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Propagation Speed for Fractional Cooperative Systems with Slowly Decaying Initial Conditions

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Abstract

The aim of this paper is to study the time asymptotic propagation for mild solutions to the fractional reaction diffusion cooperative systems when at least one entry of the initial condition decays slower than a power. We state that the solution spreads at least exponentially fast with an exponent depending on the diffusion term and on the smallest index of fractional Laplacians.

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1 Introduction

The reaction-diffusion equation with Fisher-KPP nonlinearity

$$\partial_t u + (-\triangle)^\alpha u = f(u) \tag{1.1}$$

where $(-\triangle)^\alpha$ is the fracctional laplacian with index $\alpha \in (0,1)$ appears in models of physics, chemistry and biology, when the diffusive phenomena is described by Lévy processes allowing long jumps. Concerning equation (1.1), Cabré and Roquejoffre showed in [3] that, the speed of propagation of solutions is exponential in time when the initial value decays faster than the critical power $|x|^{-d-2\alpha}$, where d is the dimension of the spatial variable.

In the case in which the initial condition decay slower than the critical power, [8] states that the level sets of the solutions move exponentially fast as time goes to infinity. Moreover, a quantitative estimate of motion of the level sets is obtained in terms of the decay of the initial condition. All these results are in great contrast with the standard case i.e. taking $\alpha =$

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1 in (1.1). Indeed, [1] shows that there exists a critical speed such that, for any compactly supported initial value, there is a linear propagation in time of the fronts.

Moreover, the work on the single equation (1.1) with standard laplacian can be extended to cooperative systems. In a series of papers, [10], [11], [14], [15], spreading speeds and traveling waves are studied for a particular class of cooperative reaction-diffusion systems, with standard diffusion. Results on single equations in the singular perturbation framework proved in [7] have also been extended in [2].

In the fractional case, the recent paper [5] studies the time asymptotic propagation of sectorial solutions to the fractional reaction-diffusion cooperative systems when the initial conditions decay faster than a power, they prove that the propagation speed is exponential in time and they find a precise exponent of propagation, which depends on the smallest index of the fractional laplacians and on the principal eigenvalue of the reaction term derivative. It is interesting to note, if we assume that each entry of the initial condition in [5] belongs to the domain of the fractional laplacian, using Theorems 2.3 and 2.6 below, it is possible to prove that the mild solution of the system studied en [5] spreads with the same speed than in the sectorial case.

Following the line, we are interested in the large time behavior of solutions $u = (u_i)_{i=1}^m$ with $m \in \mathbb{N}^*$, to the fractional reaction diffusion system:

$$\begin{cases} \partial_t u_i + (-\triangle)^{\alpha_i} u_i &= f_i(u), \qquad \forall (t,x) \in \mathbb{R}_+^* \times \mathbb{R}^d \\ u_i(0,x) &= u_{0i}(x), \qquad \forall x \in \mathbb{R}^d \end{cases} \tag{1.2}$$

where $\alpha_i \in (0,1]$ for all $i \in [\![1,m]\!] := \{1,...,m\}$ with at least one $\alpha_i \neq 1$, we note, if $\alpha_j = 1$ then the fractional laplacian becomes in the standard laplacian. Without loss of generality, we suppose $\alpha_{i+1} \leq \alpha_i$ for all $i \in [\![1,m-1]\!]$ and we set $\alpha := \alpha_m < 1$. Henceforth, we impose the nonnegative initial conditions $u_{0i} \not\equiv 0$, bounded by the constant $\Lambda > 0$ and $u_{0i} \in C_0(\mathbb{R}^d)$ for all $i \in [\![1,m]\!]$, where the Banach space $C_0(\mathbb{R}^d)$ is the set of continuous functions in $\mathbb{R}^d$ which decay to zero as $|x| \to \infty$, doted with the $L^\infty(\mathbb{R}^d)$ norm. Also, to state the main result, we need to consider that at least one entry of $u_0 = (u_{0i})_{i=1}^m$ satisfies

$$u_{0i}(x) \geq C_i |x|^{-d-\beta_i} \quad as\ |x| \to \infty, \text{ for some } \beta_i < 2\alpha_i \tag{1.3}$$

with C_i a positive constant and the other entries satisfy

$$u_{0j}(x) = O(|x|^{-d-2\alpha_j}) \quad as\ |x| \to \infty,\ i \neq j. \tag{1.4}$$

In general, the function $F = (f_i)_{i=1}^m$ satisfies

$$f_i(0) = 0, \ \ f_i \in C^1(\mathbb{R}^m)\ \forall i \in [\![1,m]\!] \ \text{ and } \ \partial_j f_i > 0 \ \ \forall i \neq j \tag{1.5}$$

i.e., the system (1.2) is cooperative. Moreover, we will make additional assumptions on the reaction term F that are not general but enable us to understand the long time behavior of a class of monotone systems, however, it is important to note that these hypothesis are compatible with strongly coupled systems.

(H1) The principal eigenvalue λ_1 of the matrix $DF(0)$ is positive,

(H2) f_i is globally Lipschitz on $\mathbb{R}^m$ for all $i \in [\![1,m]\!]$.

(H3) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \geq \Lambda$, we have $f_i(s) \leq 0$,

(H4) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \geq c_{\delta_1} {s_i}^{1+\delta_1}$,

(H5) For all $s = (s_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying $|s| \leq \Lambda$, $Df_i(0)s - f_i(s) \leq c_{\delta_2} |s|^{1+\delta_2}$,

where the constants c_{δ_1} and c_{δ_2} are positive and for all $j \in \{1,2\}$

$$\delta_j \geq \frac{2}{d+\beta} \quad \text{with} \quad \beta = \min_{i \in I}\{\beta_i\}$$

where $I = \{i |\ u_{0i}$ satisfies (1.3)$\}$. This lower bound on δ_1 and δ_2 is a technical assumption to make the supersolution and subsolution to (1.2), we construct, to be regular enough. Note that one may easily produce examples of functions F satisfying (H1) to (H5).

This paper is devote to understand the time asymptotic spread of solutions to (1.2). We consider the case, when the entries of the initial datum u_0 satisfy (1.3) or (1.4). We show in part b) of Theorem 1.1 below, that the speed of propagation for mild solutions is at least exponential in time, with an exponent depending on the smallest index β_i and of the principal eigenvalue of the matrix $DF(0)$. We prove also that this exponent is larger than the exponent founded in [5] for sectorial solutions, when u_{0i} satisfies (1.4) for all $i \in [\![1,m]\!]$.

We are now in a position to state our main theorem, which shows that the solution to (1.2) moves exponentially fast in time.

Theorem 1.1. *Let $2\alpha \geq \beta$ and assume that F satisfies (1.5) and (H1) to (H4). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3) and (1.4). Then, for all $i \in [\![1,m]\!]$, the following two facts are satisfied:*

a) *For all $t \geq 0$ and $\varepsilon_i > 0$, there exists $r_i > 0$ such that*

$$0 \leq u_i(t,x) \leq \varepsilon_i, \quad for\ all\ |x| > r_i$$

b) *There exist $\tau > 0$ large enough, $C > 0$ and $\theta_i \in (0,\Lambda)$ such that*

$$u_i(t,x) > \theta_i, \quad for\ all\ t \geq \tau\ and\ |x| \leq Ce^{\frac{\lambda_1}{d+\beta}t}.$$

It is interesting to note that, if we assume for the moment that $u_{0i} \in L^2(\mathbb{R}^d)$ for all $i \in I$, then we can find a sectorial solution u of (1.2), easily using Theorems 1 and 2 stated in [5], and we can deduce that the spread speed of u is at least exponential as $t \to \infty$, with an exponent given by $\lambda_1/(d+2\alpha)$. The aim of this paper is to improve this exponent when we consider only mild solutions, thus to establish Theorem 1.1, similarly to [5], but in this case using the fact that at least one entry of the initial condition satisfies (1.3), we state suitable sub and super solutions, in order to prove that the mild solution of (1.2), which is not necessarily classical or sectorial, spreads at least with an exponential speed with an exponent given by $\lambda_1/(d+\beta)$. Furthermore, since $2\alpha \geq \beta$, then $\lambda_1/(d+\beta) \geq \lambda_1/(d+2\alpha)$, which shows that the mild solution associated with an initial datum satisfying (1.3) and (1.4) spreads faster than the sectorial solution of the problem studied in [5].

In sake of completeness, we state the following result which shows an upper bound for the movement of the solution in a particular case.

Theorem 1.2. *Let $2\alpha \geq \beta$ and assume that F satisfies (1.5), (H1) to (H3) and (H5). Let u be the solution to (1.2) with a non negative, non identically equal to 0 and continuous initial condition u_0 satisfying (1.3) and (1.4). If $u_{0i}(x) = O(|x|^{-d-\beta})$ as $|x| \to +\infty$ for all $i \in I$, then for every positive $\varepsilon \in \mathbb{R}^m$, there exist $\tau > 0$ and $c > 0$ such that,*

$$u(t,x) < \varepsilon, \quad for\ all\ t \geq \tau\ and\ |x| > ce^{\frac{\lambda_1}{d+\beta}t}.$$

The plan to set Theorems 1.1 and 1.2 is organized as follows. First, we present some preliminaries in which we prove the existence and uniqueness of mild solutions for cooperative systems involving fractional diffusion and we state a comparison principle for mild and classical solutions, also we present some results which help us to find auxiliary classical solutions. Then, by the manipulation of some Polya integrals, we set algebraically upper and lower bounds for solutions of (1.2), which give us the space decay of the solution at any time $t > 0$. The end of this paper gives the proof of Theorems 1.1 and 1.2, that relies on the construction of explicit classical subsolutions and supersolutions.

2 Mild solutions and comparison principles

In order to state the existence of the unique solution to the system (1.2) in a Banach space X, we consider a function $G : [0,+\infty) \times X^m \to X^m$, $G = (G_i(t,u))_{i=1}^m$ that satisfies for all $i \in [\![1,m]\!]$

$$\begin{array}{l} G_i \in C^1([0,+\infty) \times X^m; X), \\ G_i(t,\cdot) \text{ is globally Lipschitz in } X^m \text{ uniformly in } t \geq 0, \end{array} \tag{2.1}$$

where X^m is the product space doted with the norm $\|u\|_{X^m} = \sum_{i=1}^m \|u_i\|_X$. Given any $T > 0$, we are interested in the nonlinear problem

$$\left\{ \begin{array}{rcl} \partial_t u + Lu &=& G(t,u), \quad in\ (0,T) \\ u(0) &=& u_0, \end{array} \right. \tag{2.2}$$

where $L = \mathrm{diag}((-\triangle)^{\alpha_1}, ..., (-\triangle)^{\alpha_m})$, $u = (u_i)_{i=1}^m$ and $u_0 \in X^m$. In the sequel, the heat kernel of the Laplace operator of order $\alpha_i \in (0,1]$ in $\mathbb{R}^d$ is denoted by p_{α_i}. It satisfies

1. $p_{\alpha_i} \in C((0,+\infty) \times \mathbb{R}^d)$, $p_{\alpha_i} > 0$ and $\int_{\mathbb{R}^d} p_{\alpha_i}(t,x)dx = 1$ for all $t > 0$,
2. $p_{\alpha_i}(t,\cdot) * p_{\alpha_i}(s,\cdot) = p_{\alpha_i}(t+s,\cdot)$ for all $(t,s) \in \mathbb{R}_+^2$,
3. If $\alpha_i \in (0,1)$, then there exists $B > 1$ such that, for $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$:

$$\frac{B^{-1}}{t^{\frac{d}{2\alpha_i}}(1 + |xt^{-\frac{1}{2\alpha_i}}|^{d+2\alpha_i})} \leq p_{\alpha_i}(t,x) \leq \frac{B}{t^{\frac{d}{2\alpha_i}}(1 + |xt^{-\frac{1}{2\alpha_i}}|^{d+2\alpha_i})}.$$

We define the map $N_{u_0} : C([0,T]; X)^m \to C([0,T]; X)^m$ by

$$N_{u_0}(u)(t) := \mathrm{T}_t u_0 + \int_0^t \mathrm{T}_{t-s} G(s,u(s))ds \tag{2.3}$$

where $\mathtt{T}_t = \mathrm{diag}(T_{t,1},...,T_{t,m})$ and $T_{t,i}w(x) = (p_{\alpha_i}(t,\cdot) * w)(x)$ is a strongly continuous semigroup of bounded linear operators for all $i \in [\![1,m]\!]$. Similarly to [16], we can prove that there exists $u \in C([0,T];X)^m$ such that

$$u = \lim_{i\to+\infty} (N_{u_0})^i(u^0), \tag{2.4}$$

where $u^0(t) = \mathtt{T}_t u_0$ for all $T > 0$. The limit u is the unique fixed point of N_{u_0}, hence u is the unique mild solution of (2.2) for all $T > 0$. By uniqueness, under assumption (2.1), the mild solution of (2.2) extends uniquely to all $t \in [0,+\infty)$, i.e., it is global in time.

Let $u = (u_i)_{i=1}^m$ be the unique mild solution of (2.2). We define

$$H_i(t,w) = G_i(t,u_1,...,u_{i-1},w,u_{i+1},...,u_m)$$

and we have

$$\begin{aligned} &H_i \in C^1([0,+\infty)\times X;X),\\ &H_i(t,\cdot) \text{ is globally Lipschitz in } X \text{ uniformly in } t \geq 0. \end{aligned} \tag{2.5}$$

Consider now the problem

$$\left\{ \begin{array}{rcll} \partial_t w + (-\triangle)^{\alpha_i} w &=& H_i(t,w), & \text{in } (0,T)\\ w(0) &=& u_{0i}. & \end{array} \right. \tag{2.6}$$

Following the computations of section 2.3 in [3], we conclude that this problem has a unique mild solution in $C([0,T];X)$, given by $w = u_i$. Thus, if the initial datum belongs to the domain $D(A_i)$ in X of $A_i = (-\triangle)^{\alpha_i}$, we have further regularity in t of the mild solution $u_i = u_i(t)$. Under hypothesis (2.5), the mild solution u_i of (2.6) satisfies

$$u_i \in C^1([0,T);X) \ \text{ and } \ u_i([0,T)) \subset D(A_i) \ \text{ if } \ u_{0i} \in D(A_i), \tag{2.7}$$

and it is a classical solution, i.e., a solution satisfying (2.6) pointwise for all $t \in (0,T)$. Doing the same procedure for all $i \in [\![1,m]\!]$ and for all $T > 0$, we conclude that $u = (u_i)_{i=1}^m$ is a classical solution of (2.2) global in time.

Now, we set a useful fact that we need in the following computations. If u is the solution of the system (2.2) with $u_0 \in X^m$ and G satisfies (2.1), then for any $l \in \mathbb{R}$, $\tilde{u}(t) = e^{lt}u(t)$ is the mild solution of the system (2.2) with $u_0 \in X^m$ and $G(t,u)$ replaced by $\widetilde{G}(t,\tilde{u}) = l\tilde{u} + e^{lt}G(t,e^{-lt}\tilde{u})$. This fact is proved in the same way as in [16].

We now consider the Banach space $X = C_0(\mathbb{R}^d)$ and set, for all $i \in [\![1,m]\!]$, $G_i(t,u)(x) := f_i(u(x))$ so that G_i satisfies (2.1). We use that $f_i \in C^1(\mathbb{R}^m)$ and $f_i(0) = 0$ to check that the map $u \in C_0(\mathbb{R}^d)^m \mapsto f_i(u) \in C_0(\mathbb{R}^d)$ is continuously differentiable. Thus, by the previous considerations, there is a unique mild solution u of (1.2) starting from $u_0 \in X^m$. Moreover, if the initial datum u_0 belongs to $\prod_{i=1}^m D_0(A_i)$, where $D_0(A_i)$ is the domain of A_i in $C_0(\mathbb{R}^d)$, then the mild solution u satisfies (2.7) for all $i \in [\![1,m]\!]$ and for all $T > 0$ and it is a classical solution global in time.

Since, for all $i \in I$, u_{0i} is not necessarily in the domain $D(A_i)$, we need to state the following results which will be helpful at the moment to find a classical auxiliary solution of (1.2).

Lemma 2.1. *Given u_{0i} satisfying (1.3), then there exists a positive $v_{0i} \in D_0(A_i)$ such that*

$$c_{1,i}|x|^{-d-\beta_i} \leq v_{0i}(x) \leq c_{2,i}|x|^{-d-\beta_i} \quad if\ |x| > 2 \tag{2.8}$$

for $\beta_i < 2\alpha_i$ and c_{1i}, c_{2i} positive constants. Moreover,

$$T_{2,i}u_{0i}(x) \geq v_{0i}(x), \qquad \forall\ x \in \mathbb{R}^d.$$

Proof. Let

$$\sigma(x) = \frac{1}{1+|x|^{d+\beta_i}} \quad \text{and} \quad v_{0i}(x) = c\int_1^2 T_{s,i}\sigma(x)ds$$

where $c > 0$ is constant and $T_{s,i}$ is the operator associated with $A_i = (-\triangle)^{\alpha_i}$. Form [3], we know that $v_{0i} \in D_0(A_i)$ since $\sigma \in C_0(\mathbb{R}^d)$. Now, we prove that, there exists $c_{1,i} > 0$ such that $v_{0i}(x) \geq c_{1,i}|x|^{-d-\beta_i}$ for $|x| > 2$. Indeed,

$$\begin{aligned} v_{0i}(x) &\geq 2^{-1/2\alpha_i}\frac{c}{B}\int_{\mathbb{R}^d}\frac{1}{1+|y|^{d+2\alpha_i}}\frac{1}{1+|x-y|^{d+\beta_i}}dy \\ &:= 2^{-1/2\alpha_i}\frac{c}{B}\sigma_1(x). \end{aligned} \tag{2.9}$$

Let's analyze σ_1. Note, we can find $C_1 > 0$ and $R \in (0,1)$ such that $\frac{1}{1+|y|^{d+2\alpha_i}} \geq C_1$ for all $|y| \leq R$. We divide the proof in two cases. We consider first $|x| > R$, thus

$$\sigma_i(x) \geq \int_{|y|\leq R}\frac{C_1}{1+|x-y|^{d+\beta_i}}dy \geq \frac{C}{1+|x|^{d+\beta_i}} \tag{2.10}$$

noting that in the last inequality, we use $|x-y| \leq |x|+|y| \leq |x|+R \leq 2|x|$. In the case in which $|x| \leq R$, we have

$$1+|x-y|^{d+\beta_i} \leq 1+(|x|+|y|)^{d+\beta_i} \leq 1+(R+1)^{d+\beta_i}, \qquad if\ |y| \leq 1$$

hence,

$$\begin{aligned} \sigma_i(x) &\geq \frac{1}{1+(R+1)^{d+\beta_i}}\int_{|y|\leq 1}\frac{1}{1+|y|^{d+2\alpha_i}}dy \geq C \\ &\geq \frac{C}{1+|x|^{d+\beta_i}}. \end{aligned} \tag{2.11}$$

Then, from (2.9), (2.10) and (2.11), we see that there exists $\widetilde{C} > 0$ such that

$$v_{0i}(x) \geq \widetilde{C}(1+|x|^{-d-\beta_i})^{-1} \quad \text{for all } x \in \mathbb{R}^d. \tag{2.12}$$

Moreover, if $|x| > 2$, we conclude that $v_{0i}(x) \geq c_{1,i}|x|^{-d-\beta_i}$ for some constant $c_{1,i} > 0$. Now, by definition of v_{0i}, it is easy to see that $v_{0i} \leq c$ in $\mathbb{R}^d$, moreover, we claim that

$$v_{0i}(x) \leq c_{2,i}|x|^{-d-\beta_i}, \quad \text{for } |x| \geq 2. \tag{2.13}$$

To prove the claim, we assume $|x| \geq 2$ and let us note that

$$\begin{aligned} v_{0i}(x) &\leq cB2^{d/2\alpha_i+1}\Big[\int_{\{|y|\leq|x|/2\}} \frac{1}{1+|y|^{d+2\alpha_i}} \frac{1}{1+|x-y|^{d+\beta_i}}dy \\ &\qquad + \int_{\{|y|>|x|/2\}} \frac{1}{1+|y|^{d+2\alpha_i}} \frac{1}{1+|x-y|^{d+\beta_i}}dy\Big] \\ &:= cB2^{d/2\alpha_i+1}(I_1+I_2). \end{aligned} \tag{2.14}$$

If $|y| \leq |x|/2$, we have $|x-y| \geq |x|-|y| \geq \frac{|x|}{2}$, then

$$\begin{aligned} I_1 &\leq \int_{\{|y|\leq|x|/2\}} \frac{1}{1+|y|^{d+2\alpha_i}} \frac{2^{d+\beta_i}}{1+|x|^{d+\beta_i}}dy \\ &\leq \frac{2^{d+\beta_i}}{|x|^{d+\beta_i}} \int_{\mathbb{R}^d} \frac{dy}{1+|y|^{d+2\alpha_i}} \leq \frac{C}{|x|^{d+\beta_i}}. \end{aligned} \tag{2.15}$$

Now, if $|y| > |x|/2$ and since $|x| \geq 2$, we see that $1+|y|^{d+2\alpha_i} \geq \frac{1}{2^{d+\beta_i}}|x|^{d+\beta_i}$, hence

$$I_2 \leq \frac{2^{d+\beta_i}}{|x|^{d+\beta_i}} \int_{\mathbb{R}^d} \frac{ds}{1+|s|^{d+\beta_i}} := \frac{C}{|x|^{d+\beta i}}.$$

Therefore, from (2.14), we get (2.13). To finalize the proof, we consider

$$T_{2,i}u_{0i}(x) = \int_{\mathbb{R}^n} H_i(2,y)u_{0i}(x-y)dy$$

where

$$H_i(t,x) = \begin{cases} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{4t}} & if\ \alpha_i = 1 \\ p_i(t,x) & if\ \alpha_i \in (0,1). \end{cases} \tag{2.16}$$

In both cases $\alpha_i \in (0,1)$ and $\alpha_i = 1$, taking $|x| > x_0$ with $x_0 > 1$ large enough such that (1.3) is satisfied, we have

$$H_i(2,\cdot) * u_{0i}(x) \geq \overline{C} \int_{|y|\leq 1} \frac{1}{|x-y|^{d+\beta_i}}dy$$

for some small constant $\overline{C} > 0$. Also, $|x-y| \leq |x|+|y| \leq |x|+1 \leq 2|x|$, so

$$H_i(t,\cdot) * u_{0i}(x) \geq \widetilde{C}|x|^{-d-\beta_i}.$$

Now, if $|x| \leq x_0$, then $H_i(2,\cdot) * u_{0i}(x) \geq \overline{C}$ for some small constant $\overline{C} > 0$. Thus, since v_{0i} satisfies (2.13), we can take $c > 0$ in the definition of v_{0i} small enough such that $T_{2,i}u_{0i} \geq v_{0i}$. □

In what follows, $Lip(f_i^j)$ denotes the Lipschitz constant of f_i^j and we define the constant

$$l = \max_{i\in[\![1,m]\!]} \{Lip(f_i)\} \tag{2.17}$$

which appears several times throughout the paper.

Lemma 2.2. *Given u_0 satisfying (1.3) and (1.4), there exists $v_0 \in \prod_{i=1}^m D_0(A_i)$ such that*

$$w(2,x) \geq v_0(x), \qquad \forall\, x \in \mathbb{R}^d \tag{2.18}$$

where w is the mild solution of

$$\begin{cases} \partial_t w_i + (-\triangle)^{\alpha_i} w_i &= l w_i + e^{lt} f_i(e^{-lt} w) \\ w_i(0,\cdot) &= u_{0i} \end{cases} \tag{2.19}$$

with $l > 0$ defined in (2.17). Moreover, v_{0i} satisfies (2.8) for all $i \in I$ and (1.4) for all $i \notin I$, also $0 < v_{0i} \leq \Lambda$ for all $i \in [\![1,m]\!]$.

Proof. By the previous computations, the solution w is the limit of the iterative process (2.4) applied to the system (2.19) with initial term given by $w^0(t,x) = \mathrm{T}_t u_0(x)$. By the choice of $l > 0$ and since F satisfies (1.5), $\widetilde{f_i}(w) := l w_i + e^{lt} f_i(e^{-lt} w)$ is nondecreasing in its second argument for all $i \in [\![1,m]\!]$. Using (2.3), (2.4), $F(0) = 0$ and the properties of $\widetilde{F}$, we can deduce that

$$w(t,x) \geq w^0(t,x), \qquad \forall\, (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{2.20}$$

If u_{0i} satisfies (1.3), by Lemma 2.1, there exists $v_{0i} \in D(A_i)$ satisfying (2.8). Moreover, if u_{0i} satisfies (1.4), we define

$$v_{0i}(x) := c_i \int_1^2 p_i(s,x) ds$$

hence, by Lemma 2.2 in [3], we have that $v_{0i} \in D(A_i)$ satisfying (1.4) and in both cases $T_{2,i} u_{0i}(x) \geq v_{0i}(x)$, thus v_{0i} is bounded by Λ. Furthermore, by (2.20), we get (2.18). □

Before stating the bounds for the solutions, we need to establish a comparison principle for mild solutions defined in any Banach space X.

Theorem 2.3. *For every $j \in \{1,2\}$, set $F^j = (f_i^j)_{i=1}^m$ where, for all $i \in [\![1,m]\!]$, f_i^j is $C^1(\mathbb{R}^m)$, satisfies (1.5) and is globally Lipschitz. Let $u^j = (u_i^j)_{i=1}^m$ be a mild solution of*

$$\partial_t u^j + L u^j = F^j(u^j),$$

with initial condition $u^j(0,\cdot) \in X$. If, for all $i \in [\![1,m]\!]$, $f_i^1 \leq f_i^2$ in $\mathbb{R}^m$ and $u_i^1(0,\cdot) \leq u_i^2(0,\cdot)$ in X, then

$$u_i^1(t,x) \leq u_i^2(t,x) \quad \text{for all} \quad (t,x) \in [0,+\infty) \times \mathbb{R}^d.$$

Proof. Taking $l = \max_{i \in [\![1,m]\!], j \in \{1,2\}} Lip(f_i^j)$, we define for $i \in [\![1,m]\!]$, $j \in \{1,2\}$ and $t \geq 0$

$$\tilde{f}_i^j(t,v) = l v_i + e^{lt} f_i^j(e^{-lt} v).$$

For $i \in [\![1,m]\!]$ and $j \in \{1,2\}$, by the choice of $l > 0$ and since $\tilde{f}_i^j$ satisfy (1.5), the function $\tilde{f}_i^j$ is nondecreasing in its second argument. Moreover, since $f_i^1 \leq f_i^2$ in $\mathbb{R}^m$, we have at any time $t \geq 0$, $\tilde{f}_i^1(t,\cdot) \leq \tilde{f}_i^2(t,\cdot)$.

For $j \in \{1,2\}$, we define $\tilde{F}^j = (\tilde{f}_i^j)_{i=1}^m$, and consider the system

$$\begin{cases} \partial_t \tilde{u}^j + L\tilde{u}^j &= \tilde{F}^j(\tilde{u}^j) \quad t>0, x\in\mathbb{R}^m, \\ \tilde{u}^j(0,\cdot) &= u_0^j, \qquad x\in\mathbb{R}^m. \end{cases} \tag{2.21}$$

By the previous section, we know that $\tilde{u}^j(t,x) = e^{lt}u^j(t,x)$ is the solution of (2.21), where u^j is the solution of (2.21) with $\tilde{F}^j$ replaced by F^j. Therefore, it is enough to prove that $\tilde{u}^1 \le \tilde{u}^2$. Consider the mapping N^j for $j = \{1,2\}$, defined by

$$N^j(w)(t,\cdot) := \mathtt{T}_t u_0^j(\cdot) + \int_0^t \mathtt{T}_{t-s}\widetilde{F}^j(s, w(s,\cdot))ds.$$

Taking $u^{0,j}(t,\cdot) = \mathtt{T}_t u_0^j(\cdot)$, we know that $\tilde{u}^j = \lim_{n\to+\infty}(N^j)^n(u^{0,j})$. Thus, using a standard induction argument, we only need to show that $(N^1)^n(u^{0,1}) \le (N^2)^n(u^{0,2})$ on $[0,+\infty)\times\mathbb{R}^d$ for all $n\in\mathbb{N}$. This fact is obvious since $u_i^1(0,\cdot) \le u_i^2(0,\cdot)$, $\tilde{f}_i^j$ is nondecreasing in its second argument and $f_i^1 \le f_i^2$ for all $i \in [\![1,m]\!]$. □

Remark 2.4. If we suppose $f_i^1 \le f_i^2$ in $\mathbb{R}_+^m$ and $0 \le u_i^1(0,\cdot) \le u_i^2(0,\cdot)$ for all $i\in[\![1,m]\!]$, we obtain the same result as in Theorem 2.3.

Remark 2.5. Since $F(0) = 0$ by the previous theorem, we conclude that the solution of (1.2), satisfies $u_i(t,x) \ge 0$ for all $(t,x)\in[0,+\infty)\times\mathbb{R}^d$ and all $i\in[\![1,m]\!]$.

Now, we state the following comparison principle for classical solutions, which is an adaptation of Theorem 2 of [5]. This result will be useful to deal with sub and super solutions. Indeed, we have not devised a mild representation for them, so we can not apply Theorem 2.3 directly.

Theorem 2.6. *Let $u = (u_i)_{i=1}^m$ and $v = (v_i)_{i=1}^m$ functions in $C^1([0,T];C_0(\mathbb{R}^d))^m$ such that, for all $i\in[\![1,m]\!]$,*

$$\partial_t u_i + (-\triangle)^{\alpha_i}u_i \le f_i(u), \quad \partial_t v_i + (-\triangle)^{\alpha_i}v_i \ge f_i(v),$$

where f_i satisfies (1.5). If for all $i\in[\![1,m]\!]$ and $x\in\mathbb{R}^d$, $u_i(0,x)\le v_i(0,x)$ and for all $t\in[0,T]$

$$u_i(t,x) = O(|x|^{-(d+\beta)}) \text{ and } v_i(t,x) = O(|x|^{-(d+\beta)}) \quad \text{as } |x|\to+\infty, \tag{2.22}$$

then

$$u(t,x) \le v(t,x) \quad \text{for all} \quad (t,x)\in[0,T]\times\mathbb{R}^d.$$

Proof. Let us define for all $i\in[\![1,m]\!]$, $w_i = u_i - v_i$. Then w_i satisfies $w_i(0,x)\le 0$ and

$$\begin{aligned} \partial_t w_i + (-\triangle)^{\alpha_i}w_i &\le f_i(u) - f_i(v) = \int_0^1 \nabla f_i(\sigma u + (1-\sigma)v)d\sigma.(u-v) \\ &= \int_0^1 \nabla f_i(\zeta_\sigma)d\sigma.w, \end{aligned} \tag{2.23}$$

where $\zeta_\sigma = \sigma u + (1-\sigma)v$. By hypothesis, for all $i\in[\![1,m]\!]$, the function w_i belongs to $C^1([0,T];C_0(\mathbb{R}^d))$ and consequently there exist positive constants $C_1(T)$ and $C_2(T)$ such that for all $(t,x)\in[0,T]\times\mathbb{R}^d$

$$|w_i(t,x)| \le C_1(T) \quad \text{and} \quad |\partial_t w_i(t,x)| \le C_2(T). \tag{2.24}$$

Moreover, by 2.22, for all $t \in [0,T]$, we have

$$w_i(t,x) = O(|x|^{-(d+\beta)}) \quad \text{as } |x| \to +\infty. \tag{2.25}$$

Thus, it is easy to see that for all $t \in [0,T]$

$$\int_{\mathbb{R}^d} |w_i(t,x)||w_j(t,x)|dx \le C_{ij}(T), \tag{2.26}$$

where $C_{ij}(T)$ are constants that depend on T. Let w_i^+ be the positive part of w_i. We want to prove that

$$\frac{d}{dt}\left[\int_{\mathbb{R}^d} (w_i^+)^2 dx\right] = \int_{\mathbb{R}^d} \partial_t \left[(w_i^+)^2\right] dx, \tag{2.27}$$

which is quite simple because $(w_i^+)^2$ and $\partial_t \left[(w_i^+)^2\right]$ are continuous in $(0,T)\times\mathbb{R}^d$ and

$$\left|\partial_t \left[(w_i^+)^2\right]\right| = 2\left|w_i^+ \partial_t w_i\right| \le 2C_2(T)\,|w_i| \le Cg(x). \tag{2.28}$$

The last inequality and the existence of the integrable function g follows from (2.24) and (2.25), thus we conclude (2.27). Now, multiplying each term of (2.23) by w_i^+ and integrating over $\mathbb{R}^d$, we have

$$\begin{aligned} 0 &\le \int_{\mathbb{R}^d} w_i^+ (-\triangle)^{\alpha_i} w_i dx \\ &\le \int_{\mathbb{R}^d} w_i^+ \int_0^1 \nabla f_i(\zeta_\sigma) d\sigma.w dx - \int_{\mathbb{R}^d} w_i^+ \partial_t w_i dx. \end{aligned} \tag{2.29}$$

By (2.26) and (2.28), we get

$$\int_{\mathbb{R}^d} w_i^+ (-\triangle)^{\alpha_i} w_i dx < \infty.$$

Now, since all the above integrals exist and having in mind that $f_i \in C^1(\mathbb{R}^m)$ for all $i \in [\![1,m]\!]$, from (2.27), (2.29) and since $\partial_j f_i(\zeta_\sigma) > 0$, we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left[\int_{\mathbb{R}^d} (w_i^+)^2 dx\right] &\le \int_{\mathbb{R}^d}\int_0^1 \partial_i f_i(\zeta_\sigma) d\sigma (w_i^+)^2 dx \\ &\quad + \sum_{j=1, j\neq i}^m \int_{\mathbb{R}^d}\int_0^1 \partial_j f_i(\zeta_\sigma) d\sigma w_i^+ w_j^+ dx \\ &\le C\sum_{j=1}^m \int_{\mathbb{R}^d} (w_j^+)^2 dx, \end{aligned}$$

where C is a constant that depends on m. Doing this procedure for each $i \in [\![1,m]\!]$ and adding, we get for $t \in [0,T]$

$$\frac{d}{dt}\left[\sum_{j=1}^m \int_{\mathbb{R}^d} (w_j^+)^2 dx\right] \le C\sum_{j=1}^m \int_{\mathbb{R}^d} (w_j^+)^2 dx.$$

By Gronwall's inequality

$$0 \le \sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+)^2 dx \le e^{Ct} \sum_{j=1}^{m} \int_{\mathbb{R}^d} (w_j^+(0,x))^2 dx = 0.$$

Thus, we conclude that for all $j \in [\![1,m]\!]$ and $(t,x) \in [0,T] \times \mathbb{R}^d$

$$w_j(t,x) \le 0.$$

$\square$

Remark 2.7. Let u be a function that satisfies the assumptions of the previous theorem. From hypothesis (H3), we deduce that the positive vector $M = \Lambda \mathtt{1}$, where $\mathtt{1}$ is the vector of size m with all entries equal to 1, is a supersolution to (1.2) since the initial condition $u_0 = (u_{0i})_{i=1}^m$ is smaller than M (in the sense that all functions u_{0i} are smaller than Λ). Thus, we can not directly apply Theorem 2.6 to prove that u is bounded from above by the constant vector M, since a constant vector is not in $C_0(\mathbb{R}^d)$. However, we can adapt the proof of this theorem to get this upper bound on u. Indeed, consider for $x \in \mathbb{R}^d$ and $t \ge 0$

$$w(t,x) = (w_i(t,x))_{i=1}^m = e^{-lt}(u(t,x) - M),$$

where $l > 0$ is defined in (2.17). Thus, for all $i \in [\![1,m]\!]$, w_i solves on $(0,+\infty) \times \mathbb{R}^d$

$$\partial_t w_i + (-\Delta)^{\alpha_i} w_i \le l(|w_i| - w_i).$$

As in the proof of Theorem 2.6, we multiply this inequality by the positive part w_i^+ of w_i, and integrate over $\mathbb{R}^d$. All the integrals converge since w_i^+ is continuous and compactly supported. Moreover, we have

$$\int_{\mathbb{R}^d} l(|w_i| - w_i) w_i^+ dx = 0,$$

which leads to the same conclusion as in Theorem 2.6. Thus, starting from $u_0 = (u_{0i})_{i=1}^m$ smaller than M, we have

$$0 \le u(t,x) \le M, \quad \text{for all } (t,x) \in [0,+\infty) \times \mathbb{R}^d.$$

3 Upper and lower estimates.

First, let consider the auxiliary initial condition $v_0 = (v_{0i})_{i=1}^m$ given in Lemma 2.2 and let v be the mild solution of (1.2) with initial condition v_0. From (H2), we know that, for $i \in [\![1,m]\!]$ and $j \in [\![1,m]\!]$

$$\left|\partial_j f_i(s)\right| \le Lip(f_i), \quad \text{for all } s \in \mathbb{R}^m,$$

where $Lip(f_i)$ is the Lipschitz constant of f_i. Taking $l > 0$ defined in (2.17), we have for all $s = (s_i)_{i=1}^m \ge 0$

$$f_i(s) = \int_0^1 Df_i(\sigma s) d\sigma \cdot s \le \left| \sum_{j=1}^{m} s_j \int_0^1 \frac{\partial f_i}{\partial s_j}(\sigma s) d\sigma \right| \le l \sum_{j=1}^{m} s_j. \tag{3.1}$$

Let us consider $\overline{v} = (\overline{v}_i)_{i=1}^m$ the mild solution of the following system

$$\begin{cases} \partial_t \overline{v} + L\overline{v} &= B\overline{v}, \quad t > 0, x \in \mathbb{R}^m \\ \overline{v}(0,\cdot) &= v_0, \quad \mathbb{R}^m, \end{cases} \tag{3.2}$$

where $B = (b_{ij})_{i,j=1}^m$ is a matrix with $b_{ij} = l$ for all $i, j \in [\![1,m]\!]$. By (3.1) and Remark 2.4, we conclude that $v \le \overline{v}$ in $\mathbb{R}^d \times [0,+\infty)$. Moreover, since v_0 belongs to the domain $\prod_{i=1}^m D_0(A_i)$, v and $\overline{v}$ are classical solutions to (1.2). Taking Fourier transforms in each term of system (3.2), we have

$$\begin{cases} \partial_t \mathfrak{F}(\overline{v}) &= (A(|\xi|) + B)\mathfrak{F}(\overline{v}), \quad \xi \in \mathbb{R}^m, t > 0 \\ \mathfrak{F}(\overline{v})(0,\cdot) &= \mathfrak{F}(v_0), \quad \mathbb{R}^m, \end{cases}$$

where $A(|\xi|) = \operatorname{diag}(-|\xi|^{2\alpha_1}, \dots, -|\xi|^{2\alpha_m})$. Thus,

$$\mathfrak{F}(\overline{v})(t,\xi) = e^{(A(|\cdot|)+B)t}.\mathfrak{F}(v_0)(\xi).$$

In what follows, we prove that for each time $t > 0$, the solution v of (1.2) with initial v_0 decay as $|x|^{-d-\beta}$ for large values of $|x|$.

Lemma 3.1. *Let $v = (v_i)_{i=1}^m$ be the classical solution of (1.2), with initial condition v_0. If $2\alpha \ge \beta$, then, there exist locally bounded functions $C_i : (0,\infty) \to \mathbb{R}_+$ such that for all $t > 0$ and $|x|$ large enough, we have*

$$v_i(t,x) \le \frac{C_i(t)}{1+|x|^{d+\beta}}, \qquad \forall i \in [\![1,m]\!]$$

Proof. Defining the Fourier Transform $\mathfrak{F}^{-1}(e^{(A(|\xi|)+B)t}) := (\eta_{ij})_{i,j=1}^m$, we have that

$$0 \le v_i(t,x) \le \overline{v} = \sum_{j=1}^m \eta_{ij}(t,\cdot) * v_{0j}(x), \qquad \forall\ i \in [\![1,m]\!]$$

moreover, by Lemma 3 of [5]

$$|\eta_{ij}(t,x)| \le \frac{C_{ij}(t)}{1+|x|^{d+2\alpha}}, \quad \forall\ t > 0,\ |x| > R$$

for some $R > 0$ and C_{ij} locally positive bounded functions in $(0,+\infty)$. Taking $R > 0$ large if necessary, there exists a constant $c > 0$ such that

$$\int_{\mathbb{R}^d} \frac{1}{1+|y|^{d+2\alpha}} \frac{1}{1+|x-y|^{d+\beta}} \le \frac{c}{|x|^{d+\beta}}, \qquad \text{if } |x| \ge 2R. \tag{3.3}$$

Moreover, from the choice of v_0, we see that, for all $j \in [\![1,m]\!]$, $v_{0j}(x) = O(|x|^{-d-\beta})$ as $|x| \to \infty$ and since v_{0j} is bounded by Λ, we have that $v_{0j}(x) \le C(1+|x|^{d+\beta})^{-1}$ for all $x \in \mathbb{R}^d$. Hence, for all $t > 0$ and $|x| \ge 2R$

$$\begin{aligned} |\eta_{ij}(t,\cdot) * v_{0j}(x)| &\le \int_{|y|<R} \frac{C|\eta_{ij}(t,y)|}{1+|x-y|^{d+\beta}} dy \\ &\quad + \int_{|y|\ge R} \frac{C_{ij}(t)}{1+|y|^{d+2\alpha}} \frac{C}{1+|x-y|^{d+\beta}} dy \\ &:= I_1 + I_2. \end{aligned}$$

Now, if $|y| < R$, we have that $|x|/2 \geq R > |y|$ and then $|x-y| \geq |x| - |y| \geq |x|/2$, thus, by Lemma 4 in [5], the first integral is bounded by

$$\begin{aligned} I_1 &\leq \frac{2^{d+\beta}C}{1+|x|^{d+\beta}} \int_{|y|<R} |\eta_{ij}(t,y)| dy \\ &\leq \frac{\overline{C}e^{ct}}{1+|x|^{d+\beta}} \left[\int_0^1 r^{d-1} e^{-r^{2\alpha_1 t}} dr + \int_1^\infty r^{d-1} e^{-r^{2\alpha t}} dr \right] \\ &\leq \frac{\overline{C}_{ij} e^{ct}}{1+|x|^{d+\beta}} \left(t^{-\frac{d}{2\alpha_1}} + t^{-\frac{d}{2\alpha}} \right). \end{aligned}$$

Moreover, by (3.3), we have $I_2 \leq \overline{C}_{ij}(t)(1+|x|^{d+\beta})^{-1}$. □

Now, for the sake of completeness, we present an alternative proof of Lemma 3.1, for the particular case in which $\alpha := \alpha_i < 1$ for all $i \in [\![1,m]\!]$. In this case, since we are working with a unique index α, we can bound directly in the iteration process (2.4), to prove that the solution of the system (1.2) decay as $|x|^{-d-\beta}$ for all $t > 0$.

Proof. From the iterative process (2.4), we have that $v = \lim_{i\to+\infty} v^i$ where v^i satisfies

$$v^{n+1}(t,x) = \mathtt{T}_t v_0(x) + \int_0^t \mathtt{T}_{t-s} F(v^n(s,x)) ds$$

with $v^0(t) = (T_t v_{0i})_{i=1}^m$. Using the semigroup properties of the operator T_t and taking $l > 0$ defined in (2.17), we have for all $i \in [\![1,m]\!]$ and $n \in \mathbb{N}$

$$|v_i^n(t,x)| \leq \left(1 + (lmt) + \frac{(lmt)^2}{2!} + \dots + \frac{(lmt)^n}{n!} \right) \sum_{j=1}^m T_t v_{0j}(x) \tag{3.4}$$

where $v^n = (v_i^n)_{i=1}^m$. Also, we know that

$$\left\| v^n - v \right\|_{C([0,\infty),X)^m} \to 0, \quad \textit{when } n \to +\infty$$

where $X = C_0(\mathbb{R}^d)$. Then, we deduce that

$$|v_i^n(t,x)| \to |v_i(t,x)| = v_i(t,x) \quad \textit{when } n \to +\infty$$

for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$ and $i \in [\![1,m]\!]$. Taking the limit when $n \to +\infty$ in (3.4), we conclude that

$$v_i(t,x) \leq e^{lmt} \sum_{j=1}^m T_t v_{0j}(x), \qquad (t,x) \in [0,\infty) \times \mathbb{R}^d. \tag{3.5}$$

Now, by definition of v_0, we have that there exist $c_i > 0$ large enough and $r_i > 1$, such that

$$v_{0i}(x) \leq c_i |x|^{-d-\beta}, \qquad \forall\ |x| \geq r_i \tag{3.6}$$

also we know that $0 \le v_{0i} \le \Lambda$. Thus, if $|x| > 2r_i$ and $t > 0$

$$\begin{aligned} T_t v_{0i}(x) &\le \int_{\mathbb{R}^d} \frac{t^{-\frac{d}{2\alpha}} B v_{0i}(x-y)}{1+(t^{-\frac{1}{2\alpha}}|y|)^{d+2\alpha}} dy \\ &\le \int_{\{|y|\le |x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B v_{0i}(x-y)}{1+(t^{-\frac{1}{2\alpha}}|y|)^{d+2\alpha}} dy \\ &\quad + \int_{\{|y|> |x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B v_{0i}(x-y)}{1+(t^{-\frac{1}{2\alpha}}|y|)^{d+2\alpha}} dy \\ &:= I_1 + I_2. \end{aligned}$$

If $|y| \le |x|/2$, we have $|x-y| \ge |x| - |y| \ge \frac{|x|}{2}$, then

$$\begin{aligned} I_1 &\le \int_{\{|y|\le |x|/2\}} \frac{t^{-\frac{d}{2\alpha}} B}{1+(t^{-\frac{1}{2\alpha}}|y|)^{d+2\alpha}} \frac{c_i}{|x-y|^{d+\beta}} dy \le \frac{2^{d+\beta} B c_i}{|x|^{d+\beta}} \int_{\mathbb{R}^d} \frac{1}{1+|s|^{d+2\alpha}} ds \\ &\le \frac{\overline{c}_i}{|x|^{d+\beta}}. \end{aligned}$$

Now, if $|y| > |x|/2$ and since $|x| > 2r_i$, we have that $1+(t^{-\frac{1}{2\alpha}}|y|)^{d+2\alpha} \ge (2^{-1}t^{-\frac{1}{2\alpha}})^{d+2\alpha}|x|^{d+\beta}$, hence

$$I_2 \le \frac{2^{d+2\alpha} B t}{|x|^{d+\beta}} \int_{\mathbb{R}^d} v_{0i}(s) ds \le \frac{\overline{c}_i t}{|x|^{d+\beta}}.$$

Therefore, we conclude that, if $|x| \ge 2r_i$ then $T_t v_{0i} \le 2\overline{c}_i(1+t)(1+|x|^{d+\beta})^{-1}$. Otherwise if $|x| < 2r_i$

$$T_t v_{0i}(x) \le \int_{\mathbb{R}^d} \frac{t^{-\frac{d}{2\alpha}} B v_{0i}(y)}{1+(t^{-\frac{1}{2\alpha}}|x-y|)^{d+2\alpha}} dy \le \Lambda B \int_{\mathbb{R}^d} \frac{1}{1+|s|^{d+2\alpha}} ds \le \frac{\overline{c}_i}{1+|x|^{d+\beta}}.$$

Thus, we conclude that

$$T_t v_{0i}(x) \le 2\overline{c}_i(1+t)(1+|x|^{d+\beta})^{-1}, \qquad \forall\ (t,x) \in (0,\infty) \times \mathbb{R}^d.$$

Using (3.5), (3.6), we get $v_i(t,x) \le C_i(t)(1+|x|^{d+\beta})^{-1}$ for all $(t,x) \in [0,\infty) \times \mathbb{R}^d$, where $C_i(t) = 2(1+t)e^{lmt}\sum_{i=1}^m \overline{c}_i$. □

The following is an important result needed to prove Theorem 1.1, which sets an algebraically lower bound for the solutions of the cooperative system (1.2).

Lemma 3.2. *Let $v = (v_i)_{i=1}^m$ be the solution of the system (1.2), with initial condition v_0 and F satisfying (1.5) and (H2). If $2\alpha \ge \beta$, there exist constants $\sigma_i > 0$, $\tau_1 > 0$ and $C_i > 0$ such that*

$$v_i(t,x) \ge \frac{C_i t e^{-\sigma_i t}}{t^{\frac{d}{\beta}+1} + |x|^{d+\beta}}, \qquad \forall i \in [\![1,m]\!]$$

for all $x \in \mathbb{R}^d$ and $t \ge \tau_1$.

Proof. It is easy to prove that $w(t,x) = e^{lt}v(t,x)$ is the mild solution of (2.19) with the initial condition v_0 and $l > 0$ defined in (2.17). Thus, following the same computations as in Lemma 2.2, we can deduce that

$$w(t,x) \geq \mathrm{T}_t v_0(x), \qquad \forall\ (t,x) \in [0,\infty)\times\mathbb{R}^d.$$

By definition of β, there exists $k \in I$ such that $\beta = \beta_k$. So, we have that

$$v_k(t,x) \geq e^{-lt}H_k(t,\cdot) * v_{0k}(x), \qquad \text{for all } t \geq 0$$

where H_k is the heat kernel defined in (2.16). In both cases $\alpha_k \in (0,1)$ or $\alpha_k = 1$, taking $|x| > 1$ and $t \geq 1$, since v_{0k} satisfies (2.12)

$$H_k(t,\cdot) * v_{0k}(x) \geq \overline{C}_k te^{-t} \int_{|y|\leq 1} \frac{1}{1+|x-y|^{d+\beta}} dy$$

for some positive constant $\overline{C}_k$. Also, $|x-y| \leq |x| + |y| \leq |x| + 1 \leq 2|x|$, so

$$1 + |x-y|^{d+\beta} \leq 2^{d+\beta} t^{\frac{d}{\beta}+1} + 2^{d+\beta}|x|^{d+\beta}$$

then $H_k(t,\cdot) * v_{0k}(x) \geq C_k te^{-t}\left(t^{\frac{d}{\beta}+1} + |x|^{d+\beta}\right)^{-1}$. Now, if $|x| \leq 1$ and $t \geq 1$

$$H_k(t,\cdot) * v_{0k}(x) \geq C_k e^{-t} \geq \frac{C_k te^{-t}}{t^{\frac{d}{\beta}+1} + |x|^{d+\beta}}$$

with $C_k > 0$ smaller if necessary. Then, taking $\sigma_k = l + 1$, we have

$$v_k(t,x) \geq \frac{C_k te^{-\sigma_k t}}{t^{\frac{d}{\beta}+1} + |x|^{d+\beta}}, \qquad \forall x \in \mathbb{R}^d, t \geq 1.$$

Now, to compute the lower bound for the other entries of the solution, we note that

$$f_i(z) = \int_0^1 Df_i(\sigma z)d\sigma \cdot z = \sum_{j=1}^{m} z_j \int_0^1 \frac{\partial f_i}{\partial z_j}(\zeta_\sigma)d\sigma \tag{3.7}$$

thus, if $z \in [0,M]$ then $\zeta_\sigma = \sigma z \in [0,M]$ and since $\frac{\partial f_i}{\partial u_j} : [0,M] \to \mathbb{R}$ is continuous for all $i, j \in [\![1,m]\!]$, using the fact that the system is cooperative, there exist constants $\gamma_{ij} > 0$ such that

$$\left|\frac{\partial f_i}{\partial u_i}(\zeta_\sigma)\right| \leq \gamma_{ii} \quad \text{and} \quad \gamma_{ij} \leq \frac{\partial f_i}{\partial u_j}(\zeta_\sigma) \ \text{ for all } \ i \neq j. \tag{3.8}$$

Now, for all $i \neq k$, by (3.7) and (3.8)

$$f_i(z) \geq \int_0^1 \frac{\partial f_i}{\partial z_k}(\zeta_\sigma)d\sigma z_k + \int_0^1 \frac{\partial f_i}{\partial z_i}(\zeta_\sigma)d\sigma z_i \geq \gamma_{ik}z_k - \delta_i z_i$$

where $\delta_i \geq \max(\gamma_{ii}, \sigma_k + 2)$. Hence, taking v_k as a fixed function and since $0 \leq v \leq M$ by Lemma 3.1 and Remark 2.7, we have for all $i \neq k$, $x \in \mathbb{R}^d$ and $t \geq 0$

$$\partial_t v_i + (-\triangle)^{\alpha_i} v_i \geq \gamma_{ik} v_k - \delta_i v_i.$$

Then, by the maximum principle of reaction diffusion equations and Duhamel's formula, we have

$$v_i(t,x) \geq e^{-\delta_i t}\left(H_i(t,\cdot) * v_{0i}(x) + \gamma_{ik}\int_0^t \int_{\mathbb{R}^d} H_i(t-s,y)v_k(s,x-y)e^{\delta_i s}dyds\right)$$

for all $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^d$. So, taking $t \geq \tau_1$ for any $\tau_1 \geq 3$

$$v_i(t,x) \geq C_k\gamma_{ik}e^{-\delta_i t}\int_1^{t-1}\int_{\mathbb{R}^d} H_i(t-s,y)\frac{se^{(\delta_i-\sigma_k)s}}{s^{\frac{d}{\beta}+1}+|x-y|^{d+\beta}}dyds.$$

To conclude, we claim that it is possible to find a constant $C > 0$ such that

$$\int_{\mathbb{R}^d}\frac{se^{-\frac{|y|^2}{4}}}{s^{\frac{d}{\beta}+1}+|x-y|^{d+\beta}}dy \geq \frac{Cse^{-s}}{s^{\frac{d}{\beta}+1}+|x|^{d+\beta}}, \qquad \forall x \in \mathbb{R}^d, s \geq 1$$

and

$$\int_{\mathbb{R}^d}\frac{1}{1+|y|^{d+2\alpha_i}}\left[\frac{s}{s^{\frac{d}{\beta}+1}+|x-y|^{d+\beta}}\right]dy \geq \frac{Cse^{-s}}{s^{\frac{d}{\beta}+1}+|x|^{d+\beta}}, \qquad \forall x \in \mathbb{R}^d, s \geq 1.$$

Thus, in both cases $\alpha_i \in (0,1)$ or $\alpha_i = 1$, by the previous inequalities, we can bound as follows

$$v_i(t,x) \geq C_i\frac{e^{-\delta_i t}}{t^{\frac{d}{\beta}}}\int_1^{t-1}\frac{se^{(\delta_i-\sigma_k-1)s}}{s^{\frac{d}{\beta}+1}+|x|^{d+\beta}}ds \geq \frac{C_i te^{-\sigma_i t}}{t^{\frac{d}{\beta}+1}+|x|^{d+\beta}}$$

for all $x \in \mathbb{R}^d$, $t \geq \tau_1$ with τ_1 larger if necessary and taking $\sigma_i := \delta_i$. □

4 Proofs of Theorems 1.1 and 1.2

Proof of Theorem 1.1. For the proof of part *a*), we analyze the limit of $u(t,x)$ as $|x| \to +\infty$ for all $t \geq 0$. Since $u_{0i} \in C_0(\mathbb{R}^d)$ for all $i \in [\![1,m]\!]$, from the construction of the mild solution u of the system (1.2) with initial datum u_0, we know that

$$u \in C([0,+\infty); C_0(\mathbb{R}^d))^m$$

thus, for each $t \geq 0$, we conclude that $u(t,\cdot) \in C_0(\mathbb{R}^d)^m$.
For the proof of part *b*), we consider the vector field

$$\underline{u} = \underline{a}e^{lt}\left(1+\underline{b}(t)|x|^{\delta_2(d+\beta)}\right)^{-\frac{1}{\delta_2}}\phi_1$$

where $\underline{a}$ is a positive constant, $l > 0$ is defined in (2.17), δ_2 as in (H5), $\underline{b}(t)$ is a time continuous function and $\phi_1 = (\phi_{1,i})_{i=1}^m \in \mathbb{R}^m$ is the normalized principal eigenvector of $DF(0)$ associated to the principal eigenvalue λ_1. Note that, since the system is cooperative, by Perron-Frobenius Theorem, we can ensure $\phi_1 > 0$. Doing a similar proof to Lemma 6 of [5], there exist a constant $D > 0$ such that

$$|(-\triangle)^{\alpha_i}\underline{u}_i| \leq D\underline{b}(t)^{\frac{2\alpha_i}{\delta_2(d+\beta)}}\underline{u}_i, \qquad \text{in } \mathbb{R}^d$$

with $\alpha_i \in (0,1]$, for all $i \in [\![1,m]\!]$. Taking for the moment $\underline{B} \leq (L\lambda_1^{-1})^{-\frac{\delta_2(d+\beta)}{\beta}}$ with any constant $L \geq \max\{D, \lambda_1\}$. We consider

$$\underline{b}(t) = (L\lambda_1^{-1} + \underline{B}^{-\frac{\beta}{\delta_2(d+\beta)}} e^{\frac{\beta\lambda_1}{d+\beta}t})^{-\frac{\delta_2(d+\beta)}{\beta}} \quad \text{and} \quad \underline{a} \leq \left(\frac{\min_{i\in[\![1,m]\!]}\{\phi_{1,i}\}\lambda_1}{2c_{\delta_2}}\right)^{\frac{1}{\delta_2}}$$

where c_{δ_2} is defined in (H5). Similarly to Lemma 8 of [5], appropriately choosing $\underline{a}$, $\underline{B}$ and by (H5), we can prove that

$$\partial_t \underline{u}_i + (-\triangle)^{\alpha_i}\underline{u}_i - \widetilde{f}_i(\underline{u}) \leq 0, \qquad \text{for all } i \in [\![1,m]\!]$$

with $\widetilde{f}_i(\underline{u}) := l\underline{u}_i + e^{lt} f_i(e^{-lt}\underline{u})$.

Moreover, it is possible to find $t_1 \geq \max\{\tau_1, 2L\lambda_1^{-1}\}$ large enough, where τ_1 was defined in Lemma 3.2, such that $e^{lt_1}v_i(t_1,x) \geq \underline{u}_i(0,x)$ for all $x \in \mathbb{R}^d$ and $i \in [\![1,m]\!]$, note that, it is possible by the lower bound stated in Lemma 3.2. Now, by Lemma 2.2, we know that $w(2,x) \geq v_0(x)$ where $w(t,x) = e^{lt}u(t,x)$ with u the solution of (1.2), hence, applying Theorem 2.6 to the system (2.19), by the previous considerations, we have for all $i \in [\![1,m]\!]$

$$w_i(t,x) \geq e^{l(t-2)}v_i(t-2,x) \geq \underline{u}_i(t-t_1-2,x), \qquad \forall\, x \in \mathbb{R}^d, t \geq t_1+2.$$

Let us define

$$\theta_i = \underline{a}\phi_{1,i}e^{l(t_1+2)}2^{-\frac{1}{\delta_2}} \quad \text{and} \quad C^{d+\beta} = e^{-\lambda_1(t_1+2)}\underline{B}^{-\frac{1}{\delta_2}}.$$

Then, if $t \geq t_1+2$ and $|x| \leq Ce^{\frac{\lambda_1}{d+\beta}t}$, we have that

$$w_i(t,x) \geq \underline{a}e^{l(t-t_1-2)}\phi_{1,i}(1+\underline{b}(t-t_1-2)|x|^{\delta_2(d+\beta)})^{-\frac{1}{\delta_2}} \geq e^{lt}\theta_i.$$

Taking $\tau := t_1+2$, we conclude $u_i(t,x) \geq \theta_i$ for all $i \in [\![1,m]\!]$. □

Proof of Theorem 1.2. We consider the function $\overline{u}$ given by

$$\overline{u} = \overline{a}\left(1+\overline{b}(t)|x|^{\delta_1(d+\beta)}\right)^{-\frac{1}{\delta_1}}\phi_1$$

with δ_1 as in (H4), $\overline{a} > 0$ and $\overline{b}$ a continuous function. Now, we choose a constant $\overline{B} < (1+D\lambda_1^{-1})^{-\frac{\delta_1(d+\beta)}{\beta}}$ and we set

$$\overline{b}(t) = (-D\lambda_1^{-1} + \overline{B}^{-\frac{\beta}{\delta_1(d+\beta)}} e^{\frac{\beta\lambda_1}{d+\beta}t})^{-\frac{\delta_1(d+\beta)}{\beta}}$$

and

$$\overline{a} \geq [(D+\lambda_1)/c_{\delta_1}]^{\frac{1}{\delta_1}} \max_{i\in[\![1,m]\!]}(1/\phi_{1,i})$$

with c_{δ_1} given in (H4). Similarly to Lemma 7 of [5], for all $i \in [\![1,m]\!]$, using the fact $b(t) \leq 1$, by (H3) and (H4), we can prove that

$$\partial_t \overline{u}_i + (-\triangle)^{\alpha_i}\overline{u}_i - f_i(\overline{u}) \geq 0. \tag{4.1}$$

Now, since $u_{0i}(x) = O(|x|^{-d-\beta})$ as $|x| \to +\infty$ for all $i \in [\![1,m]\!]$, let consider

$$\sigma(x) = \frac{1}{1+|x|^{d+\beta}} \quad \text{and} \quad v_{0i}(x) = c_i\int_1^2 T_{s,i}\sigma(x)ds$$

thus, following the same computations done in Lemma 2.1, we can prove that $v_{0i} \in D(A_i)$ is bounded and

$$v_{0i}(x) \geq c_{1,i}(1+|x|^{-d-\beta})^{-1} \quad \text{for all } x \in \mathbb{R}^d$$

with $c_{1,i}$ a multiple of the positive constant c_i for all $i \in [\![1,m]\!]$, hence, taking c_i large if necessary, we have that $u_{0i} \leq v_{0i}$ in $\mathbb{R}^d$, moreover, $v_{0i}(x) \leq c_{2,i}|x|^{-d-\beta}$ if $|x| \geq 2$ for all $i \in [\![1,m]\!]$.

Thus, we define by v the mild solution of (1.2) with initial conditions $v_0 = (v_{0i})_{i=1}^m$ and by election of v_0 we have that v is classical.

It is important to notice that v satisfies the conclusion of Lemma 3.1 and by Theorem 2.3 we have that $u \leq v$ in $\mathbb{R}^d$ for all $t \geq 0$. To end the proof, for any $t_0 > 0$ fixed, we can take $\overline{a}$ satisfying the above condition and $t_2 > t_0$ such that

$$\overline{u}_i(t_2,x) \geq v_i(t_0,x), \qquad \forall x \in \mathbb{R}^d, \forall i \in [\![1,m]\!] \tag{4.2}$$

note that, this is possible due to Lemma 3.1. Therefore, we conclude that $\overline{u}$ is a supersolution to (1.2). Thus, using (4.1), (4.2) and Theorem 2.6, we get for all $t \geq t_0$

$$\overline{u}_i(t+t_2-t_0,x) \geq v_i(t,x) \geq u_i(t,x), \qquad \forall x \in \mathbb{R}^d, \forall i \in [\![1,m]\!].$$

Now, given any $\varepsilon = (\varepsilon_i)_{i=1}^m > 0$, we define $c_i^{d+\beta} := \overline{a}\phi_{1,i}e^{\lambda_1(t_2-t_0)}\left[\varepsilon_i \overline{B}^{1/\delta_1}\right]^{-1}$. Thus, taking $c = \max_i\{c_i\}$, for all $t > t_0$ and $|x| > ce^{\frac{\lambda_1}{d+\beta}t}$

$$u_i(t,x) \leq \overline{a}\phi_{1,i}(1+\overline{b}(t+t_2-t_0)|x|^{\delta_1(d+\beta)})^{-\frac{1}{\delta_1}} < \varepsilon_i.$$

We conclude taking $\tau := t_0$. □

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References

[1] D.G. Aronson and H.F. Weinberger, Multidimensional nonlinear diffusions arising in population genetics, *Adv. Math.* **30** (1978), pp. 33-76.

[2] G. Barles and L. C. Evans and P. E. Souganidis, Wavefront propagation for reaction-diffusion systems of PDE, *Duke Math. J.* **61** (1990), pp. 835-858.

[3] X. Cabré and J. Roquejoffre, The influence of fractional diffusion in Fisher-KPP equation, *Commun. Math. Phys.* **320** (2013), pp. 679-722.

[4] X. Cabré and A. C. Coulon, and J. M. Roquejoffre, Propagation in Fisher-KPP type equations with fractional diffusion in periodic media, *C. R. Math. Acad. Sci. Paris* **350** (2012), pp. 885-890.

[5] A-C Coulon and M. Yangari, Exponential propagation for fractional reaction-diffusion cooperative systems with fast decaying initial conditions, *J. Dyn. Diff. Equat.* (2015), DOI: 10.1007/s10884-015-9479-1.

[6] A. Erdélyi, Higher Transcendental Functions. Vol. I, *New York-Toronto-London, McGraw-Hill Book Company, Inc.* (1953).

[7] L. C. Evans and P. E. Souganidis, A PDE approach to geometric optics for certain semilinear parabolic equations, *Indiana Univ. Math. J.* **45(2)** (1989), pp. 141-172.

[8] P. Felmer and M. Yangari, Fast Propagation for Fractional KPP Equations with Slowly Decaying Initial Conditions, *SIAM J. Math. Anal.* **45(2)** (2013), pp. 662-678.

[9] A.N. Kolmogorov, I.G. Petrovsky and N.S. Piskunov, Étude de l'equation de la diffusion avec croissance de la quantité de matiére et son application á un probléme biologique, *Bull. Univ. État Moscou Sér. Inter.* **A1** (1937), pp. 1-26.

[10] M. Lewis and B. Li and H. Weinberger, Spreading speed and linear determinacy for two-species competition models, *J. Math. Biol.* **45** (2002), pp. 219-233.

[11] R. Lui, Biological growth and spread modeled by systems of recursions. I. Mathematical theory, *Math. Biosci.* **93(2)** (1989), pp. 269-295.

[12] B. Li, H. Weinberger and M. Lewis, Spreading speeds as slowest wave speeds for cooperative systems, *Math. Biosci.* **196** (2005), pp. 82-98.

[13] D. Stan and J. Vázquez, The Fisher-KPP equation with nonlinear fractional diffusion, *SIAM J. Math. Anal.* (2014), DOI: 10.1137/130918289.

[14] H.F. Weinberger and M. Lewis and B. Li, Anomalous spreading speeds of cooperative recursion systems, *J. Math. Biol.* **55** (2007), pp. 207-222.

[15] H.F. Weinberger and M. Lewis and B. Li, Analysis of linear determinacy for spread in cooperative models, *J. Math. Biol.* **45** (2002), pp. 183-218.

[16] M. Yangari, Existence and uniqueness of global mild solutions for nonlocal Cauchy systems in Banach spaces, *Revista Politécnica* **35(2)** (2015), pp.149-152.