m.

Let $\{\delta_k\}_{k=1}^\infty$ be a monotone sequence of numbers, such that $\delta_k \underset{k\to\infty}{\downarrow} 0$, $0<\delta_k<T$ and $\Omega_k := \{x\in\Omega : \operatorname{dist}\{x,\partial\Omega\} > \delta_k\}$ be a domain in $\mathbb{R}^n$ for each $k\in\mathbb{N}$. Denote $I_k := (\delta_k, T]$, $Q_k := \Omega_k\times I_k$, $Q^k := \Omega\times I_k$. Note, that $Q_k\subset Q^k$, $Q_k\subset Q_{k+1}$, $Q^k\subset Q^{k+1}$ for each $k\in\mathbb{N}$; $\bigcup_{k=1}^\infty \Omega_k = \Omega$, $\bigcup_{k=1}^\infty \overline{Q}_k = Q$, $\bigcup_{k=1}^\infty \overline{Q^k} = \widetilde{Q}$.

Denote $g_r^m(x,t) := f_r^m(x,t) + g_r(x,t,w^m(x,t),w_\tau^m(x,t))$, $(x,t)\in\widetilde{Q}$, for each $r=1,\dots,M+L$, $m\in\mathbb{N}$. From continuity of functions g_r on $\widetilde{Q}\times\mathbb{R}^{M+L}\times\mathbb{R}^{M+L}$, f_r^m, w_r^m on $\widetilde{Q}$ $(r=1,\dots,M+L;\ m\in\mathbb{N})$, and estimates (5.5), (5.14) it follows that g_r^m is continuous on $\widetilde{Q}$ and for any $k\in\mathbb{N}$ following estimates hold

$$\|g_r^m\|_{C(\overline{Q^k})} \leqslant C_4, \quad r=1,\dots,M+L,\ m\in\mathbb{N}, \tag{5.15}$$

where $C_4>0$ is a constant independent of m, but it may depend on k.

From (5.6), (5.7) it follows that for each $m\in\mathbb{N}$ we have

$$p_i(x,t)\frac{\partial u_i^m(x,t)}{\partial t} - \sum_{k,l=1}^n a_{i,lk}(x,t)\frac{\partial u_i^m(x,t)}{\partial x_k\partial x_l} + \sum_{k=1}^n a_{i,k}(x,t)\frac{\partial u_i^m(x,t)}{\partial x_k}$$
$$+a_i(x,t)u_i^m(x,t) = g_i^m(x,t), \quad (x,t)\in Q,\ i=1,...,M, \tag{5.16}$$

$$q_j(x,t)\frac{\partial v_j^m(x,t)}{\partial t} + b_j(x,t)v_j^m(x,t) = g_{M+j}^m(x,t), \quad (x,t)\in\widetilde{Q}, \quad j=1,...,L, \tag{5.17}$$

and from (5.8) we obtain

$$u_i^m(x,t) = h_i^m(x,t), \quad (x,t)\in\Sigma, \quad i=1,\dots,M. \tag{5.18}$$

Since u_i^m is a classical solution of equation (5.16), which satisfies boundary condition (5.18), and from conditions $(\mathcal{B}_1)$, $(\mathcal{B}_3)$, $(\mathcal{B}_4)$ and estimates (5.14), (5.15), according to Theorem 1.1 of monograph [28, p. 476], we obtain following estimate

$$\max_{1\le i\le M} \|u_i^m\|_{\alpha,\alpha/2}^{\overline{Q^k}} \le C_5, \quad m\in\mathbb{N}, \tag{5.19}$$

where $C_5 > 0$ is a constant independent of m, but it may depend on k.

Since v_j^m is a classical solution of equation (5.17), analogically as in paper [18], following can be shown

$$\max_{1\le j\le L} \|v_i^m\|_{\alpha,\alpha/2}^{\overline{Q^k}} \le C_6, \quad m \in \mathbb{N}, \tag{5.20}$$

where $C_6 > 0$ is a constant independent of m, but it may depend on k.

Note, that from conditions $(\mathcal{A}_3)$, $(\mathcal{A}_4)$, $(\mathcal{B}_2)$, $(\mathcal{B}_4)$ and estimates (5.19), (5.20) we have

$$\max_{1\le r\le M+L} \|g_r^m\|_{\alpha,\alpha/2}^{\overline{Q^k}} \leqslant C_7, \quad m \in \mathbb{N}, \tag{5.21}$$

where $C_7 > 0$ is a constant independent of m, but it may depend on k.

From (5.14), (5.21) and conditions of the Theorem, according to Theorem 10.1 of monograph [28, p.400], for each $k \in \mathbb{N}$ we obtain

$$\max_{1\le i\le M} \|u_i^m\|_{2+\alpha,1+\alpha/2}^{\overline{Q}_k} \leqslant C_7, \quad m \in \mathbb{N}, \tag{5.22}$$

where $C_7 > 0$ is a constant independent of m, but it depend on C_5, C_6.

From condition $(\mathcal{B}_2)$ and estimates (5.14), (5.19)–(5.22), analogically as in paper [8], following can be shown

$$\max_{1\le j\le L} \|v_j^m\|_{\alpha,1+\alpha/2}^{\overline{Q^k}} \le C_8, \quad m \in \mathbb{N}, \tag{5.23}$$

where $C_8 > 0$ is a constant independent of m, but it may depend on k.

From (5.22), (5.23), Proposition 2.2 (Section 1) and Theorem on differentiation of convergent function sequence, it follows that there exists function $w = (u; v) \in [C_{\text{loc}}^{2+\alpha,1+\alpha/2}(Q)]^M \times [C_{\text{loc}}^{\alpha,1+\alpha/2}(\widetilde{Q})]^L$ and a subsequence (which we also note $\{w^m\}_{m=1}^{\infty}$) of sequence $\{w^m\}_{m=1}^{\infty}$, which converge to w in $[C^{2,1}(Q)]^M \times [C^{0,1}(\widetilde{Q})]^L$. Now, note that $h^m \to h$ when $m \to \infty$ uniformly on each compact $K \subset \Sigma$. Also note, that from continuity of g_r, f_r^m we have $g_r^m(x,t) \to f_r(x,t) + g_r(x,t,w(x,t),w_\tau(x,t))$ when $m \to \infty$ for each $(x,t) \in Q$ $(r = 1,\dots,M+L)$. According to said above, take in (5.16), (5.17), (5.18) limits with $m \to \infty$. As a result we obtain identities, which mean that function w is a classical solution of system (3.1), (3.2) and satisfies boundary condition (3.3). Fulfillment of condition (3.4) follows from (5.14). Estimates (3.11), (3.12) follow from (5.12), (5.13). □

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