Communications in **M**athematical **A**nalysis
Volume 20, Number 1, pp. 27–49 (2017) www.math-res-pub.org/cma
ISSN 1938-9787

Periodic Travelling Waves and its Inter-relation with Solitons for the 2D *abc*-Boussinesq System

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(Communicated by Marco Squasina)

Abstract

Via a variational approach involving Concentration-Compactness principle, we show the existence of x-periodic travelling wave solutions for a general 2D-Boussinesq system that arises in the study of the evolution of long water waves with small amplitude in the presence of surface tension. We also establish that x-periodic travelling waves have almost the same shape of solitons as the period tends to infinity, by showing that a special sequence of x-periodic travelling wave solutions parameterized by the period converges to a solitary wave in a appropriate sense.

AMS Subject Classification: 35Q35, 35B10, 35Q51, 35A15

Keywords: Periodic travelling waves, Solitary waves, Concentration-compactness principle, Variational principle.

1 Introduction

In this paper we study the existence of non trivial x-periodic travelling waves for the two dimensional abc-Boussinesq system related with the water wave problem

$$\begin{cases} (I-b\mu\Delta)\Phi_t + (I-a\mu\Delta)\eta + \frac{\epsilon}{p+1}\left(\Phi_x^{p+1}+\Phi_y^{p+1}\right) = 0, \\ (I-b\mu\Delta)\eta_t + (I-c\mu\Delta)\Delta\Phi + \epsilon\nabla\cdot\left(\eta\left(\Phi_x^p,\Phi_y^p\right)\right) = 0, \end{cases} \tag{1.1}$$

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where $2b-a-c=\frac{1}{3}-\sigma$, with $a,b,c>0$ and $p=\frac{p_1}{p_2}$ with p_1,p_2 integers such that $gcd(p_1,p_2)=1$ and p_2 odd. In the case $p=1$, ϵ stands for the ratio of typical wave amplitude to fluid depth (amplitude parameter or nonlinearity coefficient), $\sqrt{\mu}=h_0/L$ represents the ratio of undisturbed fluid depth to typical wave length (long-wave parameter or dispersion coefficient) and σ is associated with the surface tension (Bond number), Φ is the rescaled nondimensional velocity potential on the bottom $z=0$, and η is the rescaled free surface elevation. For $a=\sigma$, $c=\frac{2}{3}$, $b=\frac{1}{2}$ and $p=1$ the 2D Boussinesq system was obtained by J. Quintero and A. Montes in [12] to describe the propagation of long water waves with small amplitude in the presence of surface tension (see also [6]).

We will establish the existence of x-periodic travelling wave solutions of period $k>0$ for the system (1.1) with fixed positive values of p,ϵ,μ,a,b,c and wave speed ω small enough. Moreover, as in some water wave models (see for example [8], [10]), we show that a special sequence of x-periodic travelling waves is uniformly bounded and converges to a solitary wave (travelling wave in the energy space) in an appropriate norm, indicating that the shape of x-periodic travelling waves and the solitary waves are almost the same, as the period k is big enough.

Following the same variational approach as J. Quintero and A. Montes in [12], it is possible to show the existence of solitons (travelling wave solutions of finite energy) for the abc-Boussinesq system (1.1) in the energy space $\mathbb{X}=H^1\times\mathcal{V}$, where $\mathcal{V}$ is defined by the norm $\|\Phi\|_{\mathcal{V}}=\|\nabla\Phi\|_{H^1}$. We see that the abc-Boussinesq system (1.1) can be written in the Hamiltonian form

$$\begin{pmatrix}\eta_t\\ \Phi_t\end{pmatrix}=\mathcal{B}\mathcal{H}'\begin{pmatrix}\eta\\ \Phi\end{pmatrix},\quad \mathcal{B}=\begin{pmatrix}0 & (I-b\mu\Delta)^{-1}\\ (I-b\mu\Delta)^{-1} & 0\end{pmatrix}.$$

where the Hamiltonian is given by

$$\mathcal{H}\begin{pmatrix}\eta\\ \Phi\end{pmatrix}=\frac{1}{2}\int_{\mathbb{R}^2}\Big(|\nabla\Phi|^2+\eta^2+c\mu|\Delta\Phi|^2+a\mu|\nabla\eta|^2+\frac{\epsilon}{p+1}\eta(\Phi_x^{p+1}+\Phi_y^{p+1})\Big)dxdy,$$

Note that the functional $\mathcal{H}$ is well defined when $(\eta,\nabla\Phi)\in H^1\times H^1$, where $H^1=H^1(\mathbb{R}^2)$ is the usual Sobolev space.
In this paper we show the existence of travelling waves in the space

$$\mathbb{X}_k=H_k^1\times\mathcal{V}_k,$$

where $H_k^1=H_k^1(\mathbb{R}^2)$ is the usual Sobolev space of order 1 of x-periodic functions of period k and $\mathcal{V}_k$ is defined by the norm $\|\Phi\|_{\mathcal{V}_k}=\|\nabla\Phi\|_{H_k^1}$. In order to have the result, we follows a variational approach by characterizing travelling waves as a minimizer of a functional under a suitable constrain. Using an appropriated local compact embedding from the space $H_k^1\times\mathcal{V}_k$ to a special $L^q(\mathbb{R}^2)$ type space and the Lions's Concentration-Compactness Theorem, we prove that any minimizing sequence converges strongly, after an appropriate translation, to a minimizer.

J. Quintero in the case of the Benney-Luke equation for wave speed small enough (see [10]) and A. Pankov and K. Pflüger in the case of the Kadomtsev-Petviashvili equation (see [8]) proved the existence of nontrivial x-periodic travelling waves of period k that are uniformly bounded in k and bounded away from zero with respect to an appropriate norm.

They also included the limit behavior of such periodic solutions as $k \to \infty$. As done for some 1D models (see [2], [1], [9]), they proved also for those models that a special sequence of x-periodic travelling waves parameterized by the period k converges to a solitary wave in a appropriate sense, as the period $k \to \infty$. In this work, we prove that travelling wave solutions for the *abc*-Boussinesq system (1.1) share the same property for the Benney-Luke equation and Kadomtsev-Petviashvili equation, established by J. Quintero in [10] and by A. Pankov and K. Pflüger in [8], respectively, in the sense that a suitable renormalized family of solitary waves of the *abc*-Boussinesq system (1.1) converges strongly in an appropriate space to a nontrivial solitary wave for the Kadomtsev-Petviashvili equation (see [6], [12] in the case $a = \sigma$, $c = \frac{2}{3}$, $b = \frac{1}{2}$ and $p = 1$). This concordance can be explained by the fact that the Benney-Luke equation and the KP equation can be derived, up to some order with respect to ϵ and μ, from the Boussinesq system (1.1),

Other results on the Boussinesq system (1.1) including the well posedness of the Cauchy problem, Strichartz type estimates and the stability of solitary wave solutions have been studied in [7], [11], [13]. The paper is organized as follows. In section 2, we present some preliminaries related with embedding lemmas and the Concentration-Compactness Theorem. In section 3, we show the existence of x-peridic travelling wave solutions with period k. In Section 4, we prove the inter-relation between special periodic travelling waves (ground states) and solitary waves. Throughout this work C denotes a generic constant whose value may change from instance to instance.

2 Preliminaries

The periodic spaces

For $k > 0$ fixed, we define the following appropriate periodic spaces. For $Q \subset \mathbb{R}^2$, we denote $L^q(Q)$ with $1 \le q \le \infty$ as the usual Lebesgue space and $H^s(Q)$ denotes the usual Sobolev space. If we set $Q_k = \left[-\frac{k}{2}, \frac{k}{2}\right] \times \mathbb{R}$, we denote with $L^q_k(\mathbb{R}^2)$ to space of x-periodic with period k functions given by

$$L^q_k(\mathbb{R}^2) = \left\{ f : \mathbb{R}^2 \to \mathbb{R} : f \in L^q(Q_k) \text{ and } x\text{-periodic of period } k \right\},$$

with norm

$$\|u\|_{L^q_k(\mathbb{R}^2)} = \|u\|_{L^q(Q_k)}.$$

Now, let $C^\infty_{per}(\mathbb{R}^2)$ be the space of smooth functions which are x-periodic with period k and have compact support in y and define

$$\mathcal{Y}_k = \left\{ \varphi|_{Q_k} : \varphi \in C^\infty_{per}\left(\mathbb{R}^2\right) \right\}.$$

We denote by $\mathcal{V}_k$ the closure of $\mathcal{Y}_k$ with respect to the norm given by

$$\|\psi\|^2_{\mathcal{V}_k} = \int_{Q_k} \left(\psi_x^2 + \psi_y^2 + \psi_{xx}^2 + 2\psi_{xy}^2 + \psi_{yy}^2 \right) dxdy = \|\psi_x\|^2_{H^1(Q_k)} + \|\psi_y\|^2_{H^1(Q_k)}.$$

Note that $(\mathcal{V}_k, \|.\|_{\mathcal{V}_k})$ is a Hilbert space with inner product

$$(\phi, \psi)_{\mathcal{V}_k} = (\phi_x, \psi_x)_{H^1_k(\mathbb{R}^2)} + (\phi_y, \psi_y)_{H^1_k(\mathbb{R}^2)},$$

where $H^1_k(\mathbb{R}^2)$ is the Hilbert space of functions $\phi \in L^2_k(\mathbb{R}^2)$ such that $\phi_x, \phi_y \in L^2_k(\mathbb{R}^2)$. The space $H^1_k(\mathbb{R}^2)$ has the inner product given by

$$(\phi,\psi)_{H^1_k(\mathbb{R}^2)} = (\phi,\psi)_{L^2(Q_k)} + (\phi_x,\psi_x)_{L^2(Q_k)} + (\phi_y,\psi_y)_{L^2(Q_k)}.$$

We define also the Hilbert space $\mathbb{X}_k = H^1_k(\mathbb{R}^2)\times \mathcal{V}_k$ with respect to norm

$$\|(\phi,\psi)\|^2_{\mathbb{X}_k} = \|\phi\|^2_{H^1_k} + \|\psi\|^2_{\mathcal{V}_k}.$$

Embedding

The existence of periodic travelling wave solutions requires using a compact embedding result proved by J. Quintero in [10] for the Benney-Luke equation (see also A. Pankov and K. Pflüger in [8] for the KP equation). We follow the notation and the approach used in [10]. For $q \geq 2$, we define the Banach space $\mathcal{M}^q(Q)$ as the closure of $C_0^\infty(Q)$ with respect to the norm given by

$$\|\psi\|^q_{\mathcal{M}^q(Q)} = \|\psi_x\|^q_{L^q(Q)} + \|\psi_y\|^q_{L^q(Q)}.$$

Then the following embedding results hold.

Lemma 2.1. *Let $q \geq 2$. We have that the following embedding are continuous:*

$$\mathcal{V} \hookrightarrow \mathcal{M}^q(\mathbb{R}^2), \quad \mathbb{X} \hookrightarrow L^q(\mathbb{R}^2)\times \mathcal{M}^q(\mathbb{R}^2).$$

Moreover, we also have that the following embedding are compact:

$$\mathcal{V} \hookrightarrow \mathcal{M}^q_{loc}(\mathbb{R}^2), \quad \mathbb{X} \hookrightarrow L^q_{loc}(Q_T)\times \mathcal{M}^q_{loc}(Q_T).$$

We note that these results are a consequence of fact that if $q \geq 2$ then the embedding $H^1(\mathbb{R}) \hookrightarrow L^q(\mathbb{R})$ is continuous and the embedding $H^1(\mathbb{R}) \hookrightarrow L^q_{loc}(\mathbb{R})$ is compact. In the case of the space $\mathcal{V}_k$ we have a similar result. To do this, we require the use of a cut-off operator to extend any function in $\mathcal{V}_k$ to a function in $\mathcal{V}$. Let χ be a $C_0^\infty(\mathbb{R})$ cut-off function satisfying

$$\begin{aligned} \chi(s) &= 1, \ \ |s| \leq k/2, \\ \chi(s) &= 0, \ \ |s| \geq (k+1)/2, \\ \chi', \chi'' &\leq C_0. \end{aligned}$$

A. Pankov and J. Plugger considered in [8] the cut-off operator

$$(P_k v)(x,y) = \partial_x\left[\chi(x)\partial^{-1}_{x,k} v(x,y)\right], \ \ \partial^{-1}_{x,k} v(x,y) = \int_{-k/2}^{x} v(r,y)dr.$$

Using this operator, it is possible to establish an extension operator E_k from $\mathbb{X}_k$ to $\mathbb{X}$.

Lemma 2.2. *1. Let S_k be the operator defined on $\mathcal{V}_k$ by*

$$S_k v = \partial_x^{-1}\left[P_k(\partial_x v)\right], \ \ \partial_x^{-1} v(x,y) = \int_{-\infty}^{x} v(r,y)dr.$$

Then S_k is a uniformly bounded (with respect k) linear operator from $\mathcal{V}_k$ into $\mathcal{V}$ and $S_k v|_{Q_k} = v$.

2. *Let F_k be the operator defined on $H^1_k(\mathbb{R}^2)$ by*

$$F_k(u) = \chi(x)u(x,y).$$

Then F_k is a uniformly bounded (with respect k) linear operator from $H^1_k(\mathbb{R}^2)$ into $H^1(\mathbb{R}^2)$ and $F_k u|_{Q_k} = u$.

3. *$E_k = (F_k, S_k)$ defines a uniformly bounded (with respect k) linear operator from $\mathbb{X}_k$ into $\mathbb{X}$ such that $E_k(u,v)|_{Q_k} = (u,v)$.*

The first part was proved by J. Quintero in [10]. The second part is straightforward. From previous result and Lemma 2.1 we obtain the corresponding embedding in $\mathcal{V}_k$ and $\mathbb{X}_k$.

Lemma 2.3. *For $q \geq 2$ we have that*

1. *The embedding $\mathcal{V}_k \hookrightarrow \mathcal{M}^q(Q_k)$ is continuous with the embedding constants being uniformly bounded with respect to k and the embedding $\mathcal{V}_k \hookrightarrow \mathcal{M}^q_{loc}(Q_k)$ is compact.*

2. *The embedding $\mathbb{X}_k \hookrightarrow L^q(Q_k) \times \mathcal{M}^q(Q_k)$ is continuous with the constants being uniformly bounded with respect to k and the embedding $\mathbb{X}_k \hookrightarrow L^q_{loc}(Q_k) \times \mathcal{M}^{(q)}_{loc}(Q_k)$ is compact.*

Concentration-Compactness Principle

We will use the Concentration-Compactness Principle in the periodic case by P. L. Lions (see [3], [4]), to show the existence of a nontrivial x-periodic travelling wave solution of period k for tha abc-Boussinesq system (1.1). For $\zeta \in \mathbb{R}$ and $r > 0$, we define the rectangle

$$R_{r,k}(\zeta) = \left[-\frac{k}{2}, \frac{k}{2}\right] \times [\zeta - r, \zeta + r].$$

Theorem 2.4. *(P. L. Lions, [3], [4]) Suppose $\{\nu_m\}$ is a sequence of nonnegative measures on $Q_k \subset \mathbb{R}^2$ such that*

$$\lim_{m\to\infty} \int_{Q_k} d\nu_m = \mathcal{I}.$$

Then there is a subsequence of $\{\nu_m\}$ (which we denote by the same symbol) that satisfies only one of the following properties.

Vanishing. For any $r > 0$,

$$\lim_{m\to\infty} \left(\sup_{y\in\mathbb{R}} \int_{R_{r,k}(y)} d\nu_m \right) = 0. \tag{2.1}$$

Dichotomy. There exist $\theta \in (0, \mathcal{I})$ such that for any $\gamma > 0$, there are $r > 0$ and a sequence $\{y_m\}$ in $\mathbb{R}$ with the following property: Given $r' > r$ there are nonnegative measures ν^1_m, ν^2_m such that

1. *$0 \leq \nu^1_m + \nu^2_m \leq \nu_m$,*

2. $supp(v_m^1) \subset R_{r,k}(y_m), \quad supp(v_m^2) \subset Q_k \setminus R_{r',k}(y_m),$

3. $\limsup_{m\to\infty} \left(|\theta - \int_{Q_k} dv_m^1| + |(\mathcal{I}-\theta) - \int_{Q_k} dv_m^2|\right) \leq \gamma.$

Compactness. There exists a sequence $\{y_m\}$ in $\mathbb{R}$ such that for any $\gamma > 0$, there is $r > 0$ with the property that

$$\int_{R_{r,k}(\zeta_m)} dv_m \geq \mathcal{I} - \gamma, \quad \text{for all } m. \tag{2.2}$$

3 Existence of x-periodic travelling waves with period k

Let $k > 0$ be fixed. In this section we will show the existence of a x-periodic travelling wave solution of period k for the *abc*-Boussinesq system (1.1) with fixed positive values of ϵ, μ, a, b, c and wave speed ω small enough.

By a travelling wave solution for the system (1.1) we mean a solution (η, Φ) of the form

$$\eta(x,y,t) = \frac{1}{\epsilon^{\frac{1}{p}}} u\left(\frac{x-\omega t}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right), \quad \Phi(x,y,t) = \frac{\sqrt{\mu}}{\epsilon^{\frac{1}{p}}} v\left(\frac{x-\omega t}{\sqrt{\mu}}, \frac{y}{\sqrt{\mu}}\right).$$

A direct computation shows that the traveling-wave system for (1.1) takes the form

$$\begin{pmatrix} -\omega(I-b\Delta)u_x + (I-c\Delta)\Delta v + \nabla\cdot\left(u\left(v_x^p, v_y^p\right)\right) \\ -\omega(I-b\Delta)v_x + (I-a\Delta)u + \frac{1}{p+1}\left(v_x^{p+1} + v_y^{p+1}\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

We follow a variational approach by characterization travelling wave as minimizer of a functional under a suitable constrain. As happens in the case of solitons for the *abc*-Boussinesq system in [12], x-periodic travelling waves (u, v) for (3.1) are critical points of the functional $J_{\omega,k}$ given by

$$J_{\omega,k}(u,v) = I_k(u,v) + \frac{2}{p+1} G_k(u,v),$$

where the functionals I_k and G_k are defined on the space $\mathbb{X}_k$ by

$$\begin{aligned} I_k(u,v) &= \int_{Q_k} \left(u^2 + a|\nabla u|^2 + |\nabla v|^2 + c(\Delta v)^2 - 2\omega u v_x - 2b\omega u_x \Delta v\right) dxdy, \\ G_k(u,v) &= \int_{Q_k} u\left(v_x^{p+1} + v_y^{p+1}\right) dxdy. \end{aligned}$$

First we have that $I_k, G_k, J_{\omega,k} \in C^2(\mathbb{X}_k, \mathbb{R})$ and a direct computation shows that

$$J'_{\omega,k}(u,v) = 2\begin{pmatrix} u - a\Delta u - \omega(I-b\Delta)v_x + \frac{1}{p+1}\left(v_x^{p+1} + v_y^{p+1}\right) \\ \\ c\Delta^2 v - \Delta v + \omega(I-b\Delta)u_x - \nabla\cdot\left(u(v_x^p, v_y^p)\right) \end{pmatrix},$$

meaning that critical points of the functional $J_{\omega,k}$ satisfy the travelling wave system (3.1). Hereafter, we will say that solutions for (3.1) are critical points of the functional $J_{\omega,k}$. In particular, we have that

$$\left\langle J'_{\omega,k}(u,v),(u,v)\right\rangle = 2I_k(u,v) + \frac{2(p+2)}{p+1}G_k(u,v). \tag{3.2}$$

On the other hand, one can see easily that the functional G_k is well-defined on $\mathbb{X}_k$. By using Young's inequality and Lemma 2.3 we have that there is a constant $C > 0$ (not depending on k) such that

$$|G_k(u,v)| \le C(\|u\|^{p+2}_{L^{p+2}(Q_k)} + \|v\|^{p+2}_{\mathcal{M}^{p+2}(Q_k)}) \le C\|(u,v)\|^{p+2}_{\mathbb{X}_k}. \tag{3.3}$$

As in the case of solitary wave solutions, to establish the existence of a travelling wave solution in the space $\mathbb{X}_k$ for the system (1.1) we consider the following minimization problem

$$\mathcal{I}_k(\omega) := \inf\left\{I_k(u,v)\ :\ (u,v)\in \mathcal{X}_k \ \text{ with } \ G_k(u,v) = (-1)^{p+1}\right\}. \tag{3.4}$$

The first observation is that

Lemma 3.1. *Let $k > 0$ and $0 < \omega < \min\{1, \frac{a}{b}, \frac{c}{b}\}$. Then the functional I_k is nonnegative and there are positive constants $C_1(a,b,c,\omega) < C_2(a,b,c,\omega)$ such that*

$$C_1(a,b,c,\omega)I_k(u,v) \le \|(u,v)\|^2_{\mathcal{X}_k} \le C_2(a,b,c,\omega)I_k(u,v). \tag{3.5}$$

Proof. From the definition of I_k and Young's inequality we obtain that

$$\begin{aligned} I_k(u,v) &\le \int_{Q_k}\left((1+\omega)u^2 + (a+b\omega)\,u_x^2 + au_y^2 + (1+\omega)v_x^2 + v_y^2 + (c+b\omega)\,(\Delta v)^2\right)dxdy \\ &\le \max\,(1+\omega, a+b\omega, a, 1+\omega, c+b\omega)\,\|(u,v)\|^2_{\mathbb{X}_k}. \end{aligned}$$

In a similar way, we have that

$$\begin{aligned} I_k(u,v) &\ge \int_{Q_k}\left((1-\omega)u^2 + (a-b\omega)\,u_x^2 + au_y^2 + (1-\omega)v_x^2 + v_y^2 + (c-b\omega)\,(\Delta v)^2\right)dxdy \\ &\ge \min\,(1-\omega, a-b\omega, a, 1-\omega, c-b\omega)\,\|(u,v)\|^2_{\mathbb{X}_k}, \end{aligned}$$

showing that the inequality (3.5) holds. □

We see directly from Lagrange multipliers, the following result:

Lemma 3.2. *For any $k > 0$, $\mathcal{I}_k(\omega)$ is finite and positive. Moreover, If (u_0, v_0) is a minimizer for $\mathcal{I}_k(\omega)$, then $(u,v) = \beta(u_0, v_0)$ is a nontrivial solution of (3.1) for $(-\beta)^p = \left(\frac{p+1}{p+2}\right)\mathcal{I}_k$.*

Proof. It is easy to see that there is $(u,v) \in C_0^\infty(Q_k)\times C_0^\infty(Q_k)$ such that

$$G_k(u,v) = (-1)^{p+1}.$$

On the other hand, from inequality (3.3) we have that there is $C > 0$ such that for any $(u,v)\in\mathbb{X}_k$,

$$1 = |G_k(u,v)| \le C(\|u\|^{p+2}_{L^{p+2}(Q_k)} + \|v\|^{p+2}_{\mathcal{M}^{p+2}(Q_k)}) \le C\|(u,v)\|^{p+2}_{\mathbb{X}_k},$$

meaning that $I_k(u,v) \geq C$, and so that the infimum $\mathcal{I}_k(\omega)$ is finite and positive. Now, by the Lagrange Theorem there is a multiplier λ such that for any $(w,z) \in \mathbb{X}_k$,

$$\left\langle I_k'(u_0,v_0),(w,z)\right\rangle - \lambda\left\langle G_k'(u_0,v_0),(w,z)\right\rangle = 0. \tag{3.6}$$

Taking $(w,z) = (u_0,v_0)$, we obtain that

$$0 = \left\langle I_k'(u_0,v_0),(u_0,v_0)\right\rangle - \lambda\left\langle G_k'(u_0,v_0),(u_0,v_0)\right\rangle = 2I_k(u_0,v_0) - \lambda(p+2)G_k(u_0,v_0).$$

Since $G_k(u_0,v_0) = (-1)^{p+1}$ and $I_k(u_0,v_0) = \mathcal{I}_k$, we have that $\lambda = (-1)^{p+1}\left(\frac{2}{p+2}\right)\mathcal{I}_k$. Moreover, for $(u,v) = \beta(u_0,v_0)$, we see that

$$\left\langle I_k'(u,v),(u,v)\right\rangle - \lambda\beta^{-p}\left\langle G_k'(u,v),(u,v)\right\rangle = 0.$$

Then, if we choose β to be

$$-\lambda\beta^{-p} = \frac{2}{p+1} \quad\Leftrightarrow\quad (-1)^p\left(\frac{p+1}{p+2}\right)\mathcal{I}_k = \beta^p \quad\Leftrightarrow\quad \left(\frac{p+1}{p+2}\right)\mathcal{I}_k = (-\beta)^p,$$

then we have that (u,v) is a nontrivial solution of (3.1). □

Now to use the Concentration-Compactness Principle, let us define positive measures $\{\nu_m\}$ by $d\nu_m = \rho_m\, dxdy$, where ρ_m is the density defined by

$$\rho_m = u_m^2 + a|\nabla u_m|^2 + |\nabla v_m|^2 + c(\Delta v_m)^2 - 2\omega u_m \partial_x v_m - 2b\omega \partial_x u_m \Delta v_m, \tag{3.7}$$

where we are assuming that $\{(u_m,v_m)\}$ in $\mathbb{X}_k$ is a minimizing sequence for $\mathcal{I}_k(\omega)$. From Theorem 2.4, we know that there exists a subsequence of $\{\nu_m\}$ that satisfies either *vanishing*, or *dichotomy*, or *compactness*. We will see that *vanishing* and *dichotomy* can be ruled out.

Lemma 3.3. *Let $\{(u_m,v_m)\}_m$ be a bounded sequence in $\mathbb{X}_k$. If there exists $r > 0$ such that*

$$\lim_{m\to\infty}\left(\sup_{y\in\mathbb{R}}\int_{R_{r,k}(y)} d\nu_m\right) = 0.$$

Then we have that

$$\lim_{m\to\infty} G_k(u_m,v_m) = 0.$$

In particular, if $\{(u_m,v_m)\}_m$ is a minimizing sequence for $\mathcal{I}_k(\omega)$, then vanishing is ruled out.

Proof. Is easy to see that for some positive constant $C = C(a,b,c,\omega)$

$$\|u_m\|^2_{H^1(R_{r,k}(y))} + \|\nabla v_m\|^2_{H^1(R_{r,k}(y))} \leq C\int_{R_{r,k}(y)} \rho_m\, dxdy = C\int_{R_{r,k}(y)} d\nu_m.$$

Now, we need to recall that the embedding $H^1(R_{r,k}(y)) \hookrightarrow L^{p+2}(R_{r,k}(y))$ is continuous, and so we obtain that

$$\begin{aligned}\int_{R_{r,k}(y)} u_m\left((\partial_x v_m)^{p+1} + (\partial_y v_m)^{p+1}\right)dxdy \\ &\leq C\left(\|u_m\|^{p+2}_{H^1(R_{r,k}(y))} + \|\nabla v_m\|^{p+2}_{H^1(R_{r,k}(y))}\right) \\ &\leq C\left[\|u_m\|^{p+1}_{H^1(R_{r,k}(y))} + \|\nabla v_m\|^{p+1}_{H^1(R_{r,k}(y))}\right]\left(\int_{R_{r,k}(y)} d\nu_m\right)^{\frac{1}{2}}.\end{aligned}$$

Covering Q_k by a countable number of rectangles such that every point in Q_k is contained in at most three rectangles $R_{r,k}(y)$, we see that

$$\begin{aligned}\int_{Q_k} u_m\Big((\partial_x v_m)^{p+1}+(\partial_y v_m)^{p+1}\Big)dxdy\\ &\leq 3C\Big(\|u_m\|^{p+1}_{H^1(Q_k)}+\|\nabla v_m\|^{p+1}_{H^1(Q_k)}\Big)\Big(\sup_{y\in\mathbb{R}}\int_{R_{r,k}(y)} dv_m\Big)^{1/2}\\ &\leq 3C(I_k(u_m,v_m))^{\frac{p+1}{2}}\Big(\sup_{y\in\mathbb{R}}\int_{R_{r,k}(y)} dv_m\Big)^{1/2}.\end{aligned}$$

Using (3.5) and that $\{(u_m,v_m)\}_m$ is a bounded sequence in $\mathbb{X}_k$ we conclude that

$$\lim_{m\to\infty}\int_{Q_k} u_m\Big((\partial_x v_m)^{p+1}+(\partial_y v_m)^{p+1}\Big)dxdy=0.$$

Now, if $\{(u_m,v_m)\}_m$ is a minimizing sequence for $\mathcal{I}_k(\omega)$ then we have that $G(u_m,v_m)=(-1)^{p+1}$, but from the previous fact we obtain a contradiction. □

To rule out dichotomy, we will establish a splitting result for a sequence $\{(u_m,v_m)\}$ in $\mathbb{X}_k$. Fix a function $\phi\in C_0^\infty(\mathbb{R},\mathbb{R}^+)$ such that $supp\,\phi\subset[-2,2]$ and $\phi\equiv 1$ in $[-1,1]$. If $r>0$ and $y_0\in\mathbb{R}$, we define a split for $(u,v)\in\mathbb{X}_k$ given by

$$u=u_r^1+u_r^2 \text{ and } v=v_r^1+v_r^2,$$

where

$$u_r^1=u\phi_r,\ u_r^2=u(1-\phi_r),\quad v_R^1=(v-a_r)\phi_r,\ v_r^2=(v-a_r)(1-\phi_r)+a_r,$$

with

$$\phi_r(y)=\phi\Big(\frac{y-y_0}{r}\Big),$$

and

$$a_r=\frac{1}{\mathrm{vol}\,(A_r(y_0))}\int_{A_r(y_0)} v(x,y)\,dxdy,\quad A_r(y_0)=R_{2r,k}(y_0)\setminus R_{r,k}(y_0).$$

We note that the decomposition of v is non standard and reflects the nature of the space $\mathcal{V}_k$. As a consequence of the coming result we will obtain that dichotomy is not possible.

Lemma 3.4. *Let $r_m>0$ and $\{y_m\}$ in $\mathbb{R}$ be sequences. Define $A(m)=A_{r_m}(y_m)$ and $\phi_m(y)=\phi\big(\frac{y-y_m}{r_m}\big)$. If*

$$\limsup_{m\to\infty}\Big(\int_{A(m)} dv_m\Big)=0. \tag{3.8}$$

Then we have that

$$\lim_{m\to\infty}\Big[I_k(u_m,v_m)-I_k(u_m^1,v_m^1)-I_k(u_m^2,v_m^2)\Big]=0, \tag{3.9}$$

$$\lim_{m\to\infty}\Big[G_k(u_m,v_m)-G_k(u_m^1,v_m^1)-G_k(u_m^2,v_m^2)\Big]=0. \tag{3.10}$$

For the proof, we see details in the work [12] in the case of solitary wave solutions. Using this result, we have that

Lemma 3.5. *Let $\{(u_m, v_m)\}_m$ be a minimizing sequence for $\mathcal{I}_k(\omega)$. Then dichotomy is not possible.*

Proof. Assume that dichotomy occurs, then we can choose sequences $\gamma_m \to 0$ and $r_m \to \infty$ such that

$$supp(\nu_m^1) \subset R_{r_m,k}(y_m), \quad supp(\nu_m^2) \subset Q_k \setminus R_{2r_m,k}(y_m) \tag{3.11}$$

and

$$\limsup_{m\to\infty}\left(\left|\theta - \int_{Q_k} d\nu_m^1\right| + \left|(\mathcal{I}_k - \theta) - \int_{Q_k} d\nu_m^2\right|\right) = 0. \tag{3.12}$$

Using these facts, we see that

$$\limsup_{m\to\infty}\left(\int_{A(m)} d\nu_m\right) = 0. \tag{3.13}$$

Now, from (3.13) and Lemma 3.4 we have that the splitting limits (3.9) and (3.10) hold and conclude that

$$\lim_{m\to\infty}\left[I_k(u_m, v_m) - I_k(u_m^1, v_m^1) - I_k(u_m^2, v_m^2)\right] = 0,$$
$$\lim_{m\to\infty}\left[G_k(u_m, v_m) - G_k(u_m^1, v_m^1) - G_k(u_m^2, v_m^2)\right] = 0.$$

Now, let $\lambda_{m,i} = G_k(u_m^i, v_m^i)$, for $i = 1,2$. Passing to a subsequence, if necessary, we have that $\lambda_i := \lim_{m\to\infty} \lambda_{m,i}$ exists. We claim that $\lambda_i \neq 0$. To see this, suppose that $\lim_{m\to\infty} \lambda_{m,1} = 0$, then $\lim_{m\to\infty} \lambda_{m,2} = (-1)^{p+1}$ (we proceed in a similar way in the other case). Then we conclude that $\lambda_{m,2} \neq 0$, for m large enough. Now, if we define

$$(w_m, z_m) = (-1)^{\frac{p+1}{p+2}} \lambda_{m,2}^{-\frac{1}{p+2}} (u_m^2, v_m^2),$$

then, we have that

$$(w_m, z_m) \in \mathbb{X}_k, \quad G_k(w_m, z_m) = (-1)^{p+1}.$$

On the other hand, using the characterization $\mathcal{I}_k(\omega)$,

$$\begin{aligned}
\mathcal{I}_k(\omega) &= \lim_{m\to\infty}\left(I_k(u_m^1, v_m^1) + I_k(u_m^2, v_m^2)\right) \\
&\geq \lim_{m\to\infty}\left(\int_{R_{r_m,k}(y_m)} d\nu_m^1 + \lambda_{m,2}^{\frac{2}{p+2}} I_k(w_m, z_m)\right) \\
&\geq \lim_{m\to\infty}\left(\int_{Q_k} d\nu_m^1 + \lambda_{m,2}^{\frac{2}{p+2}} \mathcal{I}_k(\omega)\right) \\
&= \theta + \mathcal{I}_k(\omega).
\end{aligned}$$

In other words, $|\lambda_{m,i}| > 0$ for m large enough and $i = 1,2$. Then we are allowed to define

$$(w_{m,i}, z_{m,i}) = (-1)^{\frac{p+1}{p+2}} \lambda_{m,i}^{-\frac{1}{p+2}} (u_m^i, v_m^i), \quad i = 1,2.$$

As above, we also have that $(w_{m,i}, z_{m,i}) \in \mathbb{X}_k$ and $G_k(w_{m,i}, z_{m,i}) = (-1)^{p+1}$. Then using the same argument as above, we conclude that

$$\begin{aligned}\mathcal{I}_k(\omega) &= \lim_{m\to\infty}\Big(I_k(u_m^1, v_m^1) + I_k(u_m^2, v_m^2)\Big)\\ &= \lim_{m\to\infty}\Big(|\lambda_{m,1}|^{\frac{2}{p+2}} I_k(w_{m,1}, z_{m,1}) + |\lambda_{m,2}|^{\frac{2}{p+2}} I_k(w_{m,2}, z_{m,2})\Big)\\ &\geq \Big(|\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}}\Big)\mathcal{I}_k(\omega).\end{aligned}$$

From this fact, we conclude that

$$1 \geq |\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} \geq (|\lambda_1| + |\lambda_2|)^{\frac{2}{p+2}} \geq |\lambda_1 + \lambda_2|^{\frac{2}{p+2}} = 1.$$

Hence, $|\lambda_1| + |\lambda_2| = 1$, and also that

$$|\lambda_1|^{\frac{2}{p+2}} + |\lambda_2|^{\frac{2}{p+2}} = (|\lambda_1| + |\lambda_2|)^{\frac{2}{p+2}}, \tag{3.14}$$

which is a contradiction, since the function $f(t) = t^{\frac{2}{p+2}}$ is strictly concave on $\mathbb{R}^+$. In other words, we have ruled out dichotomy. □

Now we will show the main result in this section: the existence of a minimizer for $\mathcal{I}_k(\omega)$, which implies the existence of solutions for the system (3.1) in the space $\mathbb{X}_k$.

Theorem 3.6. *If $\{(u_m, v_m)\}$ is a minimizing sequence for $\mathcal{I}_k(\omega)$, then there is a subsequence (which we denote by the same symbol), a sequence of points $\{y_m\}$ in $\mathbb{R}$, and a minimizer $(u_0, v_0) \in \mathbb{X}_k$ for $\mathcal{I}_k(\omega)$, such that the translated functions*

$$(\tilde{u}_m(x,y), \tilde{v}_m(x,y)) = (u_m(x, y+y_m), v_m(x, y+y_m))$$

converge to (u_0, v_0) strongly in $\mathbb{X}_k$.

Proof. Let $\{(u_m, v_m)\}$ be a minimizing sequence for $\mathcal{I}_k(\omega)$. This is,

$$\lim_{m\to\infty} I_k(u_m, v_m) = \mathcal{I}_k(\omega) \quad \text{and} \quad G_k(u_m, v_m) = (-1)^{p+1}.$$

Since we ruled out vanishing and dichotomy above for a minimizing sequence of $\mathcal{I}_k(\omega)$, then by P. L. Lion's Concentration-Compactness Theorem, there exists a subsequence of $\{\nu_m\}$ (which we denote by the same symbol) satisfying compactness. Then, there exists a sequence $\{y_m\}$ in $\mathbb{R}$ such that for a given $\gamma > 0$, there exists $r > 0$ with the following property,

$$\int_{R_{r,k}(y_m)} d\nu_m \geq \mathcal{I}_k(\omega) - \gamma, \quad \text{for all } m \in \mathbb{Z}^+. \tag{3.15}$$

Using this we may localize the minimizing sequence $\{(u_m, v_m)\}_m$ around the origin by defining the translation in the y variable

$$\tilde{\rho}_m(x,y) = \rho_m(x, y+y_m), \quad (\tilde{u}_m, \tilde{v}_m)(x,y) = (u_m, v_m)(x, y+y_m).$$

Thus, we have the following localized inequality

$$\int_{R_{r,k}(0)} \tilde{\rho}_m dxdy = \int_{R_{r,k}(y_m)} d\nu_m \geq \mathcal{I}_k(\omega) - \gamma, \quad \text{for all } m \in \mathbb{Z}^+, \tag{3.16}$$

and also that

$$\lim_{m\to\infty} I_k(\tilde{u}_m,\tilde{v}_m) = \lim_{m\to\infty} I_k(u_m,v_m) = \mathcal{I}_k(\omega),\ \ G_k(\tilde{u}_m,\tilde{v}_m) = G_k(u_m,v_m) = (-1)^{p+1}. \tag{3.17}$$

From previous fact and by (3.5), we note that $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ is a bounded sequence in $\mathbb{X}_k$. On the other hand, since $\tilde{u}_m, \nabla\tilde{v}_m \in H^1(U)$ for any bounded open set $U \subset Q_k$ and the embedding $H^1(U) \hookrightarrow L^q(U)$ is compact for $q \geq 2$, then there exist a subsequence of $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ (which we denote the same) and $(u_0,v_0) \in \mathbb{X}_k$ such that for $i = 1,2$,

$$\begin{gathered}\tilde{u}_m \rightharpoonup u_0 \quad \text{in} \quad H^1(Q_k),\ \ \tilde{u}_m \rightharpoonup u_0 \quad \text{in} \quad L^2(Q_k),\\ \tilde{v}_m \rightharpoonup v_0 \quad \text{in} \quad \mathcal{V}_k,\ \ \partial_i\tilde{v}_m \rightharpoonup v_0 \quad \text{in} \quad L^2(Q_k)\end{gathered}$$

and we also have that

$$\tilde{u}_m \to u_0 \quad \text{in} \quad L^2_{loc}(Q_k),\ \ \partial_i\tilde{v}_m \to \partial_i v_0 \quad \text{in} \quad L^2_{loc}(Q_k).$$

Moreover,

$$\tilde{u}_m \to u_0 \ \text{ a.e. in } Q_k,\ \ \partial_i\tilde{v}_m \to \partial_i v_0 \ \text{ a.e in } Q_k \ \text{ for } i = 1,2.$$

Using these facts we will show that some subsequence of $\{(\tilde{u}_m,\tilde{v}_m)\}_m$ (which we denote in the same way) converges strongly in $\mathbb{X}_k$ to a nontrivial minimizer (u_0,v_0) of (3.4). We first see that

$$\tilde{u}_m \to u_0,\ \ \partial_i\tilde{v}_m \to \partial_i v_0 \ \text{ in } \ L^2(Q_k). \tag{3.18}$$

In fact, using (3.16), (3.17) and the definition of I_k we have that for $\gamma > 0$, there exists $r > 0$ such that for m large enough,

$$\int_{R_{r,k}(0)} |\tilde{u}_m|^2 dxdy \geq \int_{Q_k} |\tilde{u}_m|^2\, dxdy - 2\gamma.$$

Then we have that

$$\begin{aligned}\int_{Q_k} |u_0|^2 dxdy &\leq \liminf_{m\to\infty} \int_{Q_k} |\tilde{u}_m|^2 dxdy\\ &\leq \liminf_{m\to\infty} \int_{R_{r,k}(0)} |\tilde{u}_m|^2 dxdy + 2\gamma\\ &= \int_{R_{r,k}(0)} |u_0|^2 dxdy + 2\gamma\\ &\leq \int_{Q_k} |u_0|^2 dxdy + 2\gamma.\end{aligned}$$

Therefore

$$\liminf_{m\to\infty} \int_{Q_k} |\tilde{u}_m|^2 dxdy = \int_{Q_k} |u_0|^2 dxdy.$$

Thus, there exist a subsequence of $\{\tilde{u}_m\}$ such that $\tilde{u}_m \to u_0$ in $L^2(Q_k)$. Using a similar argument we prove the other part of (3.18). Moreover, also we can see that

$$\partial_i\tilde{u}_m \to \partial_i u_0, \quad \partial_{ij}\tilde{v}_m \to \partial_{ij} v_0 \quad \text{in} \quad L^2(Q_k). \tag{3.19}$$

Now, using (3.18)-(3.19) and that the inclusion $H^1(Q_k) \hookrightarrow L^q(Q_k)$ is continuous for $q \geq 2$, we have that

$$G_k(u_0, v_0) = \lim_{m\to\infty} G_k(\tilde{u}_m, \tilde{v}_m) = (-1)^{p+1}. \tag{3.20}$$

In fact, for $j = 1, 2$, we have that

$$\int_{Q_k} \Big(\tilde{u}_m(\partial_j \tilde{v}_m)^{p+1} - u_0(\partial_j v_0)^{p+1}\Big) dxdy$$
$$= \int_{Q_k} \Big[(\tilde{u}_m - u_0)(\partial_j \tilde{v}_m)^{p+1} + u_0\Big((\partial_j \tilde{v}_m)^{p+1} - (\partial_j v_0)^{p+1}\Big)\Big] dxdy.$$

Now, we have that

$$\int_{Q_k} (\tilde{u}_m - u_0)(\partial_j \tilde{v}_m)^{p+1}\, dxdy \leq \|\tilde{u}_m - u_0\|_{L^2(Q_k)} \|\partial_j \tilde{v}_m\|^{p+1}_{L^{2(p+1)}}$$
$$\leq C\|\tilde{u}_m - u_0\|_{L^2(Q_k)} (I_k(\tilde{u}_m, \tilde{v}_m))^{\frac{p+1}{2}}.$$

On the other hand we also have, after using Hölder inequality, that

$$\int_{Q_k} \Big|u_0\Big((\partial_j \tilde{v}_m)^{p+1} - (\partial_j v_0)^{p+1}\Big)\Big| dxdy \leq C \int_{Q_k} |u_0| \big|\partial_j \tilde{v}_m - \partial_j v_0\big| \Big(|\partial_j \tilde{v}_m|^p + |\partial_j v_0|^p\Big) dxdy$$
$$\leq C\|\partial_j(\tilde{v}_m - v_0)\|_{L^2(Q_k)} \|\partial_j(\tilde{v}_m + v_0)\|^p_{L^{4p}(Q_k)} \|u_0\|_{L^4(Q_k)}$$
$$\leq C\|\partial_j(\tilde{v}_m - v_0)\|_{L^2(Q_k)} \|\partial_j(\tilde{v}_m + v_0)\|^p_{H^1(Q_k)} \|u_0\|_{H^1(Q_k)}$$
$$\leq C\|\partial_j(\tilde{v}_m - v_0)\|_{L^2(Q_k)} (I_k(\tilde{u}_m, \tilde{v}_m) + I_k(u_0, v_0))^{\frac{p+1}{2}}.$$

Then we conclude that (3.20) holds, which implies that $(u_0, v_0) \neq (0, 0)$. On the other hand, from (3.18)-(3.19), we can see that

$$\lim_{m\to\infty} I_k(\tilde{u}_m, \tilde{v}_m) = I_k(u_0, v_0) = \mathcal{I}_k(\omega), \quad \lim_{m\to\infty} I_k(\tilde{u}_m - u_0, \tilde{v}_m - v_0) = 0.$$

Moreover, the sequence $\{(\tilde{u}_m, \tilde{v}_m)\}_m$ converges to (u_0, v_0) in $\mathbb{X}_k$, since

$$\|(\tilde{u}_m - u_0, \tilde{v}_m - v_0)\|_{\mathbb{X}_k} \leq C_1 I_k(u_m - u_0, v_m - v_0).$$

Then we concluded that $\{(\tilde{u}_m, \tilde{v}_m)\}$ converges to (u_0, v_0) in $\mathbb{X}_k$ and (u_0, v_0) is a minimizer for $\mathcal{I}_k(\omega)$. □

Combining Lemma 3.2 and Theorem 3.6 we obtain the following corollary.

Corollary 3.7. *Let $k > 0$ and $0 < \omega < \min\{1, \frac{a}{b}, \frac{c}{b}\}$. Then the abc-Boussinesq system (1.1) has a nontrivial x-periodic travelling wave solution of period k in the space $\mathbb{X}_k$.*

4 Inter-relation between periodic travelling waves and solitons

In this section we will show that a special sequence of x-periodic travelling wave solutions parameterized by the period k converges to a solitary wave in a appropriate sense as the $k \to \infty$. This result in particular shows that the shape of x-periodic travelling wave solutions are "similar" to the shape of solitons, as the period $k \to \infty$.

Before we go further, we discuss the results for solitary wave solutions,

On the solitons for the *abc*-Boussinesq system

Recently in [12] for the case $a = \sigma$, $b = \frac{1}{2}$ and $c = \frac{2}{3}$, J. Quintero and A. Montes showed the existence of solitary wave solutions (solitons) in the energy space $\mathbb{X} = H^1(\mathbb{R}^2)\times\mathcal{V}$ via the **Concentration-Compactness Principle** by P. Lions (see [3], [4]). By following the same approach, it is possible to prove the existence of solitons for the *abc*-Boussinesq system in the energy space $\mathbb{X} = H^1(\mathbb{R}^2)\times\mathcal{V}$ by considering the following minimization problem

$$\mathcal{I}(\omega) := \inf\left\{I(u,v) \ : \ (u,v)\in\mathbb{X} \ \text{with} \ G(u,v) = (-1)^{p+1}\right\}, \tag{4.1}$$

where the energy I and the constraint G are functionals defined in $\mathbb{X}$ given by

$$I(u,v) = \int_{\mathbb{R}^2}\left(u^2 + a|\nabla u|^2 + |\nabla v|^2 + c(\Delta v)^2 - 2\omega u v_x - 2b\omega u_x\Delta v\right)dxdy, \tag{4.2}$$

$$G(u,v) = \int_{\mathbb{R}^2} u\left(v_x^{p+1} + v_y^{p+1}\right)dxdy. \tag{4.3}$$

The existence of solitons for the *abc*-Boussinesq system is obtained by adapting the results in [12]. In particular, we have the following results,

Lemma 4.1. *If (u_0,v_0) is a minimizer for the problem (4.1), then $(u,v) = \beta(u_0,v_0)$ is a nontrivial solution of (3.1) for $(-\beta)^p = \left(\frac{p+1}{p+2}\right)\mathcal{I}(\omega)$.*

Lemma 4.2. *Let $0 < \omega < \min\{1, \frac{a}{b}, \frac{c}{b}\}$. Then the functional I is nonnegative and there are positive constants $C_1(a,b,c,\omega) < C_2(a,b,c,\omega)$ such that*

$$C_1(a,b,c,\omega)I(u,v) \le \|(u,v)\|_{\mathbb{X}}^2 \le C_2(a,b,c,\omega)I(u,v). \tag{4.4}$$

Furthermore, $\mathcal{I}(\omega)$ is finite and positive.

Lemma 4.3. *Let $0 < \omega < \min\{1, \frac{a}{b}, \frac{c}{b}\}$. If $\{(u_m,v_m)\}$ is a minimizing sequence for (4.1), then there is a subsequence (which we denote by the same symbol), a sequence of points $(x_m,y_m)\in\mathbb{R}^2$, and a minimizer $(u_0,v_0)\in\mathbb{X}$ of (4.1), such that the translated functions*

$$(\tilde{u}_m(x,y),\tilde{v}_m(x,y)) = (u_m(x+x_m,y+y_m), v_m(x+x_m,y+y_m))$$

converge to (u_0,v_0) strongly in $\mathbb{X}$.

A variational approach

We will consider ground state travelling waves, meaning x-periodic travelling waves (u_k,v_k) of period k for the *abc*-Boussinesq system (1.1) which minimize the problem

$$\mathbb{J}_k(\omega) = J_{\omega,k}(u_k,v_k) = \inf\left\{J_{\omega,k}(u,v) \ : \ (u,v)\in\mathbb{X}_k \ \text{with} \ \Lambda_k(u,v) = 0\right\}, \tag{4.5}$$

where

$$\Lambda_k(u,v) = \left\langle J'_{\omega,k}(u,v),(u,v)\right\rangle = 2I_k(u,v) + \frac{2(p+2)}{p+1}G_k(u,v). \tag{4.6}$$

Note that the condition $\Lambda_k(u,v) = 0$ is just a “artificial constraint” for minimizing the functional $J_{\omega,k}$ on $\mathbb{X}_k$. It is important to observe that the infimum in (4.5) is being taken in a nonempty class. To see this, we choose $\phi, \psi \in C_0^\infty(Q_1)$ such that $G_1(\phi,\psi) \neq 0$ (if p_1 is even, $G_1(\phi,\psi) < 0$). Then the x-periodic extension of (ϕ,ψ), denoted by (ϕ_k,ψ_k), belongs to $\mathbb{X}_k$ and satisfies that $G_k(\phi_k,\psi_k) = G_1(\phi,\psi)$. If we define α by

$$\alpha^p = -\frac{(p+1)I_k(\phi_k,\psi_k)}{(p+2)G_k(\phi_k,\psi_k)}.$$

we have that

$$\Lambda_k(\alpha(\phi_k,\psi_k)) = 2\alpha^2\left(I_k(\phi_k,\psi_k) + \frac{(p+2)\alpha^p}{p+1}G_k(\phi_k,\psi_k)\right) = 0.$$

We also note that a minimizer for $\mathbb{J}_k(\omega)$ corresponds to a x-periodic solution of period k for the system (3.1). More exactly,

Theorem 4.4. *If $(u_k,v_k) \in \mathbb{X}_k$ is a minimizer for (4.5), then (u_k,v_k) is a solution of the system (3.1).*

We also have the following results.

Theorem 4.5. *Let $k \geq 1$ be fixed and $0 < \omega < \min\{1, \frac{a}{b}, \frac{c}{b}\}$. If $\{(u_m,v_m)\}_m$ is a minimizing sequence for $\mathbb{J}_k(\omega)$, then there is a sequence of points $\{y_m\}$ in $\mathbb{R}$, a subsequence of $\{(u_m,v_m)\}_m$ (which we denote by the same symbol) and a minimizer $(u_k,v_k) \in \mathbb{X}_k$ for $\mathbb{J}_k(\omega)$ such that the translated functions*

$$(\tilde{u}_m(x,y), \tilde{v}_m(x,y)) = (u_m(x,y+y_m), v_m(x,y+y_m))$$

converge strongly to (u_k,v_k) in $\mathbb{X}_k$ and (u_k,v_k) is a nontrivial solution of the system (3.1). Moreover,

$$\mathbb{J}_k(\omega) = \frac{p}{p+2}\left(\frac{p+1}{p+2}\right)^{\frac{2}{p}}\left[\mathcal{I}_k(\omega)\right]^{\frac{p+2}{p}}. \tag{4.7}$$

Proof. This result is a consequence of the Lemma 3.2, Theorem 3.6, and the following argument. Let $(u,v) \in \mathbb{X}_k \setminus \{0\}$ be such $\mathbb{J}_k(\omega) = J_{\omega,k}(u,v)$ and that $\Lambda_k(u,v) = 0$, then

$$I_k(u,v) = -\frac{p+2}{p+1}G_k(u,v) = \frac{p+2}{p+1}|G_k(u,v)| = \frac{p+2}{p}J_{\omega,k}(u,v).$$

Now consider the couple

$$(z,w) = (-1)^{\frac{p+1}{p+2}}\left[G_k(u,v)\right]^{-\frac{1}{p+2}}(u,v).$$

Then $G_k(z,w) = (-1)^{p+1}$. Thus,

$$\begin{aligned}\mathcal{I}_k(\omega) \leq I_k(z,w) &= \left[G_k(u,v)\right]^{-\frac{2}{p+2}} I_k(u,v)\\ &= \left(\frac{p+2}{p+1}\right)^{\frac{2}{p+2}}\left[I_k(u,v)\right]^{\frac{p}{p+2}}\\ &= \left(\frac{p+2}{p+1}\right)^{\frac{2}{p+2}}\left[\frac{p+2}{p}J_{\omega,k}(u,v)\right]^{\frac{p}{p+2}}.\end{aligned}$$

We concluded from this that

$$\frac{p}{p+2}\left(\frac{p+1}{p+2}\right)^{\frac{2}{p}}\left[\mathcal{I}_k(\omega)\right]^{\frac{p+2}{p}} \le \mathbb{J}_k(\omega).$$

Now, let $(u,v) \neq 0$ such that $\mathcal{I}_k(\omega) = I_k(u,v)$ and that $G_k(u,v) = (-1)^{p+1}$. If we define t by

$$t^p = -\frac{(p+1)I_k(u,v)}{(-1)^{p+1}(p+2)}.$$

we have that

$$\Lambda_k(t(u,v)) = 2t^2\left(I_k(u,v) + \frac{(p+2)t^p}{p+1}G_k(u,v)\right) = 0.$$

In this case,

$$t^2 = \left(\left(\frac{p+1}{p+2}\right)I_k(u,v)\right)^{\frac{2}{p}}.$$

Then we see that

$$\mathbb{J}_k(\omega) \le J_{\omega,k}(tu,tv) = t^2\left(I_k(u,v) + \frac{2t^p}{p+1}G_k(u,v)\right) = \frac{p}{p+2}\left(\frac{p+1}{p+2}\right)^{\frac{2}{p}}\left[\mathcal{I}_k(\omega)\right]^{\frac{p+2}{p}}.$$

Hence, as desired (4.7) holds. Now, the first part follows by noting that if $(u_m,v_m) \in \mathcal{X}_k$ is such that $\mathbb{J}_k(\omega) = \lim_{m\to\infty} J_{\omega,k}(u_m,v_m)$ with $\Lambda_k(u_m,v_m) = 0$, then

$$I_k(u_m,v_m) = -\frac{p+2}{p+1}G_k(u_m,v_m) = \frac{p+2}{p}J_{\omega,k}(u_m,v_m).$$

Moreover, we also have that

$$(z_m,w_m) = (-1)^{\frac{p+1}{p+2}}\left[G_k(u_m,v_m)\right]^{-\frac{1}{p+2}}(u_m,v_m).$$

is such that $G_k(z_m,w_m) = (-1)^{p+1}$ and $\mathcal{I}_k(\omega) = \lim_{m\to\infty} I_k(z_m,w_m)$, which follows by using formula (4.7). In other words, $(u_m,v_m) \in \mathbb{X}_k$ is a minimizing sequence for $\mathcal{I}_k(\omega)$. So, the conclusion follows by Theorem 3.6, after going back to the minimizing sequence $(u_m,v_m) \in \mathcal{X}_k$ for $\mathbb{J}_k(\omega)$. □

Theorem 4.6. *For $k \ge 1$, we have that the family $\{\mathbb{J}_k(\omega)\}_{k\ge 1}$ is bounded below and above for positive constants independent of k.*

Proof. Let $(u_k,v_k) \in \mathbb{X}_k$ be a family such that

$$J_{\omega,k}(u_k,v_k) = \mathbb{J}_k(\omega), \quad \Lambda_k(u_k,v_k) = 0.$$

Then, using (3.3)-(3.5) and (4.6), we have that

$$I_k(u_k,v_k) = -\frac{p+2}{p+1}G_k(u_k,v_k) \le C(p)\|(u_k,v_k)\|_{\mathbb{X}_k}^{p+2} \le C(p)\left[I_k(u_k,v_k)\right]^{\frac{p+2}{2}}.$$

This implies that there is $C_3 = C_3(p) > 0$ such that

$$\mathbb{J}_k(\omega) = J_{\omega,k}(u_k,v_k) = \frac{p}{p+2}I_k(u_k,v_k) \ge C_3.$$

On the other hand, for $k \geq 1$ we can choose $\phi \in C_0^\infty(Q_1)$ such that $G_1(\phi,\phi) \neq 0$ (if p_1 is even, $G_1(\phi,\phi) < 0$). Since $supp\phi \subset Q_1 \subset Q_k$, then we can define a periodic extension of ϕ as follows

$$\phi_k(x,y) = \phi(x,y), \quad (x,y) \in Q_1$$
$$\phi_k(x,y) = 0, \qquad (x,y) \in Q_k \setminus Q_1.$$

Then (ϕ_k,ϕ_k) belongs to $\mathbb{X}_k$ and satisfies that $G_k(\phi_k,\phi_k) = G_1(\phi,\phi) \neq 0$ (if p_1 is even, $G_1(\phi,\phi) < 0$). If we set α by

$$\alpha^p = -\frac{(p+1)I_k(\phi_k,\phi_k)}{(p+2)G_k(\phi_k,\phi_k)},$$

then a direct computation shows that

$$\Lambda_k(\alpha(\phi_k,\phi_k)) = 2\alpha^2\left(I_k(\phi_k,\phi_k) + \frac{(p+2)\alpha^p}{p+1}G_k(\phi_k,\phi_k)\right) = 0.$$

Hence, we conclude that

$$\mathbb{J}_k(\omega) \leq J_{\omega,k}(\alpha(\phi_k,\phi_k)) = J_1(\alpha(\phi,\phi)) = C_4(\phi).$$

This implies that there are positive constants C_3, C_4 (not depending on k) such that

$$C_3 \leq \mathbb{J}_k(c) \leq C_4, \quad k \geq 1. \tag{4.8}$$

□

Now we establish some technical results. The first one is related with the characterization of "vanishing sequences" in $\mathbb{X}_k$. Define ρ on $\mathbb{X}_k$ be defined as

$$\rho(u,v) = u^2 + a|\nabla u|^2 + |\nabla v|^2 + c(\Delta v)^2 - 2\omega u v_x - 2\omega b u_x \Delta v, \tag{4.9}$$

and for $r > 0$ denote by $R_r(\zeta)$ the closed square centered at the point $\zeta \in \mathbb{R}^2$.

Lemma 4.7. *Let $k \geq 1$ and assume that $\{(u_k,v_k)\}_{k\geq 1}$ is a sequence of x-periodic functions such that $(u_k,v_k) \in \mathbb{X}_k$ and $\|(u_k,v_k)\|_{\mathbb{X}_k} \leq C$ for all k. If there exist $r > 0$ such that*

$$\lim_{k\to\infty} \sup_{\zeta\in\mathbb{R}^2} \int_{R_r(\zeta)} \rho(u_k,v_k)\,dxdy = 0. \tag{4.10}$$

Then, for $q \geq 2$ we have that

$$\lim_{k\to\infty} \|v_k\|_{\mathcal{M}^q(Q_k)} = \lim_{k\to\infty} \|u_k\|_{L^q(Q_k)} = 0.$$

Proof. First suppose that $\{w_k\}_k$ is a sequence such that $w_k \in H^1(Q_k)$ and $\|w_k\|_{H^1(Q_k)} \leq C$. Assume that there is a positive constant $r > 0$ such that

$$\lim_{k\to\infty} \sup_{\zeta\in\mathbb{R}^2} \int_{R_r(\zeta)} w_k^2\,dxdy = 0. \tag{4.11}$$

We will see that $\lim_{k\to\infty}\|w_k\|_{L^q(Q_k)} = 0$. In fact, if $q \geq 2$ we have that

$$\|w_k\|^q_{L^q(R_r(\zeta))} \leq \|w_k\|_{L^2(R_r(\zeta))}\|w_k\|^{q-1}_{L^{2(q-1)}(R_r(\zeta))} \leq \|w_k\|_{L^2(R_r(\zeta))}\|w_k\|^{q-1}_{H^1(Q_k)}.$$

Covering Q_k by a countable number of squares such that every point in Q_k is contained in at most three squares $R_r(\zeta)$, we obtain that

$$\|w_k\|^q_{L^q(Q_k)} \leq 3 \sup_{\zeta\in\mathbb{R}^2} \|w_k\|_{L^2(R_r(\zeta))}\|w_k\|^{q-1}_{H^1(Q_k)}.$$

Using (4.11) and that there is $C > 0$ such that $\|w_k\|_{H^1(Q_k)} \leq C$, we conclude that

$$\lim_{k\to\infty} \|w_k\|_{L^q(Q_k)} = 0.$$

Now suppose that $\|(u_k, v_k)\|_{X_k} \leq C$ and that (4.10) holds. Then $u_k, \partial_x v_k, \partial_y v_k \in H^1(Q_k)$. Hence, for w_k being defined as either u_k, $\partial_x v_k$, or $\partial_y v_k$, we see that w_k satisfies in each case the condition (4.11). By the previous observation, we conclude for $q \geq 2$ that

$$\lim_{k\to\infty} \|w_k\|_{L^q(Q_k)} = 0.$$

In other words, we have for $q \geq 2$ that

$$\lim_{k\to\infty} \|u_k\|_{L^q(Q_k)} = \lim_{k\to\infty} \|v\|_{\mathcal{M}^q(Q_k)} = 0.$$

□

Now we prove a result related with the behavior of the bounded sequence $\{(u_k, v_k)\}_k$ of critical points for $J'_{\omega,k}$.

Lemma 4.8. *Let $k \geq 1$ and assume that $\{(u_k, v_k)\}_{k\geq1}$ is a sequence of x-periodic functions such that $(u_k, v_k) \in \mathbb{X}_k$, $\|(u_k, v_k)\|_{\mathbb{X}_k} \leq C$ and $J'_{\omega,k}(u_k, v_k) = 0$. Then*

1. $\lim_{k\to\infty} \|(u_k, v_k)\|_{\mathbb{X}_k} = 0$, *or*
2. *There are positive constants r, η and a sequence $\zeta_k \in \mathbb{R}^2$ such that*

$$\lim_{k\to\infty} \int_{R_r(\zeta_k)} \rho(u_k, v_k)dxdy > \eta.$$

Proof. Note that

$$I_k(u_k, v_k) + \frac{p+2}{p+1} G_k(u_k, v_k) = 0.$$

Then from (3.3)-(3.5) we conclude that

$$\|(u_k, v_k)\|^2_{\mathbb{X}_k} \leq CI_k(u_k, v_k) \leq C(p)\,|G_k(u_k, v_k)| \leq C(p)\left(\|u_k\|^3_{L^3(Q_k)} + \|v_k\|^3_{\mathcal{M}^3(Q_k)}\right).$$

If condition (2) does not hold, then from previous result we conclude that the right hand side tends to zero, which implies that the condition (1) holds. □

Finally we establish the main result in this section. We define J_w in the space $\mathbb{X}$ with values in $\mathbb{R}$ as

$$J_\omega(u,v) = I(u,v) + \frac{2}{p+1} G(u,v),$$

and denote with Λ to

$$\Lambda(u,v) = \langle J'_\omega(u,v),(u,v)\rangle .$$

Theorem 4.9. *Let $k \geq 1$ and let $(u_k, v_k) \in \mathbb{X}_k$ be a minimizer for $\mathbb{J}_k(\omega)$. Then there exists a sequence $\zeta_k \in \mathbb{R}^2$ and a function $(u_0, v_0) \in \mathbb{X}$ such that $S_k(u_k(\cdot+\zeta_k), v_k(\cdot+\zeta_k))$ converges weakly to $(u_0, v_0) \in \mathbb{X}$ along a subsequence. We also have that (u_0, v_0) is a nontrivial solution of the system (3.1) and a minimizer for $\mathbb{J}(\omega)$, where*

$$\mathbb{J}(\omega) = J_\omega(u_0, v_0) = \inf\{J_\omega(u,v) \ : \ (u,v) \in \mathbb{X} \ \textit{with} \ \ \Lambda(u,v) = 0\},$$

Moreover,

$$\lim_{k\to\infty} \|(u_k, v_k) - (u_0(\cdot+\zeta_k), v_0(\cdot+\zeta_k))\|_{\mathbb{X}_k} = 0. \tag{4.12}$$

Proof. Using (3.5), (4.8) and the equality $J_{\omega,k}(u_k, v_k) = \frac{p}{p+2} I_k(u_k, v_k)$ we can conclude that

$$0 < C_3 \leq \|(u_k, v_k)\|_{\mathbb{X}_k} \leq C_4,$$

implying that the condition (1) in Lemma 4.8 does not hold. In other words, there exists a sequence $\zeta_k \in \mathbb{R}^2$ such that the shifted sequence $(\tilde{u}_k, \tilde{v}_k) = (u_k(\cdot+\zeta_k), v_k(\cdot+\zeta_k))$, for appropriate choice of $r, \eta > 0$, satisfies

$$\int_{R_r(0)} \rho(\tilde{u}_k, \tilde{v}_k)\, dxdy = \int_{R_r(\zeta_k)} \rho(u_k, v_k)\, dxdy > \frac{\eta}{2}.$$

Clearly $(\tilde{u}_k, \tilde{v}_k)$ is also a minimizing sequence for $\mathbb{J}_k(\omega)$. Now observe that the sequence $E_k(\tilde{u}_k, \tilde{v}_k) \in \mathbb{X}$ is bounded since

$$\|E_k(\tilde{u}_k, \tilde{v}_k)\|_{\mathbb{X}} \leq C_5 \|(\tilde{u}_k, \tilde{v}_k)\|_{\mathbb{X}_k}.$$

Thus, there exists a subsequence of $\{E_k(\tilde{u}_k, \tilde{v}_k)\}$ (denoted by the same symbol) and $(u_0, v_0) \in \mathbb{X}$ such that

$$E_k(\tilde{u}_k, \tilde{v}_k) \rightharpoonup (u_0, v_0), \ \text{ as } k \to \infty \quad \text{(weakly in } \mathbb{X}).$$

We claim that (u_0, v_0) is a non trivial solitary wave for the system (3.1). Let $Z = (U, V) \in C_0^\infty(\mathbb{R}^2) \times C_0^\infty(\mathbb{R}^2)$. Then, for sufficiently large k we have that $K = supp Z \subset Q_k \times Q_k$. Hence, Z can be considered as an element of $\mathbb{X}_k$ for a large k just by defining its periodic extension.

Thus, we see that

$$
\begin{aligned}
&\langle I'(u_0,v_0),Z\rangle\\
&=2\int_K (u_0U+a\nabla u_0\cdot\nabla U+\nabla v_0\cdot\nabla V+c\Delta v_0\Delta V)\,dxdy\\
&-\omega\int_K (2u_0V_x+2(v_0)_xU+2b(u_0)_x\Delta V+2b\Delta v_0U_x)\,dxdy\\
&=2\lim_{k\to\infty}\int_{Q_k}(F_k(\tilde u_k)U+a\nabla F_k(\tilde u_k)\cdot\nabla U+\nabla S_k(\tilde v_k)\cdot\nabla V+c\Delta S_k(\tilde v_k)\Delta V)\,dxdy\\
&-\omega\lim_{k\to\infty}\int_{Q_k}(2F_k(\tilde u_k)V_x+2(S_k(\tilde v_k))_xU+2b(F_k(\tilde u_k))_x\Delta V+2b\Delta S_k(\tilde v_k)U_x)\,dxdy\\
&=2\lim_{k\to\infty}\int_{Q_k}(\tilde u_kU+a\nabla\tilde u_k\cdot\nabla U+\nabla\tilde v_k\cdot\nabla V+c\Delta\tilde v_k\Delta V)\,dxdy\\
&-\omega\lim_{k\to\infty}\int_{Q_k}(2\tilde u_kV_x+2(\tilde v_k)_xU+2b(\tilde u_k)_x\Delta V+2b\Delta\tilde v_kU)_x dxdy\\
&=\lim_{k\to\infty}\left\langle I_k'(\tilde u_k,\tilde v_k),Z\right\rangle.
\end{aligned}
$$

In a similar way, noting that the sequences $\left\{(\partial_i S_k(\tilde v_k))^{p+1}\right\}_k$ and $\{F_k(\tilde u_k)(\partial_i S_k(\tilde v_k))^p\}_k$ are bounded in $L^2(\mathbb{R}^2)$ (taking a subsequence, if necessary), we have that

$$\langle G'(u_0,v_0),Z\rangle=\lim_{k\to\infty}\left\langle G_k'(\tilde u_k,\tilde v_k),Z\right\rangle.$$

In other words, we have shown that

$$\langle J_\omega'(u_0,v_0),Z\rangle=\lim_{n\to\infty}\left\langle J_{\omega,k}'(\tilde u_k,\tilde v_k),Z\right\rangle=0.$$

Then, by a using a density argument, we conclude that

$$J_\omega'(u_0,v_0)=0.$$

In other words, (u_0,v_0) is a non trivial soliton (a travelling wave solution in the energy space $\mathbb{X}$) of the system (1.1). Finally we want to establish that

$$J_\omega(u_0,v_0)=\mathbb{J}_\omega=\inf\{J_\omega(u,v)\in\mathbb{X}\ :\ \Lambda(u,v)=0\},\quad \lim_{k\to\infty}\|(\tilde u_k,\tilde v_k)-(u_0,v_0)\|_{\mathbb{X}_k}=0.$$

First we notice that if $(w,z)\in\mathbb{X}$ and $\Lambda(w,z)=0$, then there exists a sequence $\{(w_k,z_k)\}_k$ such that $(w_k,z_k)\in C_0(Q_k)\times C_0(Q_k)$ such that

$$\lim_{k\to\infty}\|(w_k,z_k)-(w,z)\|_{\mathbb{X}}=0.$$

By the assumption on (w,z) we have that $I(w,z)+\frac{p+2}{p+1}G(w,z)=0$. Then, it follows that $G(w,z)<0$. Hence, for k large enough we have that $G(w_k,z_k)<0$. Moreover, $\Lambda(t_kw_k,t_kz_k)=0$ for $t_k\in(0,1)$ given by $t_k^p=-\frac{(p+1)I(w_k,z_k)}{(p+2)G(w_k,z_k)}$. Note that the continuity of functionals I and G and that $\Lambda(w,z)=0$ imply that

$$t_k=\left(-\frac{(p+1)I(w_k,z_k)}{(p+2)G(w_k,z_k)}\right)^{\frac1p}\to\left(-\frac{(p+1)I(w,z)}{(p+2)G(w,z)}\right)^{\frac1p}=1,\ \text{ as }\ k\to\infty.$$

On the other hand, $t_k(w_k, z_k) \to (w, z)$ in $\mathbb{X}$. Then the continuity of the functional J_ω implies that

$$J_\omega\,(t_k w_k, t_k z_k)) \to J_\omega(z, w), \quad \text{as} \quad k \to \infty.$$

Thus, given $\epsilon > 0$, there exists k_ϵ such that if $k > k_\epsilon$,

$$J_\omega(t_k w_k, t_k z_k) \le J_\omega(w, z) + \epsilon.$$

This implies that $\limsup_{k\to\infty} \mathbb{J}_k(\omega) \le J_\omega(w, z) + \epsilon$ for any $(w, z) \in \mathbb{X}$ with $\Lambda(w, z) = 0$ and any $\epsilon > 0$. Therefore

$$\limsup_{k\to\infty} \mathbb{J}_k(\omega) \le \mathbb{J}(\omega).$$

Now, we recall that

$$\mathbb{J}_k(\omega) = J_{\omega,k}(\tilde{u}_k, \tilde{v}_k) = \frac{p}{p+2} I_k(\tilde{u}_k, \tilde{v}_k) = \frac{p}{p+2} \int_{Q_k} \rho(\tilde{u}_k, \tilde{v}_k)\, dxdy.$$

Note that for a given bounded domain $\Omega \subset \mathbb{R}^2$, we have that $\Omega \subset Q_k$ for k large enough. Then

$$\mathbb{J}_k(\omega) = J_{\omega,k}(\tilde{u}_k, \tilde{v}_k) \ge \frac{p}{p+2} \int_{\Omega} \rho(\tilde{u}_k, \tilde{v}_k)\, dxdy.$$

Hence, due to the local compactness result, we get that

$$\liminf_{k\to\infty} \mathbb{J}_k(\omega) \ge \liminf_{k\to\infty} \left(\frac{p}{p+2} \int_{\Omega} \rho(\tilde{u}_k, \tilde{v}_k)\, dxdy \right) = \frac{p}{p+2} \int_{\Omega} \rho(u_0, v_0)\, dxdy.$$

Thus, since Ω is arbitrary, we have shown that

$$\liminf_{k\to\infty} \mathbb{J}_k(\omega) \ge \frac{p}{p+2} \int_{\mathbb{R}^2} \rho(u_0, v_0)\, dxdy = \frac{p}{p+2} I(u_0, v_0).$$

Since $\langle J'_\omega(u_0, v_0), (u_0, v_0)\rangle = 0$. Then $J_c(u_0, v_0) = \frac{p}{p+2} I(u_0, v_0)$. So that

$$\liminf_{k\to\infty} \mathbb{J}_k(\omega) \ge \mathbb{J}(\omega).$$

In other words,

$$\liminf_{k\to\infty} \mathbb{J}_k(\omega) = \mathbb{J}(\omega) = \frac{p}{p+2} I(u_0, v_0) = J_\omega(u_0, v_0),$$

which is equivalent to say that (u_0, v_0) is a ground state solution. It was also proved that

$$\lim_{k\to\infty} I_k(\tilde{u}_k, \tilde{v}_k) = I(u_0, v_0).$$

It remains to prove that

$$\lim_{k\to\infty} \|(\tilde{u}_k, \tilde{v}_k) - (u_0, v_0)\|_{\mathbb{X}_k} = 0. \tag{4.13}$$

To see this, let $(w_k, z_k) \in C_0^\infty(Q_k) \times C_0^\infty(Q_k)$ such that $(w_k, z_k) \to (u_0, v_0)$ in $\mathbb{X}$. We will show that

$$\lim_{k\to\infty} \|(\tilde{u}_k, \tilde{v}_k) - (w_k, z_k)\|_{\mathbb{X}_k} = 0.$$

This implies that the limit (4.13) holds. We consider the operators A and A_k defined by

$$A((u,v),\cdot) = \langle I'(u,v),\cdot\rangle,\ (u,v)\in\mathbb{X};\quad A_k((u,v),\cdot) = \left\langle I_k'(u,v),\cdot\right\rangle,\ (u,v)\in\mathbb{X}_k.$$

Then we have that

$$\begin{aligned}&I_k\left[(\tilde u_k,\tilde v_k)-(w_k,z_k)\right]\\ &= I_k(\tilde u_k,\tilde v_k)+I_k(w_k,z_k)-A_k\left((\tilde u_k,\tilde v_k),(w_k,z_k)\right)\\ &= I_k(\tilde u_k,\tilde v_k)+I_k(w_k,z_k)-A_k\left((\tilde u_k,\tilde v_k),(u_0,v_0)\right)-A_k\left((\tilde u_k,\tilde v_k),(w_k-u_0,z_k-v_0)\right).\end{aligned}$$

Since (w_k,z_k) converges strongly to (u_0,v_0) in $\mathbb{X}$ and $||(\tilde u_k,\tilde v_k)||_{\mathbb{X}_k}$ is bounded, we conclude as $k\to\infty$ that

$$|A_k\left((\tilde u_k,\tilde v_k),(w_k,z_k)-(u_0,v_0)\right)| \le C||(\tilde u_k,\tilde v_k)||_{\mathbb{X}_k}||(w_k,z_k)-(u_0,v_0)||_{\mathbb{X}}\to 0.$$

But, we have that

$$A_k\left((\tilde u_k,\tilde v_k),(u_0,v_0)\right)\to A\left((u_0,v_0),(u_0,v_0)\right) = 2I(u_0,v_0).$$

So, taking limit as $k\to\infty$,

$$||(\tilde u_k,\tilde v_k)-(w_k,z_k)||^2_{\mathbb{X}_k} \le CI_k\left((\tilde u_k,\tilde v_k)-(w_k,w_k)\right)\to 0.$$

Then, as desired

$$||(u_k,v_k)-(u_0(\cdot+\zeta_k),v_0(\cdot+\zeta_k))||_{\mathbb{X}_k}\to 0.$$

□

Acknowledgments. J. R. Quintero was supported by the Mathematics Department at Universidad del Valle (Colombia) under the project C.I. 71007. A. M. Montes was supported by the Universidad del Cauca (Colombia) under the project I.D. 3982. J. Q. and A. M. are supported by Colciencias grant N. 42878.

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