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Abstract

In this paper, we prove L^p estimates of a class of parabolic maximal functions provided that their kernels are in L^q. Using the obtained estimates, we prove the boundedness of the maximal functions under very weak conditions on the kernel. In particular, we prove the L^p-boundedness of our maximal functions when their kernels are in $L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})$ or in the block space $B_q^{0,-1/2}(\mathbb{S}^{n-1})$, $q > 1$.

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1 Introduction

Let $\mathbb{R}^n$, $n \geq 2$ be the $n-$dimensional Euclidean space and let $\mathbb{S}^{n-1}$ be the unit sphere in $\mathbb{R}^n$ equipped with the normalized induced Lebesgue measure $d\sigma$. For nonzero $x \in \mathbb{R}^n$, we set $x' = x/|x|$. Let $(\mathbb{R}^n, \rho)$ be the metric space of Fabes and Rivière [9]. It is shown in [9] that ρ is the unique solution to the equation

$$F(x,\rho) = \sum\nolimits_{j=1}^{n} x_j^2 \rho^{-2\alpha_j} = 1.$$

Here, $\alpha_1, \ldots, \alpha_n$ are fixed real numbers in the interval $[1,\infty)$. The space $(\mathbb{R}^n, \rho)$ is commonly known by the mixed homogeneity space related to $\{\alpha_j\}_{j=1}^n$. It is evident that if $\alpha_1 = \ldots = \alpha_n = 1$, then $\rho(x) = |x|$, the standard Euclidean metric on $\mathbb{R}^n$.

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For $\lambda > 0$, let D_λ be the $n \times n$ diagonal matrix with diagonal entries $\lambda^{\alpha_1}, ..., \lambda^{\alpha_n}$, i.e.,

$$D_\lambda = diag(\lambda^{\alpha_1}, ..., \lambda^{\alpha_n}) = \begin{bmatrix} \lambda^{\alpha_1} & & 0 \\ & \ddots & \\ 0 & & \lambda^{\alpha_n} \end{bmatrix}.$$

A measurable function $\Omega : \mathbb{R}^n \to \mathbb{R}$ is said to be homogenous of degree zero with respect to D_λ if

$$\Omega(D_\lambda x) = \Omega(x), \lambda > 0. \tag{1.1}$$

Using the change of variables

$$\begin{aligned} x_1 &= \rho^{\alpha_1} \cos\varphi_1 \ldots \cos\varphi_{n-2} \cos\varphi_{n-1}, \\ x_2 &= \rho^{\alpha_2} \cos\varphi_1 \ldots \cos\varphi_{n-2} \sin\varphi_{n-1}, \\ &\vdots \\ x_{n-1} &= \rho^{\alpha_{n-1}} \cos\varphi_1 \sin\varphi_2, \\ x_n &= \rho^{\alpha_n} \sin\varphi_1, \end{aligned}$$

it follows that

$$dx = \rho^{\alpha-1} J(\varphi_1, ..., \varphi_{n-1}) d\rho d\sigma$$

where $J(\varphi_1, ..., \varphi_{n-1})$ is the Jacobian of the above transformation.

For a suitable function $\varphi : \mathbb{R}_+ \to (0, \infty)$, we let $S(\varphi)$ be the surface introduced by Al-Salman [1]. More precisely, we let $S(\varphi)$ be the image of the mapping

$$D_{\varphi(\rho(.))}(.) : \mathbb{R}^n \times \mathbb{S}^{n-1} \to \mathbb{R}^n,$$

where $D_{\varphi(\rho(.))}(.)$ is given by

$$D_{\varphi(\rho(x))}(x') = ((\varphi(\rho(x)))^{\alpha_1} x'_1, ..., (\varphi(\rho(x)))^{\alpha_n} x'_n). \tag{1.2}$$

Let $L^2(\mathbb{R}_+, r^{-1}dr)$ be the Hilbert space of all measurable functions $h : \mathbb{R}_+ \to \mathbb{R}$ satisfying the integrability condition

$$||h||_{L^2(\mathbb{R}_+, r^{-1}dr)} = \left(\int_0^\infty \frac{|h(t)|^2}{t} dt \right)^{\frac{1}{2}} < \infty. \tag{1.3}$$

Consider the operator

$$\mathcal{M}_{\Omega,P,\varphi}(f)(x) = \sup_{||h||_{L^2(\mathbb{R}_+, r^{-1}dr)} \leq 1} \left| \int_{\mathbb{R}^n} e^{iP(y)} f(x - D_{\varphi(\rho(y))}(y')) \frac{\Omega(y') h(\rho(y)) dy}{\rho(y)^{\alpha-1}} \right|, \tag{1.4}$$

where $\alpha = \sum_{j=1}^n \alpha_j$, $P : \mathbb{R}^n \to \mathbb{R}$ is a polynomial mapping, and Ω is a homogenous function of degree zero (with respect to D_λ) that is integrable on $\mathbb{S}^{n-1}$ and satisfies the cancellation property

$$\int_{\mathbb{S}^{n-1}} \Omega(y') J(y') d\sigma(y') = 0. \tag{1.5}$$

By specializing to the case $\rho(x)=|x|$ and $D_{\varphi(\rho(x))}(x')=x$, the resulting operator was considered by Al-Salman in [3]. In fact, Al-Salman studied the operator

$$\mathcal{M}_{\Omega,P}(f)(x)=\sup_{\|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)}\le 1}\left|\int_{\mathbb{R}^n} e^{iP(y)}f(x-y))\frac{\Omega(y')h(|y|)dy}{|y|^{n-1}}\right|. \tag{1.6}$$

By establishing suitable L^p bounds of $\mathcal{M}_{\Omega,P}$ when $\Omega\in L^q(\mathbb{S}^{n-1})$, $q>1$, Al-Salman proved that the operator $\mathcal{M}_{\Omega,P}$ is bounded on L^p provided that Ω is in $L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})$ or in the block space $B_q^{0,-1/2}(\mathbb{S}^{n-1})$, $q>1$ (see definition in Section 6 below).

On the other hand, when $P=0$, the resulting operator $\mathcal{M}_{\Omega,\varphi}=\mathcal{M}_{\Omega,0,\varphi}$ was considered in [1]. It is proved in [1] that $\mathcal{M}_{\Omega,\varphi}$ is bounded on L^p for certain values of p provided that φ is a polynomial mapping and Ω satisfies certain size conditions introduced by Grafakos and Stefanov in [11]. It is worth pointing out here that the conditions introduced in [11] are distinct from the condition $\Omega\in L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})$ or the condition $\Omega\in B_q^{0,-1/2}(\mathbb{S}^{n-1})$, $q>1$. Also, it is known that the spaces $L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})$ and $B_q^{0,-1/2}(\mathbb{S}^{n-1})$ are distinct in the sense $L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})\not\subseteq B_q^{0,-1/2}(\mathbb{S}^{n-1})$ and $B_q^{0,-1/2}(\mathbb{S}^{n-1})\not\subseteq L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})$. Moreover, $\bigcup_{r>1} L^r(\mathbb{S}^{n-1})\sqsubset L\log L^{\frac{1}{2}}(\mathbb{S}^{n-1})\cap B_q^{0,-1/2}(\mathbb{S}^{n-1})$.

The main concern of this paper is to consider the general parabolic operator $\mathcal{M}_{\Omega,P,\varphi}$ and to prove results analogous to those proved in [3]. Our results will generalize as well as improve previously obtained results.

Our main results are the following:

Theorem 1.1. *Suppose that* $\Omega\in L^q(\mathbb{S}^{n-1})$, $q>1$, and *satisfy the conditions* (1.1) and (1.5) with $\|\Omega\|_1\le 1$. *Let* $\varphi:[0,\infty)\to\mathbb{R}$ *be a real valued polynomial of degree d. Let* $\mathcal{M}_{\Omega,\varphi}$ *be given by (1.4) with* $P=0$. *Then*

$$\left\|\mathcal{M}_{\Omega,\varphi}(f)\right\|_p\le\{1+\log^{\frac{1}{2}}(e+\|\Omega\|_q)\}C_{p,q}\|f\|_p, \tag{1.7}$$

for all $p\ge 2$ *where* $C_{p,q}=\frac{2^{1/q'}}{2^{1/q'}-1}C_p$. *Here* $1/q'=1-1/q$ *and* C_p *is a constant that may depend on the degree of the polynomial* φ *but it is independent of the function* Ω *and the index* q.

Theorem 1.2. *Suppose* $\Omega\in L^q(\mathbb{S}^{n-1})$, $q>1$, and satisfy the conditions (1.1) and (1.5) with $\|\Omega\|_1\le 1$. *Suppose also that* $\varphi:[0,\infty)\to\mathbb{R}$ *is a real valued polynomial of degree d such that* $D_{\varphi(\rho(x))}$ is an odd polynomial. *Then*

$$\left\|\mathcal{M}_{\Omega,P,\varphi}(f)\right\|_p\le\{1+\log^{\frac{1}{2}}(e+\|\Omega\|_q)\}C_{p,q}\|f\|_p, \tag{1.8}$$

for all $p\ge 2$ *where* $C_{p,q}=\frac{2^{1/q'}}{2^{1/q'}-1}C_p$. *Here* $1/q'=1-1/q$ *and* C_p *is a constant that may depend on the degree of the polynomial* φ *but it is independent of the function* Ω, *the index* q, *and the coefficients of the polynomial* P.

Combining Theorem 1.1, Theorem 1.2, and suitable decomposition of the function Ω, we have the following two results.

Theorem 1.3. *Suppose that* $\Omega \in L(Log)^{1/2}(\mathbb{S}^{n-1})$ *and* satisfy the conditions (1.1) and (1.5). *Let* $\varphi : [0,\infty) \to \mathbb{R}$ *be a real valued polynomial of degree d with the property that* $D_{\varphi(\rho(x))}$ *is an odd polynomial whenever* $\deg(P) > 0$. *Then*

$$\left\|\mathcal{M}_{\Omega,P,\varphi}(f)\right\|_p \leq C_p \|f\|_p, \tag{1.9}$$

for all $p \geq 2$. *Here* C_p *is a constant that may depend on the degree of the polynomial* φ *but it is independent of the function* Ω *and the coefficients of the polynomial* P.

Theorem 1.4. *Suppose that* $\Omega \in B_q^{0,-\frac{1}{2}}(\mathbb{S}^{n-1}), q > 1$ *and* satisfy the conditions 1.1 and 1.5. *Let* $\varphi : [0,\infty) \to \mathbb{R}$ *be a real valued polynomial of degree d with the property that* $D_{\varphi(\rho(x))}$ *is an odd polynomial whenever* $\deg(P) > 0$. *Then*

$$\left\|\mathcal{M}_{\Omega,P,\varphi}(f)\right\|_p \leq C_p \|f\|_p, \tag{1.10}$$

for all $p \geq 2$. *Here* C_p *is a constant that may depend on the degree of the polynomial* φ *but it is independent of the function* Ω *and the coefficients of the polynomial* P.

In order to prove our results, we need to invest new ideas in addition to some ideas from [1], [3], and [10]. The argument in this paper has a significant difference from that used in [3], where suitable decompositions are needed (See Lemma 3.1 and the proofs of Theorems 1.1 and 1.2.). It should be remarked here that the method presented in this paper is general enough to enable us to study more general operators. Finally, we would like to point out that this work is part of master thesis of done by the first author under the supervision of the second author[14].

Throughout this paper, the letter C will denote a constant that may vary at each occurrence, but it is independent of the essential variables.

2 Introductory estimates

This section is devoted to prove necessary estimates that we shall need in the proofs of the main results. We start by recalling the following lemma due to Van der Corput:

Lemma 2.1 (van der Corput). *Suppose that* φ *is a real valued function which is smooth in* (a,b), *and that* $\left|\varphi^{(k)}(x)\right| \geq 1$ *for all* $x \in (a,b)$. *Then*

$$\left|\int_a^b e^{i\lambda\varphi(x)}\psi(x)dx\right| \leq c_k \lambda^{\frac{-1}{k}} \left[|\psi(b)| + \int_a^b \left|\psi'(x)\right| dx\right]. \tag{2.1}$$

holds when:
1. $k \geq 2$ *or*
2. $k = 1$ *and* $\varphi'(x)$ *is monotonic.*
The bound c_k *is independent of* φ *and* λ.

The following three lemmas will be useful:

Lemma 2.2 ([10]). *Let $P(x) = \sum_{|\alpha|=d} a_\alpha x^\alpha$ be a real valued homogeneous polynomial of degree d in $\mathbb{R}^n$ where d is an odd integer. Then*

$$\underset{\omega\in R}{Sup} \int_{\mathbb{S}^{n-1}} |P(x)-\omega|^{-\epsilon}\, d\sigma(x) < A_{\epsilon,n,d} \left(\sum_{|\alpha|=d} |a_\alpha| \right)^{-\epsilon} \tag{2.2}$$

for each $\epsilon < \frac{1}{2d}$. The bound $A_{\epsilon,n,d}$ does not depend on the coefficients $\{a_\alpha\}$.

Lemma 2.3 ([15]). Let $P : \mathbb{R}^n \to \mathbb{R}^d$ *be a polynomial mapping and let* $\Omega \in L^1(\mathbb{S}^{n-1})$ *be a homogeneous function of degree zero on $\mathbb{R}^n$. For every $1 < p \le \infty$ there exists a positive constant C_p such that* the maximal function

$$M_{\Omega,P} f(x) = \sup_{r>0} \frac{1}{r^n} \left| \int_{|y|<r} f(x - P(y))\, \Omega(y) dy \right| \tag{2.3}$$

satisfies

$$\|M_P f\|_p \le C_p \|\Omega\|_1 \|f\|_p, \tag{2.4}$$

for $f \in L^p\left(\mathbb{R}^d\right)$. The constant C_p may depend on the degree of the polynomial P. But, it is independent of the coefficients of the polynomials.

Lemma 2.4. ([10]). *Let l and n be positive integers. Let $V_l(n)$ be the space of real-valued homogeneous polynomials of degree l on $\mathbb{R}^n$. Let $U_l(n)$ be a subspace of $V_l(n)$ with $|x|^l \notin U_l(n)$. Let $\Omega \in L^q(\mathbb{S}^{n-1}), q > 1$* and that $s = \min\{2, q\}$. *Then there exists a positive constant A independent of Ω such that*

$$\int_{2^k}^{2^{k+1}} \left| \int_{\mathbb{S}^{n-1}} e^{iF(ry')} \Omega(y') d\sigma(y') \right| \frac{dr}{r} \le A \|\Omega\|_{L^q} (2^{kl} \|P_l\|)^{-\frac{1}{4s'l}}$$

for all $k \in \mathbb{Z}$ and function $F : \mathbb{R}^n \to \mathbb{R}$ of the form

$$F(x) = \sum_{j=0}^{l} P_j + W(|x|)$$

where P_j is a homogeneous polynomial of degree j, $0 \le j \le m, P_l \in U_l(n)$, and W is an arbitrary function. The constant A may depend on the subspace $U_l(n)$ if l is even, but it is independent of $U_l(n)$ if l is odd. Here, $\|P_l\| = \sum_{|\alpha|=l} \left|a_\alpha\right|$ where $P_l(y) = \sum_{|\alpha|=l} a_\alpha y^\alpha$.

By following the argument in [1], we prove the following proposition:

Proposition 2.5. *Suppose that φ is a polynomial of degree d. Then there are linear transformations $L_j, 1 \le j \le d_\varphi$ where $d_\varphi = d \max\{\alpha_j : 1 \le j \le n\}$such that*

$$D_{\varphi(r)}(x').\xi = \sum_{i=1}^{d_\varphi} (L_{j,\varphi}(\xi).x') r^j \tag{2.5}$$

Proof. For $1 \le j \le n$, let

$$(\varphi(r))^{\alpha i} = \sum_{j=1}^{d\,\alpha_i} b_{ji} r^j.$$

For $1 \le i \le n$, set $b_{ji} = 0$ for $j > d\alpha_i$. Then

$$(\varphi(r))^{\alpha i} = \sum_{j=1}^{d_\varphi} b_{ji} r^j \tag{2.6}$$

and

$$D_{\varphi(r)}(x').\xi = \sum_{i=1}^{n} (\varphi(r))^{\alpha_i} x'_i \xi_i = \sum_{i=1}^{d_\varphi} (L_j(\xi).x') r^j$$

where $L_j(\xi) = (b_{j1}\xi_1, ..., b_{jn}\xi_n)$. This completes the proof.

Throughout this paper, we shall let

$$a_{q,k} = 2^{\log(e+\|\Omega\|_q)k} \tag{2.7}$$

and

$$C_q(\Omega) = \log(e + \|\Omega\|_q). \tag{2.8}$$

Now, we prove the following lemma.

Lemma 2.6. *Let $\Omega \in L^q(\mathbb{S}^{n-1})$ with $\|\Omega\|_1 \leqslant 1, q > 1$. Let φ be a polynomial of degree d. For $k \in Z$, let*

$$J_{k,\Omega}(\xi) = \int_1^{2^{2C_q(\Omega)}} \left| \int_{\mathbb{S}^{n-1}} \Omega(y^{'}) J(y') e^{iD_{\varphi(\rho a_{q,k+1})}(y').\xi} d\sigma(y') \right|^2 \frac{d\rho}{\rho}. \tag{2.9}$$

Then, $J_{k,\Omega}(\xi)$ satisfies

$$sup_\xi J_{k,\Omega}(\xi) \le CC_q(\Omega) \left| \left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi}(\xi) \right|^{-\frac{\epsilon}{4q' C_q(\Omega)}}, \tag{2.10}$$

where $L_{d_\varphi}(\xi)$ is a linear transformation, C is constant independent of the function Ω, the parameter k, and the index q.

Proof. Using the boundedness of J and the fact that $\|\Omega\| \le 1$, we get

$$J_{k,\Omega}(\xi') \le CC_q(\Omega). \tag{2.11}$$

Next, by making use of the observation

$$\begin{aligned} & e^{iD_{\varphi(\rho a_{q,k+1})}(y').\xi'} e^{-iD_{\varphi(\rho a_{q,k+1})}(z').\xi'} \\ = \; & e^{i[D_{\varphi(\rho a_{q,k+1})}(y') - D_{\varphi(\rho a_{q,k+1})}(z')].\xi'} = e^{i[D_{\varphi(\rho a_{q,k+1})}(y'-z')].\xi'}, \end{aligned}$$

it follows that

$$\begin{aligned} J_{k,\Omega}(\xi) &= \int_1^{2^{2C_q(\Omega)}} \iint_{\mathbb{S}^{n-1}} \overline{\Omega(z')J(z')}\Omega(y')J(y')e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\, d\sigma(y',z')\frac{d\rho}{\rho} \\ &\leq \iint_{\mathbb{S}^{n-1}} \left|\overline{\Omega(z')J(z')}\Omega(y')J(y')\right| \left| \int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\frac{d\rho}{\rho} d\sigma(y',z')\right|. \end{aligned}$$

By Hölder's inequality, we have

$$\begin{aligned} & J_{k,\Omega}(\xi) \\ \leq\ & \left[\iint_{\mathbb{S}^{n-1}} \left|\overline{\Omega(z')J(z')}\Omega(y')J(y')\right|^q d\sigma(y',z')\right]^{1/q} \times \\ & \left[\iint_{\mathbb{S}^{n-1}} \left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\frac{d\rho}{\rho}\right|^{q'} d\sigma(y',z')\right]^{1/q'} \\ \leq\ & \|\Omega\|_q^2 M \left[\iint_{\mathbb{S}^{n-1}} \left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\frac{d\rho}{\rho}\right|^{q'} d\sigma(y',z')\right]^{1/q'}. \end{aligned}$$

Thus,

$$(J_{k,\Omega}(\xi'))^{q'} \leq C\|\Omega\|_q^{2q'} \iint_{\mathbb{S}^{n-1}} \left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi'}\ \frac{d\rho}{\rho}\right|^{q'} d\sigma(y',z'). \tag{2.12}$$

Now, we claim that

$$\left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\ \frac{d\rho}{\rho}\right| \leq C\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi}(\xi).(y'-z')\right|^{-\epsilon}. \tag{2.13}$$

In order to see (2.13), notice that

$$D_{\varphi(t)}(y'-z') = \sum_{i=1}^{n} (\varphi(t))^{\alpha_i}(y'_i - z'_i)\xi_i.$$

It should be noticed here that $D_{\varphi(t)}(y'-z').\xi$ is a polynomial of degree d_φ. Thus by Proposition 2.5, there exist linear transformations $L_j, 1 \leq j \leq d_\varphi$ that satisfy

$$D_{\varphi(\rho a_{q,k+1})}(y'-z').\xi = \sum_{j=1}^{d_\varphi} (L_j(\xi).(y'-z')(\rho a_{q,k+1})^j.$$

Therefore, by Lemma 2.1, we have

$$\left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\ \frac{d\rho}{\rho}\right| \le C\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi}(\xi).(y'-z')\right|^{-\epsilon}. \tag{2.14}$$

Thus, by (2.14) and the trivial estimate

$$\left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\ \frac{d\rho}{\rho}\right| \le CC_q(\Omega), \tag{2.15}$$

we get

$$\left|\int_1^{2^{2C_q(\Omega)}} e^{iD_{\varphi(\rho a_{q,k+1})}(y'-z').\xi}\frac{d\rho}{\rho}\right| \le \left[C_q(\Omega)\right]^{1-\frac{1}{4q'}} C\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi}(\xi).(y'-z')\right|^{-\frac{\epsilon}{4q'}}. \tag{2.16}$$

By (2.16) and (2.12), we get

$$\begin{aligned}
& J_{k,\Omega}(\xi) \\
\le\ & C\|\Omega\|_q^2\left[C_q(\Omega)\right]^{1-\frac{1}{4q'}}\left(\iint_{\mathbb{S}^{n-1}} C\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi,\varphi}(\xi).(y'-z')\right|^{-\frac{\epsilon}{4}} d\sigma(y',z')\right)^{\frac{1}{q'}} \\
=\ & C\|\Omega\|_q^2\left[C_q(\Omega)\right]^{1-\frac{1}{4q'}}\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi}(\xi)\right|^{-\frac{\epsilon}{4q'}}\left(\iint_{\mathbb{S}^{n-1}}\left|(y'-z').(L_{d_\varphi}(\xi))'\right|^{-\frac{\epsilon}{4}} d\sigma\right)^{\frac{1}{q'}}.
\end{aligned}$$

Since $0<\epsilon<1$, the last integral is bounded in $\xi\in\mathbb{S}^{n-1}$. In fact, by Lemma 2.2, we have

$$J_{k,\Omega}(\xi)\le C\|\Omega\|_q^2\left|L_{d_\varphi}(\xi)\left(a_{q,k+1}\right)^{d_\varphi}\right|^{-\epsilon/4q'}\left[C_q(\Omega)\right]^{1-1/4q'}. \tag{2.17}$$

By interpolation between (2.11) and (2.17), we get

$$J_{k,\Omega}(\xi)\le C\|\Omega\|_q^{2\theta}\left|L_{d_\varphi}(\xi)\left(a_{q,k+1}\right)^{d_\varphi}\right|^{-\epsilon\theta/4q'}\left[C_q(\Omega)\right]^{1-\theta/4q'}.$$

for any $0<\theta<1$. By choosing $\theta=\frac{1}{C_q(\Omega)}$, we get

$$J_{k,\Omega}(\xi)\le CC_q(\Omega)\left|\left(a_{q,k+1}\right)^{d_\varphi} L_{d_\varphi,q}(\xi)\right|^{-\epsilon/4q'C_q(\Omega)}.$$

Now, we assume that the polynomial P is given by $P(y)=\sum_{|\alpha|\le d} a_\alpha y^\alpha$. Then

$$D_{\varphi(\rho)}(y') = ((\sum_{i=1}^{d_\varphi} b_{1i}\rho^i)y'_1,\ldots,(\sum_{i=1}^{d_\varphi} b_{ni}\rho^i)y'_n) = \varphi(\rho)\otimes y'$$

where

$$\varphi(\rho) = (\sum_{i=1}^{d_\varphi} b_{1i}\rho^i, ..., \sum_{i=1}^{d_\varphi} b_{ni}\rho^i)$$

and b_{ji} are as in the proof of Proposition 2.5. Here, we use the notation

$$x \otimes y = (x_1 y_1, ..., x_n y_n).$$

Now, for $1 \le s \le d_\varphi$, we let $\sigma_{s,\rho}$ be the measure defined by

$$\int f d\sigma_{s,\rho} = \int_{\mathbb{S}^{n-1}} e^{iP(D_\rho y')} f(\Gamma_{s,\rho}(y'))\Omega(y')J(y')d\sigma(y') \tag{2.18}$$

where $\Gamma_{s,\rho}(y') = \varphi_s(\rho) \otimes y'$ and

$$\varphi_s(\rho) = (\sum_{i=1}^{s} b_{1i}\rho^i, ..., \sum_{i=1}^{s} b_{ni}\rho^i). \tag{2.19}$$

For later use, it is worth observation that

$$\hat{\sigma}_{s,\rho}(\xi) = \int_{\mathbb{S}^{n-1}} e^{iP(D_\rho(y'))}\Omega(y')J(y')e^{-i\xi.\Gamma_{s,\rho}(y')}d\sigma(y'). \tag{2.20}$$

Also, we let

$$\varphi_{s,d}(y',\xi,\rho). = P(D_\rho(y')) - \xi.\Gamma_{s,\rho}(y') = \sum_{|\alpha|\le d} a_\alpha \rho^{|\alpha|} y'^\alpha - \sum_{i=1}^{s} (L_i(\xi).y')\rho^i$$

Now, we have the following proposition:

Proposition 2.7. *Let $J_{s,d,k}$ be given by*

$$J_{s,d,k}(\xi) = \int_{a_{q,k}}^{a_{q,k+1}} \left| \int_{\mathbb{S}^{n-1}} e^{i\varphi_{s,d}(y',\xi,\rho)} \Omega(y') J(y') d\sigma(y') \right|^2 \frac{d\rho}{\rho}.$$

Then

$$J_{s,d,k}(\xi) \le CC_q(\Omega) 2^{-\frac{k}{4q'}} (\sum_{|\alpha|=d} |a_\alpha|)^{\frac{-1}{4dq' C_q(\Omega)}} \tag{2.21}$$

for $d > s$ and

$$J_{s,d,k}(\xi) \le CC_q(\Omega) (2^{\frac{k}{4q'} C_q(\Omega)} |L_s(\xi)|)^{\frac{-\epsilon}{C_q(\Omega)}} \tag{2.22}$$

for $1 < d \le s$. Here, for the case $d = s$, we assume all terms of $D_{\varphi(\rho)}(y')$ are odd.

Proof. Notice that

$$J_{s,d,k}(\xi) \le M^2 \iint_{\mathbb{S}^{n-1}} |\Omega(y')| |\Omega(z')| \left| \int_{a_{q,k}}^{a_{q,k+1}} e^{i(\varphi_{s,d}(y',\xi,\rho) - \varphi_{s,d}(z',\xi,\rho))} \frac{d\rho}{\rho} \right| d\sigma. \tag{2.23}$$

By Lemma 2.1 and the observing

$$\begin{aligned}&\varphi_{s,d}(y',\xi,\rho)-\varphi_{s,d}(z',\xi,\rho)\\ =\ &\sum_{|\alpha|=d}a_\alpha(y'^\alpha-z'^\alpha)\rho^d+\sum_{|\alpha|<d}a_\alpha(y'^\alpha-z'^\alpha)\rho^{|\alpha|}-\sum_{i=1}^{s}(L_i(\xi)\cdot(y'-z'))\rho^i,\end{aligned}$$

we get

$$\left|\int_{a_{q,k}}^{a_{q,k+1}} e^{i(\varphi_{s,d}(y',\xi,\rho)-\varphi_{s,d}(z',\xi,\rho))}\frac{d\rho}{\rho}\right| \le C(a_{q,k}^d\left|\sum_{|\alpha|=d}a_\alpha(y'^\alpha-z'^\alpha)\right|)^{\frac{-1}{d}}. \tag{2.24}$$

Here, $L_{j,\varphi}(\xi)$ are the linear transformations given by Proposition 2.5. By Combining the estimate (2.24) and the trivial estimate

$$\left|\int_{a_{q,k}}^{a_{q,k+1}} e^{i(\varphi_{s,d}(y',\xi,\rho)-\varphi_{s,d}(z',\xi,\rho))}\frac{d\rho}{\rho}\right| \le CC_q(\Omega),$$

we get

$$\left|\int_{a_{q,k}}^{a_{q,k+1}} e^{i(\varphi_{s,d}(y',\xi,\rho)-\varphi_{s,d}(z',\xi,\rho))}\frac{d\rho}{\rho}\right| \le CC_q(\Omega)(a_{q,k}^d\left|\sum_{|\alpha|=d}a_\alpha(y'^\alpha-z'^\alpha)\right|)^{\frac{-1}{4dq'}}.$$

Thus, by the last estimate, the estimate (2.23), and Lemma 2.2, we get

$$J_{s,d,k}(\xi)\le\le M^2 2^{\frac{-kC_q(\Omega)}{4q'}}\|\Omega\|_q^2 C_q(\Omega)\{\sum_{|\alpha|=d}|a_\alpha|\}^{\frac{-1}{4dq'}};$$

which when combined with the estimate $J_{s,d,k}(\xi)\le CC_q(\Omega)$, imply that

$$J_{s,d,k}(\xi)\le C2^{\frac{-k}{4q'}}C_q(\Omega)(\sum_{|\alpha|=d}|a_\alpha|)^{\frac{-1}{4dC_q(\Omega)q'}}.$$

Next, for $d=s$, we observe that

$$\varphi_{s,d}(y',\xi,\rho)=(\sum_{|\alpha|=s}a_\alpha y'^\alpha-L_s(\xi).y')\rho^s+\ldots=\sum_{j=1}^{s}\psi_{j,\xi}(y')$$

where

$$\psi_{j,\xi}(y)=\sum_{|\alpha|=j}a_\alpha y^\alpha-(L_j(\xi).y)|y|^{j-1},1\le j\le s.$$

It is clear that $\psi_{j,\xi}$ is a homogeneous polynomial and that

$$\left\|\psi_{j,\xi}\right\|\ge\sum_{|\alpha|=j}|a_\alpha|+|L_s(\xi)|\ge|L_s(\xi)|$$

(note that $d=s>1$). Thus by Lemma 2.4, we get

$$J_{s,d,k}(\xi)\le C\|\Omega\|_q(2^{kC_q(\Omega)}|L_s(\xi)|)^{\frac{-\epsilon}{4s}}$$

for some $\epsilon > 0$. By interpolation with the estimate $J_{s,d,k}(\xi) \leq CC_q(\Omega)$, we get

$$J_{s,d,k}(\xi) \leq CC_q(\Omega)(2^{kC_q(\Omega)}|L_s(\xi)|)^{\frac{-\epsilon}{4sC_q(\Omega)}}.$$

Finally, for $d < s$, we argue as for the case $d > s$. Notice that

$$\begin{aligned} & \varphi_{s,d}(y',\xi,\rho) - \varphi_{s,d}(z',\xi,\rho) \\ = & \sum_{|\alpha|\leq d} a_\alpha(y'^\alpha - z'^\alpha)\rho^d - L_s(\xi)(y'-z')\rho^s + \sum_{i=1}^{s-1}(L_i(\xi).(z'-y')\rho^i. \end{aligned}$$

By Lemma 2.1 and interpolation, we get

$$J_{s,d,k}(\xi) \leq M^2 C_q(\Omega)\|\Omega\|_q^2 \left|2^{kC_q(\Omega)}L_s(\xi)\right|^{\frac{-1}{4sq'}};$$

which when combined with the estimate $J_{s,d,k}(\xi) \leq CC_q(\Omega)$ imply

$$J_{s,d,k}(\xi) \leq CC_q(\Omega)\left|2^{kC_q(\Omega)}L_s(\xi)\right|^{\frac{-1}{4sC_q(\Omega)q'}}.$$

This completes the proof.

As a consequence of Proposition 2.7, we can prove the following:

Proposition 2.8. *Let $\{\sigma_{s,\rho} : \rho > 0, 1 \leq s \leq d_\varphi\}$ be as in (2.18). Let L_s be the transformation as given by Proposition 2.5. Suppose that $D_{\varphi(\rho)}(y')$ is odd polynomial. Then*

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,\rho}(\xi) - \hat{\sigma}_{s-1,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \leq CC_q(\Omega)\left|a_{q,k}L_s(\xi)\right|^{\frac{1}{C_q(\Omega)}} \tag{2.25}$$

and

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \leq CC_q(\Omega)\left|a_{q,k}L_s(\xi)\right|^{\frac{-\epsilon}{C_q(\Omega)}}. \tag{2.26}$$

for $d \leq s \leq M$ and

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{d-1,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \leq C_q(\Omega)\left|a_{q,k}\right|^{\frac{-\epsilon}{C_q(\Omega)}}. \tag{2.27}$$

Now we prove the following:

Proposition 2.9. *Let $\{\sigma_{s,\rho} : \rho > 0, 1 \leq s \leq d_\varphi\}$ be given by (2.18). Let $\Omega \in L^q(\mathbb{S}^{n-1})$, $q > 1$, with $\|\Omega\|_1 \leqslant 1$. Let $\sigma^*_{s,q}$ be the operator defined by*

$$\sigma^*_{s,q}(f)(x) = \sup_k \int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \left\|\sigma_{s,\rho}\right| * f(x)\right| \frac{d\rho}{\rho}.$$

Then

$$\left\|\sigma^*_{s,q}(f)\right\|_p \le C_p C_q(\Omega)\|f\|_p$$

for $1 < p < \infty$. *The constants* C_p *are independent of the essential variables.*

Proof. Notice that

$$\begin{aligned}\sigma^*_{s,q}(f)(x) &\le M\sup_k \int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}}\int_{\mathbb{S}^{n-1}} \left|f(x-\Gamma_{s,\rho}(y'))\right|\left|\Omega(y')\right| d\sigma(y')\frac{d\rho}{\rho} \\ &\le M\left(C_q(\Omega)\right)\left(\sup_j \int_{2^j\le|y|<2^{j+1}} \left|f(x-\Gamma_{s,\rho}(y'))\right|\left|\Omega(y')\right|\frac{dy}{|y|^n}\right).\end{aligned}$$

Thus, by Lemma 2.3, the proof is complete.

3 A main lemma

This section is devoted to prove a lemma which will be the corner stone in the argument of the proofs of the main results.

Lemma 3.1. *Let* $\{w_k\}_{k\in\mathbb{Z}}$ *be a collection of* C^∞ *functions defined on* $(0,\infty)$ *that satisfy the following*

$$supp(w_k) \subseteq [\frac{1}{a_{q,k+1}}, \frac{1}{a_{q,k-1}}], 0 \le w_k \le 1, \sum_{k\in\mathbb{Z}} w_k(u) = 1. \tag{3.1}$$

Let

$$\eta_\infty(u) = \sum_{k=-\infty}^{0} w_k(u).$$

Let $\eta_{\Omega,p,\infty}$ *be the operator defined by*

$$\eta_{\Omega,p,\infty}(f)(x) = \left[\int_1^\infty \left|\eta_\infty \int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega J d\sigma\right|^2 \frac{d\rho}{\rho}\right]^{1/2}. \tag{3.2}$$

Then

$$\left\|\eta_{\Omega,p,\infty}(f)\right\|_p \le C_p C_q(\Omega)\|f\|_p \text{ for } p \ge 2.$$

Proof. Let $\{\sigma_{s,\rho} : \rho > 0, 1 \le s \le d_\varphi\}$ be given by (2.20). By Lemma 6.1 in [10], for $d \le s \le d_\varphi$, $n_s = rank(L_s)$, $\exists$ two nonsingular linear transformations $H_s : \mathbb{R}^{n_s} \longrightarrow \mathbb{R}^{n_s}$, $G_s : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ such that

$$\left|H_s\Pi^n_s G_s(\xi)\right| \le |L_s(\xi)| \le A_s\left|H_s\Pi^n_s G_s(\xi)\right|, \xi \in \mathbb{R}^n.$$

Let $\varphi \in C_0^\infty(\mathbb{R})$ be such that

$$\varphi(t) = \begin{cases} 1, |t| \le \frac{1}{2} \\ 0, |t| \ge 1 \end{cases}$$

Let $\phi(t) = \varphi(t^2)$ and let

$$\hat{\tau}_{s,\rho}(\xi) = \hat{\sigma}_{s,\rho}(\xi) \prod_{s<j\leq d_\varphi} \varphi(\left|a_{q,k}H_s\Pi_s^n G_s(\xi)\right|) - \hat{\sigma}_{s-1,\rho}(\xi) \prod_{s-1<j\leq d_\varphi} \varphi(\left|a_{q,k}H_s\Pi_s^n G_s(\xi)\right|)$$

Then it follows that

$$\begin{aligned}\sum_{s=d}^{d_\varphi} \hat{\tau}_{s,\rho}(\xi) &= \sum_{s=d}^{d_\varphi} \hat{\sigma}_{s,\rho}(\xi)\pi_s - \sum_{s=d}^{d_\varphi} \hat{\sigma}_{s-1,\rho}(\xi)\pi_{s-1} \\ &= \sum_{s=d}^{d_\varphi} \hat{\sigma}_{s,\rho}(\xi)\pi_s - \sum_{s=d-1}^{d_\varphi-1} \hat{\sigma}_{s,\rho}(\xi)\pi_s \\ &= \hat{\sigma}_{d_\varphi,\rho}(\xi)\pi_{d_\varphi} - \hat{\sigma}_{d-1,\rho}(\xi)\pi_{d-1}, \end{aligned} \tag{3.3}$$

where

$$\pi_s = \prod_{s<j\leq d_\varphi} \varphi(\left|a_{q,k+1}H_s\Pi_s^n G_s(\xi)\right|).$$

Notice that

$$\pi_{d_\varphi} = \prod_{d_\varphi<j\leq d_\varphi} \varphi = \prod_{j\in\emptyset} = 1.$$

Thus,

$$\sum_{s=d}^{d_\varphi} \hat{\tau}_{s,\rho}(\xi) = \hat{\sigma}_{d_\varphi,\rho}(\xi) - \hat{\sigma}_{d-1,\rho}(\xi)\pi_{d-1}. \tag{3.4}$$

By Proposition 2.8, it can be shown that

$$\left\|\hat{\tau}_{s,\rho}\right\| \leq CC_q(\Omega) \tag{3.5}$$

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\tau}_{s,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \leq C_q(\Omega) \left|2^{k\log(e+\|\Omega\|_q)} L_s(\xi)\right|^{\frac{-\epsilon}{C_q(\Omega)}}$$

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\tau}_{s,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \leq C_q(\Omega) \left|2^{k\log(e+\|\Omega\|_q)} L_s(\xi)\right|^{\frac{1}{C_q(\Omega)}}. \tag{3.6}$$

For $d \leq s \leq d_\varphi$, let

$$\eta^{(s)}_{\Omega,p,\infty}(f)(x) = \left[\int_1^\infty \left| \eta_\infty(\rho) \left(\tau_{s,\rho} * f(x)\right)\right|^2 \frac{d\rho}{\rho}\right]^{1/2}$$

and

$$\eta^{(d-1)}_{\Omega,p,\infty}(f)(x) = \left[\int_1^\infty \left| \eta_\infty(\rho) \left(\sigma_{d-1,\rho} * \varphi_{d-1} * f(x)\right)\right|^2 \frac{d\rho}{\rho}\right]^{1/2},$$

where $(\varphi_{d-1}\hat{)} = \pi_{d-1}$. Thus,

$$\eta_{\Omega,p,\infty}(f)(x) \le \sum_{s=d}^{d_\varphi} \eta^{(s)}_{\Omega,p,\infty}(f)(x) + \eta^{(d-1)}_{\Omega,p,\infty}(f)(x). \tag{3.7}$$

Now, we estimate $\left\|\eta^{(s)}_{\Omega,p,\infty}(f)\right\|_p$ for $d \le s \le d_\varphi$. By the argument on page 818 in [10], we may assume that L_s is a projection, i.e., $L_s(\xi) = \pi_{n_s}(\xi)$.

Let $\{\omega_j\}_j$ be a smooth partition of unity adapted to the intervals $E_j = [\frac{1}{a_{q,j+1}}, \frac{1}{a_{q,j-1}}]$ and satisfy

$$\omega_j \in C^\infty, 0 \le \omega_j \le 1, \sum_{k=-\infty}^{\infty} \omega_j(\rho) = 1, supp(\omega_j) \subset E_j, \text{ and } \left|\frac{d}{dt}\omega_j(t)\right| \le \frac{C}{t^j}$$

where C is a constant independent of the sequence $a_{q,j}$. Let Ψ_j be such that

$$(\Psi_j\hat{)}(\xi) = \omega_j(\left|\pi_{n_s}(\xi)\right|^2).$$

Then,

$$\begin{aligned} \eta^{(s)}_{\Omega,p,\infty}(f)(x) &= \left[\int_0^\infty \left|\sum_{j=-\infty}^{\infty}\sum_{k=-\infty}^{0} w_k(\rho)(\Psi_{j+k} * \tau_{s,\rho} * f(x))\right|^2 \frac{d\rho}{\rho}\right]^{\frac{1}{2}} \\ &\le \sum_{j=-\infty}^{\infty} \eta^{(j,s)}_{\Omega,p,\infty}(f)(x) \end{aligned}$$

where

$$\eta^{(j,s)}_{\Omega,p,\infty}(f)(x) = \left[\int_0^\infty \left|\sum_{k=-\infty}^{0} w_k(\rho)\left(\Psi_{j+k} * \tau_{s,\rho} * f(x)\right)\right|^2 \frac{d\rho}{\rho}\right]^{\frac{1}{2}}.$$

Notice that

$$\begin{aligned} \left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_2^2 &\le C \sum_{k=-\infty}^{0} \int_{\mathbb{R}^n} \int_{a_{q,k}}^{a_{q,k+1}} \left|\Psi_{j+k} * \tau_{s,\rho} * f(x)\right|^2 \frac{d\rho}{\rho} dx \\ &\le C \sum_{k=-\infty}^{0} \int_{E_{j+k}} \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\tau}_{s,\rho}(\xi)\right|^2 \left|\hat{f}(\xi)\right|^2 \omega_{j+k}(\left|\pi_{n_s}(\xi)\right|^2) \frac{d\rho}{\rho} d\xi. \end{aligned}$$

By the estimates (3.5) -(3.6), we have

$$\left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_2^2 \le C\left(C_q(\Omega)\right)^2 2^{-2|j|} \sum_{k=-\infty}^{0} \int_{E_{j+k}} \left|\hat{f}(\xi)\right|^2 d\xi.$$

Thus,

$$\left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_2 \le C\left(C_q(\Omega)\right) 2^{-|j|} \|f\|_2 . \tag{3.8}$$

By similar argument as in [10] and Proposition 2.9, we have

$$\left\|\tau^*_{s,q}(f)\right\|_p \leq C_p\left(C_q(\Omega)\right)\|f\|_p \tag{3.9}$$

for $1 < p < \infty$. The constants C_p are independent of the essential variables. Here,

$$\tau^*_{s,q}(f)(x) = \sup_k \int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \left\|\tau_{s,\rho}\right| * f(x)\right| \frac{d\rho}{\rho}.$$

Next, for $p \geq 2$, choose $g \in L^{(p/2)'}$ with $\|g\|_{(p/2)'} = 1$ such that

$$\begin{aligned}
&\left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_p^2 \\
= &\int_{\mathbb{R}^n}\int_0^\infty \left|\sum_{k=-\infty}^{0} w_k(\rho)\left(\Psi_{j+k} * \tau_{s,\rho} * f(x)\right)\right|^2 \frac{d\rho}{\rho}|g(x)|\,dx \\
\leq &\int_{\mathbb{R}^n}\int_0^\infty \sum_{k=-\infty}^{0} w_k(\rho)\left|\Psi_{j+k} * \tau_{s,\rho} * f(x)\right|^2 \frac{d\rho}{\rho}|g(x)|\,dx \\
\leq &\left(C_q(\Omega)\right)\int_{\mathbb{R}^n}\int_0^\infty \sum_{k=-\infty}^{0} w_k(\rho)\left|\Psi_{j+k} * f\right|^2 * \left|\tau_{s,\rho}\right| \frac{d\rho}{\rho}|g(x)|\,dx \\
\leq &\left(C_q(\Omega)\right)\int_{\mathbb{R}^n}\left(\sum_{k=-\infty}^{0}\left|\Psi_{j+k} * f\right|^2\right)\tau^*_{s,q}(\tilde{g})(-x)dx \\
\leq &\left(C_q(\Omega)\right)\left\|\sum_{k=-\infty}^{0}\left|\Psi_{j+k} * f\right|^2\right\|_{p/2}\left\|\tau^*_{s,q}(\tilde{g})\right\|_{(p/2)'} \\
\leq &\left(C_q(\Omega)\right)^2\|f\|_p^2.
\end{aligned}$$

Thus,

$$\left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_p \leq C_p\left(C_q(\Omega)\right)\|f\|_p, \tag{3.10}$$

where the last inequality follows by (3.9) and Littlewood-Paley theory. By interpolation between (3.8) and (3.10), we get

$$\left\|\eta^{(j,s)}_{\Omega,p,\infty}(f)\right\|_p \leq C\left(C_q(\Omega)\right)2^{-\theta_p|j|}\|f\|_p. \tag{3.11}$$

Now, we estimate $\left\|\eta^{(d-1)}_{\Omega,p,\infty}(f)\right\|_p$. Notice that

$$\begin{aligned}
\eta^{(d-1)}_{\Omega,p,\infty}(f)(x) &= \left[\int_0^\infty \left|\,\eta_\infty(\rho)\left(\sigma_{d-1,\rho} * \varphi_{d-1} * f(x)\right)\right|^2 \frac{d\rho}{\rho}\right]^{\frac{1}{2}} \\
&\leq \sum_{k=-\infty}^{0} \eta^{(d-1)}_{\Omega,p,\infty,k}(f)(x) \qquad (3.12)
\end{aligned}$$

where

$$\eta^{(d-1)}_{\Omega,p,\infty,k}(f)(x) = \left[\int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \left|\sigma_{d-1,\rho} * \varphi_{d-1} * f(x)\right|^2 \frac{d\rho}{\rho} \right]^{1/2}.$$

By Plancherel's theorem, we get

$$\begin{aligned} \left\|\eta^{(d-1)}_{\Omega,p,\infty,k}(f)\right\|_2^2 &\leq \int_{\mathbb{R}^n} \left|\hat{f}(\xi)\right|^2 \left(\int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \left|\hat{\sigma}_{d-1,\rho}(\xi)\right|^2 \frac{d\rho}{\rho} \right) d\xi \\ &\leq C_q(\Omega) 2^{\varepsilon k} \|f\|_2^2. \end{aligned}$$

Thus

$$\left\|\eta^{(d-1)}_{\Omega,p,\infty,k}(f)\right\|_2 \leq \sqrt{C_q(\Omega)} 2^{\frac{\varepsilon}{2}k} \|f\|_2. \tag{3.13}$$

Finally, for $p > 2$, choose $g \in L^{(p/2)'}$ with $\|g\|_{(p/2)'} = 1$ such that

$$\begin{aligned} &\left\|\eta^{(d-1)}_{\Omega,p,\infty,k}(f)\right\|_p^2 \\ = &\int_{\mathbb{R}^n} \int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \left| \int_{\mathbb{S}^{n-1}} e^{iP(D_r \cdot y')} \varphi_{d-1} * f(x - \Gamma_{d-1,\rho}(y')) \Omega(y') J(y') d\sigma(y') \right|^2 \frac{dr}{r} |g(x)| dx \\ \leq &\; C \|\Omega\|_1 \int_{\mathbb{R}^n} \int_{\frac{1}{a_{q,k+1}}}^{\frac{1}{a_{q,k-1}}} \int_{\mathbb{S}^{n-1}} \left|\varphi_{d-1} * f(z)\right|^2 \left|\Omega(y')\right| d\sigma(y') \frac{d\rho}{\rho} \left|g(z + \Gamma_{d-1,\rho}(y'))\right| dz \\ \leq &\; C \int_{\mathbb{R}^n} \left|\varphi_{d-1} * f(z)\right|^2 \sigma^*_{s,q}(g)(z + \Gamma_{d-1,\rho}(y')) dz \\ \leq &\; C C_q(\Omega) \left\|\varphi_{d-1} * f\right\|_p^2 \left\|\sigma^*_{s,q}(g)\right\|_{(p/2)'} \leq C C_q(\Omega)^2 \|f\|_p^2 \end{aligned}$$

Thus,

$$\left\|\eta^{(d-1)}_{\Omega,p,\infty,k}(f)\right\|_p \leq C_q(\Omega) \|f\|_p. \tag{3.14}$$

By interpolation between (3.13) and (3.14), we obtain

$$\left\|\eta^{(d-1)}_{\Omega,p,\infty,k}(f)\right\|_p \leq C_q(\Omega) 2^{\frac{\epsilon_p}{2}k} \|f\|_p \tag{3.15}$$

for $p \geq 2$. Hence, by (3.6), (3.11), (3.12), and (3.15), the proof is complete.

4 L^p estimates of the classical operator

This section is devoted to the proof of Theorem 1.1. The proof is very involved and contains various estimates. The details are as follows:

Proof (of Theorem 1.1). We start by making a suitable decomposition of the operator $\mathcal{M}_{\Omega,\varphi}$. In order to do so, we choose a collection of C^∞ functions $\{w_k\}$, $k \in \mathbb{Z}$ on $(0,\infty)$ with the properties

$$\begin{aligned} supp(w_k) &\subseteq [\frac{1}{a_{q,k+1}}, \frac{1}{a_{q,k-1}}], 0 \leqslant w_k \leqslant 1, \\ \sum_{k\in\mathbb{Z}} w_k &= 1, \left|\frac{d^s w_k(u)}{du^s}\right| \leqslant C_s u^{-s}, \end{aligned} \tag{4.1}$$

where $supp(w_k)$ is the support of the function w_k and C_s is independent of $C_q(\Omega)$. For $k \in \mathbb{Z}$, let G_k be the operator defined by

$$(G_k(f))\hat{}(\xi) = w_k(|\xi|)\hat{f}(\xi). \tag{4.2}$$

Then

$$\mathcal{M}_{\Omega,\varphi}(f)(x) \leq \left[\int_0^\infty \left|\int_{\mathbb{S}^{n-1}} f(x - D_{\varphi(\rho)}(y'))J(y')\Omega(y')d\sigma(y')\right|^2 \frac{d\rho}{\rho}\right]^{1/2}. \tag{4.3}$$

For $\rho \in (0,\infty)$, define the measure σ_ρ by

$$\int f d\sigma_\rho = \int_{\mathbb{S}^{n-1}} f(D_{\varphi(\rho)}(y'))J(y')\Omega(y')d\sigma(y').$$

Thus

$$\mu_{\Omega,0,\varphi}(f)(x) = \left(\int_0^\infty \left|\sigma_\rho * f(x)\right|^2 \frac{d\rho}{\rho}\right)^{1/2}.$$

By making use of Fourier transform and the observation

$$\hat{\sigma}_\rho(\xi) = \int_{\mathbb{S}^{n-1}} e^{-i\xi.(D_{\varphi(\rho)}(y'))} J(y')\Omega(y')d\sigma(y'), \tag{4.4}$$

one can show that

$$\int_{\mathbb{S}^{n-1}} f(x - D_{\varphi(\rho)}(y'))J(y')\Omega(y')d\sigma(y') = \sigma_\rho * f(x). \tag{4.5}$$

Now, we write

$$\begin{aligned} \sigma_\rho * f(x) &= \sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} (\Psi_{k+j} * \sigma_\rho * f(x))\chi_{[a_{q,k}, a_{q,k+1}]}(\rho) \\ &= \sum_{j\in\mathbb{Z}} F_j(f,\rho) \end{aligned}$$

where

$$F_j(f,\rho)=\sum_{k\in\mathbb{Z}}(\Psi_{k+j}*\sigma_\rho*f(x))\chi_{[a_{q,k},a_{q,k+1}]}(\rho). \tag{4.6}$$

Thus

$$\begin{aligned}\mu_{\Omega,0,\varphi}(f)(x) &\leq \left(\int_0^\infty\left|\sum_{j\in\mathbb{Z}}F_j(f,\rho)\right|^2\frac{d\rho}{\rho}\right)^{1/2}\\ &\leq \sum_{j\in\mathbb{Z}}\left(\left|\int_0^\infty F_j(f,\rho)\right|^2\frac{d\rho}{\rho}\right)^{1/2}=\sum_{j\in\mathbb{Z}}E_j(f)(x)\end{aligned}$$

where

$$E_j(f)(x)=\left[\int_0^\infty\left|\sum_{k\in\mathbb{Z}}(\Psi_{k+j}*\sigma_\rho*f(x))\chi_{[a_{q,k},a_{q,k+1}]}(\rho)\right|^2\frac{d\rho}{\rho}\right]^{1/2}. \tag{4.7}$$

Using the observation

$$\left|\sum_{k\in\mathbb{Z}}(\Psi_{k+j}*\sigma_\rho*f(x))\chi_{[a_{q,k},a_{q,k+1}]}(\rho)\right|^2=\sum_{k\in\mathbb{Z}}\left|(\Psi_{k+j}*\sigma_\rho*f(x))\right|^2\chi_{[a_{q,k},a_{q,k+1}]}(\rho),$$

it follows that

$$E_j(f)(x)=\left[\sum_{k\in\mathbb{Z}}\int_{a_{q,k}}^{a_{q,k+1}}\left|(\Psi_{k+j}*\sigma_\rho*f(x)\right|^2\frac{d\rho}{\rho}\right]^{1/2}.$$

Now, we estimate $\left\|E_j(f)\right\|_p$ for $2<p<\infty$. For $p>2$, choose $g\in L$ with $\|g\|_{(p/2)'}=1$ such that

$$\begin{aligned}&\left\|E_j(f)\right\|_p^2\\ =&\int_{\mathbb{R}^n}\sum_{k\in\mathbb{Z}}\int_{a_{q,k}}^{a_{q,k+1}}\left|\int_{\mathbb{S}^{n-1}}\Omega JG_{k+j}(f)(x-D_{\varphi(\rho a_{q,k})}(y'))d\sigma\right|^2\frac{d\rho}{\rho}|g(x)|dx\end{aligned} \tag{4.8}$$

Using the estimates $\|\Omega\|_1\leq 1$ and $|J(y')|\leq C$, we can see that

$$\left\|\Omega J^2\right\|_{L^1(\mathbb{S}^{n-1})}=\int_{\mathbb{S}^{n-1}}\left|\Omega(y')\right|\left|J(y')\right|^2d\sigma(y')\leq C.$$

By last inequality and an application of Cauchy -Schwartz inequality, we get

$$\begin{aligned}
&\left\|E_j(f)\right\|_p^2 \\
\leq\ & \left\|\Omega J^2\right\|_{L^1} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_1^{2^{2C_q(\Omega)}} \int_{\mathbb{S}^{n-1}} |\Omega| \left|G_{k+j}(f)(z)\right|^2 \left|g(z+D_{\varphi(\rho a_{q,k})}(y'))\right| d\sigma \frac{d\rho}{\rho} dz \\
\leq\ & C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{a_{q,k}}^{a_{q,k+1}} \left|G_{k+j}(f)(z)\right|^2 \int_{\mathbb{S}^{n-1}} |\Omega| \left|g(z+D_{\varphi(\rho a_{q,k})}(y'))\right| d\sigma \frac{d\rho}{\rho} dz \\
\leq\ & C \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{\mathbb{S}^{n-1}} \left|G_{k+j}(f)(z)\right|^2 \left|\Omega(y')\right| \int_1^{2^{2C_q(\Omega)}} \left|g(z+D_{\varphi(\rho a_{q,k+1})}(y'))\right| \frac{d\rho}{\rho} d\sigma dz.
\end{aligned}$$

By noticing that

$$\begin{aligned}
& \int_1^{2^{2C_q(\Omega)}} \left|g(z+D_{\varphi(\rho a_{q,k+1})}(y'))\right| \frac{d\rho}{\rho} \\
=\ & \int_0^{2\ln 2C_q(\Omega)} \left|g(z+D_{\varphi(e^r a_{q,k+1})}(y'))\right| dr \\
\leq\ & 2\ln 2C_q(\Omega) M_{y',\varphi}(\tilde{g})(-z)
\end{aligned}$$

where $M_{y',\varphi}(\tilde{g})(-z)$ is as in Proposition 2.9, it follows that

$$\begin{aligned}
&\left\|E_j(f)\right\|_p^2 \\
\leq\ & CC_q(\Omega) \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{\mathbb{S}^{n-1}} \left|G_{k+j}(f)(z)\right|^2 \left|\Omega(y')\right| M_{y',\varphi}(\tilde{g})(-z) d\sigma dz \\
\leq\ & CC_q(\Omega) \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left|G_{k+j}(f)(z)\right|^2 M_{y',\varphi}(\tilde{g})(-z) dz.
\end{aligned}$$

By Hőlder's inequality and Littlewood-Paley theory, we get

$$\begin{aligned}
\left\|E_j(f)\right\|_p^2 &\leq CC_q(\Omega) \left\| \sum_{k\in\mathbb{Z}} \left|G_{k+j}(f)\right|^2 \right\|_{p/2} \left\|M_{y',\varphi}(\tilde{g})\right\|_{(p/2)'} \\
&\leq CC_q(\Omega) \|f\|_p^2 \left\|M_{y',\varphi}(\tilde{g})\right\|_{(p/2)'}. \qquad (4.9)
\end{aligned}$$

On the other hand, by Proposition 2.9, we have

$$\left\|M_{y',\varphi}(\tilde{g})\right\|_{(p/2)'} \leq C\|g\|_{(p/2)'} \leq C. \qquad (4.10)$$

Therefore,

$$\left\|E_j(f)\right\|_p \leq C\sqrt{C_q(\Omega)}\,\|f\|_p. \qquad (4.11)$$

Now, we claim that

$$\left\|E_j(f)\right\|_2 \le C2^{-\alpha|j|/q'} \sqrt{C_q(\Omega)}\,\|f\|_2 \tag{4.12}$$

where

$$(E_j(f))^2 = \sum_{k\in\mathbb{Z}} \left\| \int_{\mathbb{S}^{n-1}} \Omega J G_{k+j}(f)(x - D_{\varphi(\rho a_{q,k})}(y'))d\sigma(y') \right\|^2_{L^2([1,a_{q,1}],\frac{d\rho}{\rho})} . \tag{4.13}$$

Let d_φ and b_{ij} be as in the proof of Proposition 2.5. For $1 \le s \le d_\varphi$, let φ_s be given by (2.19). For given $1 \le s \le d_\varphi$ and $1 \le l \le n$, define $\varphi_{s,l}(\rho)$ by

$$\vec{\varphi}_{s,l}(\rho) = \sum_{j=1}^{s} a_{j,l}\rho^j \tag{4.14}$$

and

$$\tilde{D}_{\varphi_s(\rho)}(x') = (\vec{\varphi}_{s,1}(\rho)x'_1, ..., (\vec{\varphi}_{s,n}(\rho)x'_n). \tag{4.15}$$

Then

$$\tilde{D}_{\varphi_M(\rho)}(x') = D_{\vec{\varphi}(\rho)}(x'). \tag{4.16}$$

Let $\{\sigma_{s,j} : 1 \le s \le d_\varphi\}$ be a family of measures defined by

$$\begin{aligned}&(\sigma_{s,j} * f)(x) \\ &= \int_{\mathbb{S}^{n-1}} \Omega(y')\,J(y')G_j(f)(x - \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y'))d\sigma.\end{aligned} \tag{4.17}$$

Thus,

$$\begin{aligned}&(\sigma_{s,j+k} * f)\hat{\ }(\xi) \\ &= \int_{\mathbb{R}^n} e^{-ix.\xi} \int_{\mathbb{S}^{n-1}} \Omega(y')\,J(y')G_{k+j}(f)(x - \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y'))d\sigma dx\end{aligned} \tag{4.18}$$

and

$$\hat{\sigma}_{s,j}(\xi) = w_j(|\xi|) \int_{\mathbb{S}^{n-1}} \Omega(y')\,J(y')\, e^{-i\tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y').\xi} d\sigma. \tag{4.19}$$

Now, we claim that $\sigma_{s,j}(\xi)$ satisfies the following properties

$$\left\|\sigma_{s,j}\right\|_1 \leqslant C. \tag{4.20}$$

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \frac{d\rho}{\rho} \leqslant CC_q(\Omega) \left|2^{kC_q(\Omega)l_s} L_s(\xi)\right|^{-\varepsilon_{s,1}/C_q(\Omega)} \tag{4.21}$$

and

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi) - \hat{\sigma}_{s-1,j}(\xi)\right|^2 \frac{d\rho}{\rho} \leqslant CC_q(\Omega) \left|2^{kC_q(\Omega)l_s} L_s(\xi)\right|^{\varepsilon_{s,2}/C_q(\Omega)} \tag{4.22}$$

for $\xi \in \mathbb{R}^n$ and $1 \leq s \leq M$.

To see (4.20), notice that

$$\begin{aligned}\left\|\sigma_{s,j}\right\|_1 &= \left\| w_j(|\xi|) \int_{\mathbb{S}^{n-1}} \Omega(y')\, J(y')\, e^{-i\tilde{D}_{\vec{\varphi}_s(\rho a_{q,k+1})}(y').\xi} d\sigma(y') \right\|_\infty \\ &\leqslant C.\end{aligned}$$

To see (4.21), notice that

$$\begin{aligned}&\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \frac{d\rho}{\rho} \\ &= \int_{a_{q,k}}^{a_{q,k+1}} \left| w_j(|\xi|) \int_{\mathbb{S}^{n-1}} \Omega(y')\, J(y')\, e^{-i\tilde{D}_{\vec{\varphi}_s(\rho a_{q,k+1})}(y').\xi} d\sigma(y') \right|^2 \frac{d\rho}{\rho}.\end{aligned}$$

Let

$$V_{k,q,s}(\xi, y', z') = e^{i\left\{\tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(z').\xi - \tilde{D}_{\vec{\varphi}_{s-1}(\rho a_{q,k})}(y').\xi\right\}}.$$

Then

$$\begin{aligned}&\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \frac{d\rho}{\rho} \\ &\leq \int_{a_{q,k}}^{a_{q,k+1}} \iint_{\mathbb{S}^{n-1}} \Omega(y')\, J(y')\, \overline{\Omega(z')} J(z')\, V_{k,q,s}(\xi, y', z')\, d\sigma(y', z') \frac{d\rho}{\rho} \\ &= \iint_{\mathbb{S}^{n-1}} \Omega(y')\, J(y')\, \overline{\Omega(z')} J(z') \left[\int_{a_{q,k}}^{a_{q,k+1}} V_{k,q,s}(\xi, y', z')\, \frac{d\rho}{\rho} \right] d\sigma(y', z') \\ &\leq \left[\iint_{\mathbb{S}^{n-1}} \left| \Omega(y')\, J(y')\, \overline{\Omega(z')} J(z') \right|^q d\sigma(y', z') \right]^{\frac{1}{q}} \times \\ &\quad \left[\iint_{\mathbb{S}^{n-1}} \left| \int_{a_{q,k}}^{a_{q,k+1}} V_{k,q,s}(\xi, y', z') \frac{d\rho}{\rho} \right|^{q'} d\sigma(y', z') \right]^{\frac{1}{q'}} \\ &\leq C \|\Omega\|_q^2 \left[\iint_{\mathbb{S}^{n-1}} \left| \int_{a_{q,k}}^{a_{q,k+1}} V_{k,q,s}(\xi, y', z')\, \frac{d\rho}{\rho} \right|^{q'} \right]^{\frac{1}{q'}} d\sigma(y', z').\end{aligned}$$

Let I be the integral inside the double integral over $\mathbb{S}^{n-1}$, i.e.,

$$I = \left| \int_{a_{q,k}}^{a_{q,k+1}} V_{k,q,s}(\xi, y', z') \, \frac{d\rho}{\rho} \right|.$$

Then it is clear that

$$I \leq C C_q(\Omega) \tag{4.23}$$

On the other hand, notice that

$$\begin{aligned}
& \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(z').\xi - \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y').\xi \\
= \; & \left\{ \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(z') - \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y') \right\}.\xi = \tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(z' - y').\xi \\
= \; & \sum_{i=1}^{n} \sum_{j=1}^{s} b_{j,i} (\rho a_{q,k})^j ((z'_i - y'_i).\xi_i) \\
= \; & a_{q,k} \sum_{i=1}^{n} \sum_{j=1}^{s} b_{j,i} \rho^j (a_{q,k})^{j-1} ((z'_i - y'_i).\xi_i) \\
= \; & a_{q,k} \omega_{k,s,n}(\rho).
\end{aligned}$$

Since

$$\begin{aligned}
& \left| \frac{d^s}{d\rho^s} (\omega_{k,s,n})(\rho) \right| = \left| \omega_{k,s,n}^{(s)}(\rho) \right| \\
= \; & \left| s! 2^{(s-1)\log(e + \|\Omega\|_q)(k)} \sum_{i=1}^{n} b_{s,i} ((z'_i - y'_i).\xi_i) \right| \\
\geq \; & \left| 2^{(s-1)(k)\log(e+\|\Omega\|_q)} (L_s(\xi).(z' - y') \right|,
\end{aligned}$$

it follows by Lemma 2.1 that

$$\begin{aligned}
I \; & \leq \; C_s \left| a_{q,k} \left(a_{q,k} \right)^{s-1} (L_s(\xi).(z' - y') \right|^{-1/s} \\
& = \; C_s \left| \left(a_{q,k} \right)^{s} (L_s(\xi).(z' - y') \right|^{-1/s}.
\end{aligned} \tag{4.24}$$

By interpolation between (4.23) and (4.24), we get

$$I \leq C_s \left| \left(a_{q,k} \right)^{s} (L_s(\xi).(z' - y') \right|^{-1/4q's} (C_q(\Omega))^{1 - \frac{1}{4q'}}.$$

Therefore,

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \frac{d\rho}{\rho} \le C\,\|\Omega\|_q^2 \times \left[\iint_{\mathbb{S}^{n-1}} I^{q'}\right]^{\frac{1}{q'}}$$

$$\le \quad C\left(C_q(\Omega)\right)^{1-\frac{1}{4q'}} \|\Omega\|_q^2 \left[\iint_{\mathbb{S}^{n-1}} \left|\left|\left(a_{q,k}\right)^s (L_s(\xi).(z'-y')\right|^{-1/4s}\right|\right]^{\frac{1}{q'}} d\sigma(y',z')$$

$$\le \quad C\left(C_q(\Omega)\right)^{1-\frac{1}{4q'}} \|\Omega\|_q^2 \left|\left(a_{q,k}\right)^s L_s(\xi)\right|^{\frac{-1}{4sq'}} \times$$

$$\left[\iint_{\mathbb{S}^{n-1}} \left|\left|\left(\frac{L_s(\xi)}{|L_s(\xi)|}.(z'-y')\right|^{-1/4s}\right|\right]^{\frac{1}{q'}} d\sigma(y',z');$$

which when combined with Lemma 2.2 imply that

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \frac{d\rho}{\rho} \le C_q\left(C_q(\Omega)\right)^{1-\frac{1}{4q'}} \|\Omega\|_q^2 \left|\left(a_{q,k}\right)^s L_s(\xi)\right|^{\frac{-1}{4sq'}}. \tag{4.25}$$

Thus, the estimate (4.21) follows by interpolation between (4.25) and the estimate

$$\int_{2^{\log(e+\|\Omega\|_q)k}}^{2^{\log(e+\|\Omega\|_q)(k+1)}} \left|\hat{\sigma}_{s,j}(\xi)\right|^2 \rho^{-1} d\rho \leqslant C C_q(\Omega).$$

To see (4.22), we set

$$E_{k,q,s}(\xi,y') = e^{-i\tilde{D}_{\varphi_s(\rho a_{q,k})}(y').\xi} - e^{-i\tilde{D}_{\varphi_{s-1}(\rho a_{q,k})}(y').\xi}.$$

Then

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi)-\hat{\sigma}_{s-1,j}(\xi)\right|^2 \frac{d\rho}{\rho}$$

$$= \quad \int_{a_{q,k}}^{a_{q,k+1}} \left|w_j(|\xi|)\int_{\mathbb{S}^{n-1}} \Omega(y')\,J(y')E_{k,q,s}(\xi,y')d\sigma(y')\right|^2 \frac{d\rho}{\rho}$$

$$\le \quad C\left[\int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \left[\int_{a_{q,k}}^{a_{q,k+1}} \left|E_{k,q,s}(\xi,y')\right|^2 \frac{d\rho}{\rho}\right]^{1/2} d\sigma(y')\right]^2.$$

Notice that

$$\begin{aligned} & (\tilde{D}_{\vec{\varphi}_s(\rho a_{q,k})}(y') - \tilde{D}_{\vec{\varphi}_{s-1}(\rho a_{q,k})}(y')).\xi \\ = & \sum_{i=1}^{n}\sum_{j=1}^{s} b_{j,i}(\rho a_{q,k})^j((y_i').\xi_i) - \sum_{i=1}^{n}\sum_{j=1}^{s-1} b_{j,i}(\rho a_{q,k})^j((y_i').\xi_i) \\ = & \sum_{i=1}^{n} b_{s,i}(\rho a_{q,k})^s((y_i').\xi_i) = (\rho a_{q,k})^s L_s(\xi).y'. \end{aligned}$$

Therefore,

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi) - \hat{\sigma}_{s-1,j}(\xi)\right|^2 \frac{d\rho}{\rho} \le CC_q(\Omega)\left|\left(a_{q,k}\right)^s L_s(\xi)\right|^{-\frac{\epsilon}{4q' C_q(\Omega)}}. \tag{4.26}$$

By interpolation between (4.26) and the trivial estimate

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\sigma}_{s,j}(\xi) - \hat{\sigma}_{s-1,j}(\xi)\right|^2 \frac{d\rho}{\rho} \le CC_q(\Omega),$$

we obtain (4.22).

Now, by a similar argument as in [10], there exist family of measures $\{\lambda_{s,j,o}\ ,\ 1 \le s \le d_\varphi\}$ such that for $\xi \in \mathbb{R}^n$ and $1 \le s \le d_\varphi$, we have

$$\sup\left\|\lambda_{s,j,o}\right\| \le C, \tag{4.27}$$

$$\int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho} \le C_s C_q(\Omega) \times \min\{\left|\left(a_{q,k}\right)^s (L_s(\xi))\right|^{\frac{2}{C_q(\Omega)}}, \left|\left(a_{q,k}\right)^s (L_s(\xi))\right|^{\frac{-1}{4sC_q(\Omega)q'}}\}, \tag{4.28}$$

and

$$\sigma_{s,j} = \sum_{o=1}^{d_\varphi} \lambda_{s,j,o}. \tag{4.29}$$

Thus by (4.29) and Minkowski's inequality, we get

$$\begin{aligned} E_j(f)(x) &= \left[\sum_{k\in\mathbb{Z}} \int_{a_{q,k}}^{a_{q,k+1}} \left|(\sigma_{s,j} * \Psi_{k+j} * f)(x)\right|^2 \frac{d\rho}{\rho}\right]^{1/2} \\ &= \left[\sum_{k\in\mathbb{Z}} \int_{a_{q,k}}^{a_{q,k+1}} \left|(\sum_{o=1}^{d_\varphi} \lambda_{s,j,o} * \Psi_{k+j} * f)(x)\right|^2 \frac{d\rho}{\rho}\right]^{1/2} \\ &\le \sum_{o=1}^{d_\varphi} \left[\sum_{k\in\mathbb{Z}} \int_{a_{q,k}}^{a_{q,k+1}} \left|(\lambda_{s,j,o} * \Psi_{k+j} * f)(x)\right|^2 \frac{d\rho}{\rho}\right]^{1/2}. \end{aligned}$$

By Plancherel's theorem and Fubini's theorem, we get

$$\begin{aligned}
&\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \int_{a_{q,k}}^{a_{q,k+1}} \left|(\lambda_{s,j,o} * \Psi_{k+j} * f)(x)\right|^2 \frac{d\rho}{\rho} dx \\
= &\int_{\mathbb{R}^n} \sum_{k\in\mathbb{Z}} \left|w_{k+j}(|\xi|)\hat{f}(\xi)\right|^2 \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho} d\xi.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|E_j(f)(x)\right\|_2^2 \\
= &\int_{\mathbb{R}^n} \left|E_j(f)(\xi)\right|^2 d\xi \\
\leq &\int_{\mathbb{R}^n} \left|\sum_{o=1}^{M} \left(\sum_{k\in\mathbb{Z}} \left|w_{k+j}(|\xi|)\hat{f}(\xi)\right|^2 \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho}\right)^{1/2}\right|^2 d\xi.
\end{aligned}$$

Let

$$S_{j,s,o} = \left[\sum_{k\in\mathbb{Z}} \left|w_{k+j}(|\xi|)\hat{f}(\xi)\right|^2 \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho}\right]^{\frac{1}{2}}. \tag{4.30}$$

Then

$$\begin{aligned}
&\left\|E_j(f)(x)\right\|_2^2 \\
\leq &\sum_{o=1}^{d_\varphi} \int_{\mathbb{R}^n} \left|\sum_{k\in\mathbb{Z}} \left|\hat{w}_{k+j}(|\xi|)\hat{f}(\xi)\right|^2 \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho}\right| d\xi \\
\leq &\sum_{o=1}^{d_\varphi} \left\|S_{j,s,o}\right\|_2^2.
\end{aligned}$$

Now, it is clear that

$$\left\|S_{j,s,o}\right\|_2^2 \leq \left|\sum_{k\in\mathbb{Z}} \int_{\mathbb{I}_{k+j}} \left|\hat{f}(\xi)\right|^2 \int_{a_{q,k}}^{a_{q,k+1}} \left|\hat{\lambda}_{s,j,o}(\xi)\right|^2 \frac{d\rho}{\rho}\right| d\xi,$$

where

$$\mathbb{I}_k = \{x \in \mathbb{R}^n : \frac{1}{a_{q,k+1}} \leq \left|\pi^n_{m_s}(x)\right| \leq \frac{1}{a_{q,k-1}}\}.$$

By the estimates (4.27) and (4.28), we get

$$\left\|S_{j,s,o}\right\|_2 \leq C_s \sqrt{C_q(\Omega)} 2^{-\alpha|j|} \|f\|_2. \tag{4.31}$$

Therefore,

$$\begin{aligned}\left\|E_j(f)\right\|_2^2 &\leq \sum_{o=1}^{d_\varphi}\left\|S_{j,s,o}\right\|_2^2 \leq \sum_{o=1}^{d_\varphi} C_s C_q(\Omega)2^{-\alpha 2|j|}\|f\|_2^2\\ &\leq C_s C_q(\Omega)2^{-\alpha 2|j|}\|f\|_2^2;\end{aligned}$$

which implies that

$$\left\|E_j(f)\right\|_2 \leq C_s\sqrt{C_q(\Omega)}2^{-\alpha|j|}\|f\|_2. \tag{4.32}$$

By interpolation between (4.11) and (4.32), we have

$$\left\|E_j(f)\right\|_p \leq C_s\sqrt{C_q(\Omega)}2^{-\epsilon|j|/q'}\|f\|_p \tag{4.33}$$

for some $\epsilon > 0$ and for all $2 \leq p < \infty$, and $j \in \mathbb{Z}$ with constant C independent of Ω and j.

Now since

$$\mu_{\Omega,0,\phi}(f) \leq \sum_{j\in\mathbb{Z}} E_j(f)(x)$$

we have

$$\begin{aligned}\left\|\mu_{\Omega,0,\phi}(f)\right\|_p &\leq \sum_{j\in\mathbb{Z}}\left\|E_j(f)\right\|_p\\ &\leq \sum_{j\in\mathbb{Z}} C_s\sqrt{C_q(\Omega)}2^{-\epsilon|j|/q'}\|f\|_p \leq C_s\sqrt{C_q(\Omega)}\|f\|_p\sum_{j\in\mathbb{Z}}2^{-\epsilon|j|/q'}\\ &\leq C_s\sqrt{C_q(\Omega)}\|f\|_p\left\{\frac{2^{1/q'}}{2^{1/q'}-1}\right\}.\end{aligned}$$

This completes the proof.

5 L^p estimates of the general operator

In this section, we present the proof of Theorem 1.2.

Proof.(of Theorem 1.2). Let $\Omega \in L^q(\mathbb{S}^{n-1}), 1 < q$,with $\|\Omega\|_1 \leqslant 1$ and satisfying (1.1)-(1.5). Let $\mathcal{M}_{\Omega,P,\varphi}$ be given by (1.4). If $\deg(P) = 0$, then by Theorem 1.1, the result holds. Therefore, assume that $deg(P) = d \geq 1$. We shall argue by induction on d. Assume the result holds for all polynomials of degree less than or equal to d. Let $P(x) = \sum_{|\alpha|\leq d+1} a_\alpha x^\alpha$ be a polynomial of degree $d+1$. We may assume that P does not contain $|x|^{d+1}$as one of its terms. By dilation invariance, we may also assume that $\sum_{|\alpha|=d+1}|a_\alpha| = 1$. We choose a collection of C^∞ functions $\{w_k\}_{k\in\mathbb{Z}}$ defined on $(0,\infty)$ that satisfy the following:

$$supp(w_k) \subseteq [\frac{1}{a_{q,k+1}}, \frac{1}{a_{q,k-1}}], 0 \leq w_k \leq 1, \sum_{k\in\mathbb{Z}} w_k(u) = 1.$$

Notice that

$$\begin{aligned}
&\mathcal{M}_{\Omega,P,\varphi}(f)(x)\\
&= \sup_{\|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)}\leq 1}\left|\int_{\mathbb{R}^n} e^{iP(y)} f(x-D_{\varphi(\rho(y))}(y'))\frac{\Omega(y)h(\rho(y))dy}{\rho(y)^{\alpha-1}}\right|\\
&= \sup_{\|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)}\leq 1}\left|\int_0^\infty\int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega h(\rho)Jd\sigma d\rho\right|\\
&= \sup_{\|h\|_{L^2(\mathbb{R}_+,r^{-1}dr)}\leq 1}\left|\int_0^\infty h(\rho)\int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega Jd\sigma d\rho\right|
\end{aligned}$$

which is by duality implies that

$$\mathcal{M}_{\Omega,P,\varphi}(f)(x)=\left[\int_0^\infty\left|\int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega Jd\sigma\right|^2\frac{d\rho}{\rho}\right]^{1/2}.$$

Set

$$\eta_\infty(u)=\sum_{k=-\infty}^{0} w_k(u)\quad ,\eta_0(u)=\sum_{k=1}^{\infty} w_k(u).$$

Then

$$\begin{aligned}
\eta_\infty(u)+\eta_0(u) &= 1,\\
supp(\eta_0(u)) &\subseteq (0,1], supp(\eta_\infty(u))\subseteq[1,\infty).
\end{aligned}$$

Define the operators $\eta_{\Omega,p,0},\eta_{\Omega,p,\infty}$ as follows:

$$\eta_{\Omega,p,0}(f)(x)=\left[\int_0^\infty\left|\eta_0(\rho)\int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega(y')Jd\sigma\right|^2\frac{d\rho}{\rho}\right]^{1/2}$$

$$\eta_{\Omega,p,\infty}(f)(x)=\left[\int_0^\infty\left|\eta_\infty(\rho)\int_{\mathbb{S}^{n-1}} e^{iP(D_\rho.y')} f(x-D_{\varphi(\rho)}(y'))\Omega Jd\sigma\right|^2\frac{d\rho}{\rho}\right]^{1/2}.$$

By Minkowski's inequality, it follows that

$$\mathcal{M}_{\Omega,P,\varphi}(f)(x)\leq\eta_{\Omega,p,0}(f)+\eta_{\Omega,p,\infty}(f). \tag{5.1}$$

Now, we estimate $\left\|\eta_{\Omega,p,0}(f)\right\|_p$. Let $Q(x)=\sum_{|\alpha|\leq d} a_\alpha x^\alpha$. Assume that $deg(Q)=l$, where $0\leq l\leq d$. Define the operators $\eta^{(1)}_{\Omega,p,0}$ and $\eta^{(2)}_{\Omega,p,0}$ by

$$\eta^{(1)}_{\Omega,p,0}(f) = \left[\int_0^1 \left|\int_{\mathbb{S}^{n-1}} \left(e^{iP(D_\rho.y')} - e^{iQ(D_\rho.y')}\right) f(x - D_{\varphi(\rho)}(y'))\Omega J d\sigma\right|^2 \frac{d\rho}{\rho}\right]^{1/2} \tag{5.2}$$

$$\eta^{(2)}_{\Omega,p,0}(f) = \left[\int_0^1 \left|\int_{\mathbb{S}^{n-1}} e^{iQ(D_\rho.y')} f(x - D_{\varphi(\rho)}(y'))\Omega J d\sigma\right|^2 \frac{d\rho}{\rho}\right]^{1/2}. \tag{5.3}$$

By induction assumption, it follows that

$$\left\|\eta^{(2)}_{\Omega,p,0}(f)\right\|_p \leq C_p \left\{\frac{2^{1/q'}}{2^{1/q'} - 1}\right\} \log^{1/2}(e + \|\Omega\|_q) \|f\|_p \tag{5.4}$$

for all $1 < p < \infty$.

On the other hand, we notice that

$$\begin{aligned}
& \left|e^{iP(D_\rho.y')} - e^{iQ(D_\rho.y')}\right| \\
= & \left|e^{iQ(D_\rho.y')}(e^{iP(D_\rho.y') - iQ(D_\rho.y')} - 1)\right| \\
= & \left|e^{i\sum_{|\alpha|=d+1} a_\alpha (D_\rho.y')^\alpha} - 1\right| \leq \left|\sum_{|\alpha|=d+1} a_\alpha (D_\rho.y')^\alpha\right| \leq \rho^{2(d+1)}.
\end{aligned}$$

Thus,

$$\begin{aligned}
& \eta^{(1)}_{\Omega,p,0}(f) \\
= & \left[\int_0^1 \left|\int_{\mathbb{S}^{n-1}} \left(e^{iP(D_\rho.y')} - e^{iQ(D_\rho.y')}\right) f(x - D_{\varphi(\rho)}(y'))\Omega(y') J d\sigma\right|^2 \frac{d\rho}{\rho}\right]^{1/2} \\
\leq & \|\Omega\|_1^{\frac{1}{2}} \left[\int_0^1 \int_{\mathbb{S}^{n-1}} \left|e^{iP(D_\rho.y')} - e^{iQ(D_\rho.y')}\right|^2 \left|f(x - D_{\varphi(\rho)}(y'))\right|^2 |\Omega| |J|^2 d\sigma \frac{d\rho}{\rho}\right]^{1/2} \\
\leq & C\left[\int_0^1 \left(\int_{\mathbb{S}^{n-1}} \rho^{2d+1} \left|f(x - D_{\varphi(\rho)}(y'))\right|^2 \left|\Omega(y')\right| d\sigma(y')\right) d\rho\right]^{1/2} \\
\leq & C\left[\int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \int_0^1 \rho^{2d+1} \left|f(x - D_{\varphi(\rho)}(y'))\right|^2 d\rho d\sigma\right]^{1/2} \\
\leq & C\left[\int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \sum_{j=-\infty}^{-1} \int_{2^j}^{2^{j+1}} \rho^{2d+1} \left|f(x - D_{\varphi(\rho)}(y'))\right|^2 d\rho d\sigma\right]^{1/2} \\
\leq & C\left[\int_{\mathbb{S}^{n-1}} \left|\Omega(y')\right| \sum_{j=-\infty}^{-1} 2^{(2d+2)j} \int_{2^j}^{2^{j+1}} \left|f(x - D_{\varphi(\rho)}(y'))\right|^2 d\rho d\sigma\right]^{1/2}.
\end{aligned}$$

Thus,

$$\eta^{(1)}_{\Omega,p,0}(f) \le C\|\Omega\|_1^{\frac{1}{2}} (M_{y',\varphi}(|f|^2)^{1/2}) \tag{5.5}$$

where

$$M_{y',\varphi}(|f|)(x) = \sup_j \frac{1}{2^j} \int_{2^j}^{2^{j+1}} \left| f(x - D_{\varphi(\rho)}(y')) \right| d\rho.$$

By (5.5) and Proposition on page 477 in [15], we get

$$\left\| \eta^{(1)}_{\Omega,p,0}(f) \right\|_p \le C_p \|f\|_p \tag{5.6}$$

for all $1 < p < \infty$. Therefore by (5.4) and (5.6), we get

$$\left\| \eta_{\Omega,p,0}(f) \right\|_p \le C_p \log^{1/2}\left(e + \|\Omega\|_q\right) \{ \frac{2^{1/q'}}{2^{1/q'} - 1} \} \|f\|_p . \tag{5.7}$$

Finally, by Lemma 3.1, we have

$$\left\| \eta_{\Omega,p,\infty}(f) \right\|_p \le C_q \log^{1/2}(e + \|\Omega\|_q) \|f\|_p .$$

This complete the proof.

6 Proofs of results concerning $L(\log L)^{1/2}(\mathbb{S}^{n-1})$ and $B_q^{0,-\frac{1}{2}}(\mathbb{S}^{n-1})$

In this section, we present the proofs of Theorem 1.3 and Theorem 1.4.

Proof (of Theorem 1.3). Given $\Omega \in L(\log L)^{1/2}(\mathbb{S}^{n-1})$. Then $\Omega = \sum_{m=0}^{\infty} \Omega_m$ where that

$$\int_{\mathbb{S}^{n-1}} \Omega_m(y') J(y') d\sigma(y') = 0,$$

$$\Omega_0 \in L^2(\mathbb{S}^{n-1}),$$

$$\|\Omega_m\|_1 \le C \text{ and } \|\Omega_m\|_\infty \le 2^{4m} C \text{ for } m = 1, 2, ...,$$

$$\sum_{m=1}^{\infty} \sqrt{m} \|\Omega_m\|_1 \le \|\Omega\|_{L(Log)^{1/2}(\mathbb{S}^{n-1})} C.$$

Thus, we have the following

$$\mathcal{M}_{\Omega,P,\varphi}(f)(x) \le \mathcal{M}_{\Omega_0,P,\varphi}(f)(x) + \sum_{m=1}^{\infty} \|\Omega_m\|_1 \mathcal{M}_{\Omega_m,P,\varphi}(f)(x).$$

By the observation that

$$Log^{1/2}(e + \|\Omega_m\|_2) \le \sqrt{m}$$

and Theorem 1.1 and Theorem 1.2, we get

$$\begin{aligned}\left\|\mathcal{M}_{\Omega,P,\varphi}(f)\right\|_p &\leq \{1+\sum_{m=1}^{\infty}\sqrt{m}\|\Omega_m\|_1\}C_p\|f\|_p \\ &\leq C_p\|f\|_p\end{aligned}$$

for all $p \geq 2$. This completes the proof.

In order to present a proof of Theorem 1.4, we recall the definition of block spaces $B_q^{0,-\frac{1}{2}}\left(\mathbb{S}^{n-1}\right), q > 1$. A function Ω is in $B_q^{0,-\frac{1}{2}}\left(\mathbb{S}^{n-1}\right)$ if

$$\Omega = \sum_{\mu=1}^{\infty} c_\mu b_\mu,$$

where for each μ, c_μ *is a complex number and* b_μ is a function defined on $I = B(x_0', \theta_0) = \{x' \in \mathbb{S}^{n-1} : \left|x' - x_0'\right| < \theta_0\}$ and satisfies $\|b\|_{L^q} \leq |I|^{-\frac{1}{q'}}$ and

$$M_q\left(\left\{c_\mu\right\}\right) = \sum_{\mu=1}^{\infty}\left|c_\mu\right|\left(1+\phi\left(\left|I_\mu\right|\right)\right) < \infty,$$

where $\phi(t) \sim \log^{-\frac{1}{2}}\left(t^{-1}\right)$ as $t \to 0$. Here, $x_0' \in \mathbb{S}^{n-1}$ *and* $0 < \theta_0 \leq 2$. It should remarked here that block spaces are introduced by Jiang and Lu [12] in their study of singular integral operators.

It is known that

$$C^1(S^{n-1}) \subset L^q(\mathbb{S}^{n-1}) \subset B_q^{0,-\frac{1}{2}}\left(\mathbb{S}^{n-1}\right).$$

Proof (of Theorem 1.4). The proof follows by the decomposition of $B_q^{0,-\frac{1}{2}}(\mathbb{S}^{n-1}), q > 1$ and similar argument as in the proof of Theorem 1.3. We omit details.

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