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FIXED POINT THEOREMS FOR POSITIVE MAPS AND APPLICATIONS

ABDELHAMID BENMEZAI *
Faculty of Mathematics, USTHB Algeirs, Algeria

SALIMA MECHROUK†
Faculty of Sciences, University M'Hmed Bouguera
Boumerdes, Algeria

JOHNNY HENDERSON‡
Department of Mathematics, Baylor University
Waco, Texas 76798-7328, USA

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Abstract

We prove in this article new fixed point theorems for positive maps having approximative minorant and majorant at 0 and ∞ in specific classes of operators. Then, the new fixed point theorems are used to obtain existence results for positive solutions to boundary value problems involving a generalized $p(t)$-Laplacian operator.

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1 Introduction and abstract background

The problem of seeking positive solutions for boundary value problems (bvps for short) having positive nonlinearities, is usually converted to that of finding solutions in the cone of nonnegative functions C of some functional space X, to the fixed point equation, $u = Tu$ where $T : C \to C$ is completely continuous.

This formulation has motivated many works, where existence results of fixed points for operators leaving invariant a cone in a Banach space, have been proved; see [1], [2], [7], [8] and [12]. Krasnosel'skii's theorems of compression and expansion of a cone in a Banach

*E-mail address: aehbenmezai@gmail.com
†E-mail address: mechrouk@gmail.com
‡E-mail address: Johnny_Henderson@baylor.edu

space (see Theorems 4.10, 4.11, 4.12, 4.14 and 4.16 in [11] and Theorems 2.3.3 and 2.3.4 in [10]), are among the best known and most used in the literature.

Inspired by the works in [2], [5], [6], [7] and [8], we prove in this paper new fixed point theorems for maps leaving invariant a cone in a Banach space, and as in Krasnosel'kii's theorems, the main assumptions are on the behavior of the considered mapping at 0 and ∞. More precisely, we will assume that the mapping has an approximative minorant at 0 and an approximative majorant at ∞, or vice versa; existence of the fixed point is obtained under additional conditions: it is required that the positive spectrums of the approximative minorant and majorant are oppositely located with respect to 1.

We present at the end of this paper an application of the main results, to obtain existence results for at least one positive solution to bvps involving a ϕ-Laplacian or a $p(t)$-Laplacian operator.

We will use extensively in this work cones and the fixed point index theory, so let us recall some facts related to these two tools. Let X be a Banach space. A nonempty closed convex subset C of X is said to be a cone if $(tC) \subset C$ for all $t \geq 0$ and $C \cap (-C) = \{0_X\}$. It is well known that a cone C induces a partial order in the Banach space X. We write for all $x, y \in X$: $x \leq y$ if $y - x \in C$, $x \prec y$ if $y - x \in C$, $y \neq x$ and $x \nleq y$ if $y - x \notin C$. Notations $\geq$, $\succ$ and $\ngeq$ denote respectively the reverse situations.

Let C be a cone in X and let $N : X \to X$. The mapping N is said to be positive if $N(C) \subset C$. In this case, a nonnegative constant μ is said to be a positive eigenvalue of N if there exists $u \in C \setminus \{0_X\}$ such that $Nu = \mu u$.

Let $N : X \to X$ be a positive mapping. N is said to be

i) increasing if for all $u, v \in X$, $u \leq v$ implies $Nu \leq Nv$,

ii) lower bounded on C, if there exists a positive constant m such that for all $u \in C$, $\|Nu\| \geq m\|u\|$. For such an operator N, we denote $N_C^- = \inf\{\|Nu\| / \|u\|,\ u \in C\}$,

iii) upper bounded on C, if there exists a positive constant M such that for all $u \in K$, $\|Nu\| \leq M\|u\|$. For such an operator N, we denote $N_C^+ = \sup\{\|Nu\| / \|u\|,\ u \in C\}$,

Let $N_1, N_2 : X \to X$ be positive maps. We write $N_1 \leq N_2$ if for all $x \in C$, $N_1 x \leq N_2 x$.

A function $f : \Omega \subset X \to X$ is said to be bounded, if it maps bounded sets into bounded sets, and it is said to be completely continuous, if it is continuous and maps bounded sets into relatively compact sets.

At the end, let us recall some lemmas providing fixed point index computations. Let C be a cone in X. Let for $R > 0$, $C_R = C \cap B(0_X, R)$ where $B(0_X, R)$ is the open ball of radius R centred at 0_X, ∂C_R be its boundary and consider a compact mapping $f : \overline{C_R} \to C$.

Lemma 1.1. *If $fx \neq \lambda x$ for all $x \in \partial C_R$ and $\lambda \geq 1$, then $i(f, C_R, C) = 1$.*

Lemma 1.2. *If $fx \ngeq x$ for all $x \in \partial C_R$, then $i(f, C_R, C) = 1$.*

Lemma 1.3. *If $fx \nleq x$ for all $x \in \partial C_R$, then $i(f, C_R, C) = 0$.*

A detailed presentation of the fixed point index theory can be found in [10].

2 Main results

2.1 Preliminaries

In all this section E is a real Banach space, K, P are two cones in E with $P \subset K$ (it may happen that $K = P$). We set

$$N_K^P(E) = \{N : E \to E,\ N \text{ is continuous and } N(K) \subset P\}, \text{ and}$$
$$Q_K^P(E) = \left\{N \in N_K^P(E) : N \text{ is completely continuous}\right\}.$$

Now, for $N \in N_K^P(E)$ we define the subsets

$$\Lambda_P^N = \{\lambda \geq 0 : \text{ there exist } u \in P \setminus \{0_E\} \text{ such that } Nu \preceq \lambda u\},$$

$$\Theta_P^N = \{\theta \geq 0 : \text{ there exist } u \in P \setminus \{0_E\} \text{ such that } Nu \succeq \theta u\}.$$

Remark 2.1. Note that
i) $0 \in \Theta_P^N$ and if $\theta \in \Theta_P^N$, then $[0, \theta] \subset \Theta_P^N$.
ii) If $\lambda \in \Lambda_P^N$ then $[\lambda, +\infty) \subset \Lambda_P^N$.
iii) $\Lambda_P^N \subset \Lambda_K^N$ and $\Theta_P^N \subset \Theta_K^N$.
iv) If μ is positive eigenvalue of N, then $\mu \in \Theta_P^N \cap \Lambda_P^N$.
v) If $N^{-1}(0_E) \cap K = \{0_E\}$, then $\Lambda_P^N = \Lambda_K^N$ and $\Theta_P^N = \Theta_K^N$.

In all this paper, we set for $N \in N_K^P(E)$,

$$\theta_P^N = \sup \Theta_P^N$$

and when Λ_P^N is nonempty

$$\lambda_P^N = \inf \Lambda_P^N.$$

Lemma 2.2 ([8])**.** *Let $N \in Q_K^P(E)$ and assume that N is upper bounded on K. Then the subset Λ_P^N is nonempty.*

Lemma 2.3. *Let $N \in N_K^P(E)$ and assume that the cone P is solid. Then the subset Λ_P^N is nonempty.*

Proof. Let $u_0 \in int(P)$, then we have from Lemma 3.7 in [13] that there is $\alpha_0 > 0$ such that $\alpha_0 N u_0 \preceq u_0$, proving that $\alpha_0^{-1} \in \Lambda_P^N$. □

Lemma 2.4 ([8])**.** *Let $N \in N_K^P$ be upper bounded on K and assume that the cone K is normal with a constant n. Then $\theta_P^N < \infty$.*

Observe that if $N \in Q_K^P(E)$, then for all $R > 0$, the permanence property of the fixed point index implies that $i(N, K_R, K) = i(N, P_R, P)$.

Lemma 2.5. *Let $N \in Q_K^P(E)$ and let γ, R be positive real numbers. We have*
i) *$i(\gamma N, P_R, P) = 1$, if $\gamma \theta_P^N < 1$.*
ii) *$i(\gamma N, P_R, P) = 0$, if the subset Λ_P^N is nonempty and $\gamma \lambda_P^N > 1$.*

Proof. **i)** Suppose that $\gamma\theta_P^N < 1$ and let $u \in \partial P_R$ such that $\gamma Nu \geq u$. This implies that $1/\gamma \in \Theta_P^N$ and $1/\gamma \leq \theta_P^N$ which contradicts $\gamma\theta_P^N < 1$. Thus, the hypothesis of Lemma 1.2 holds and $i(\gamma N, P_R, P) = 1$.

ii) Suppose that the subset Λ_P^N is nonempty and $\gamma\lambda_P^N > 1$ and let $u \in \partial P_R$ such that $\gamma Nu \leq u$. This implies that $1/\gamma \in \Lambda_P^N$ and $1/\gamma \geq \inf \Lambda_p^N = \lambda_P^N$ which contradicts $\gamma\lambda_P^N > 1$. Thus, the hypothesis of Lemma 1.3 holds and $i(\gamma N, P_R, P) = 0$. This completes the proof. □

Lemma 2.6. *Let $N \in Q_K^P(E)$ and assume that the subset Λ_P^N is nonempty. Then we have $\lambda_P^N \leq \theta_P^N$.*

Proof. The case $\theta_P^N = +\infty$ is obvious and if $\theta_P^N = 0$, then $i(\gamma N, P_R, P) = 1$ for all $\gamma > 0$ proving that $\lambda_P^N = 0$.

Now, to the contrary suppose that $\lambda_P^N > \theta_P^N > 0$ and let $\gamma \in \left(1/\lambda_P^N, 1/\theta_P^N\right)$. We have from i) and ii) of Lemma 2.5, the contradiction

$$i(\gamma N, K_R, K) = \begin{cases} 1, \text{ since } \gamma\theta_P^N < 1, \\ 0, \text{ since } \gamma\lambda_P^N > 1. \end{cases}$$

This ends the proof. □

Corollary 2.7. *Let $N \in Q_K^P(E)$ and assume that the subset Λ_P^N is nonempty and $\lambda_P^N > 0$. Then N admits at least one positive eigenvalue.*

Proof. Let $\gamma_0 > 0$ be such that $\gamma_0 > 1/\lambda_P^N$ and let $R > 0$. We have from ii) of Lemma 2.5 that $i(\gamma N, K_R, K) = 0 \neq 1$ and we deduce from Lemma 1.1 that there is $\mu > 1$ and $u \in \partial P_R$ such that $\gamma_0 Nu = \mu u$. This proves that μ/γ_0 is a positive eigenvalue of N. □

Lemma 2.8. *Let $N \in N_K^P(E)$ be lower bounded on P and assume that the cone K is normal and the subset Λ_P^N is nonempty. Then we have $\theta_P^N \geq \lambda_P^N > 0$.*

Proof. Let $\lambda > 0$ and $u \in P \setminus \{0_E\}$ be such that $Nu \preceq \lambda u$ and let n_K be the constant of normality of the cone K. We have then

$$N_P^- \|u\| \leq \|Nu\| \leq \lambda n_K \|u\|$$

leading to $\lambda \geq N_P^-/n_K$ and $\lambda_P^N \geq N_P^-/n_K > 0$. □

Lemma 2.9. *Let $N \in N_K^P(E)$ be upper bounded on K and assume that the cone K is normal. Then we have $\theta_P^N < +\infty$.*

Proof. Let $\lambda > 0$ and $u \in P \setminus \{0_E\}$ be such that $Nu \succeq \lambda u$ and let n_K be the constant of normality of the cone K. We have then

$$\lambda \|u\| \leq n_K \|Nu\| \leq n_K N_K^+ \|u\|$$

leading to $\lambda \leq n_K N_K^+$ and $\theta_P^N \leq n_K N_K^+ < +\infty$. □

Proposition 2.10. *Let $N_1, N_2 \in N_K^P(E)$ and assume that $N_1 \leq N_2$. Then $\lambda_P^{N_1} \leq \lambda_P^{N_2}$ and $\theta_P^{N_1} \leq \theta_P^{N_2}$.*

Proof. Indeed, we have

$$\Theta_P^{N_1} \subset \Theta_P^{N_2} \text{ and } \Lambda_P^{N_2} \subset \Lambda_P^{N_1}$$

leading to

$$\lambda_P^{N_1} \le \lambda_P^{N_2} \text{ and } \theta_P^{N_1} \le \theta_P^{N_2}.$$

□

2.2 Fixed point theorems for positive maps

Throughout this subsection, we let $T : K \to K$ be a completely continuous mapping and we want to obtain properties for the existence of fixed points of T.

Theorem 2.11. *Assume that the cone K is normal and there exists operators $N_1, N_2 \in Q_K^K(E)$, $\tilde{N}_1 \in Q_K^{P_1}(E)$, $\tilde{N}_2 \in Q_K^{P_2}(E)$, where P_1, P_2 are two cones contained in K, $\alpha > 0$ and three functions $G_1, G_2; G_3 : K \to K$ such that $\theta_{P_1}^{\tilde{N}_1} < 1 < \lambda_{P_2}^{\tilde{N}_2}$, N_1, N_2 are uniformly continuous on $\bar{B}(0_E, 2)$, $\tilde{N}_1, \tilde{N}_2$ are increasing and for all $u \in K$,*

$$\begin{aligned} &Tu \le N_1(u + G_1(u)),\\ &N_2(u - G_2(u)) \le T(u) \le \alpha N_2(u + G_3(u)). \end{aligned} \tag{2.1}$$

If either

$$\begin{cases} G_1(u) = \circ(\|u\|) \text{ near } 0 \text{ and } G_i(u) = \circ(\|u\|) \text{ near } \infty \text{ for } i = 2,3,\\ \lim_{t\to 0} \dfrac{N_1(tu)}{t} = \tilde{N}_1(u) \text{ and } \lim_{t\to+\infty} \dfrac{N_2(tu)}{t} = \tilde{N}_2(u) \text{ uniformly in } \partial B(0_E,1), \end{cases} \tag{2.2}$$

or

$$\begin{cases} G_1(u) = \circ(\|u\|) \text{ near } \infty \text{ and } G_i(u) = \circ(\|u\|) \text{ near } 0 \text{ for } i = 2,3,\\ \lim_{t\to+\infty} \dfrac{N_1(tu)}{t} = \tilde{N}_1(u) \text{ and } \lim_{t\to 0} \dfrac{N_2(tu)}{t} = \tilde{N}_2(u) \text{ uniformly in } \partial B(0_E,1), \end{cases} \tag{2.3}$$

then T admits at least one fixed point.

Proof. We present the proof in the case where (2.2) holds, the other case is checked similarly. We have to prove the existence of $0 < r < R$ such that

$$i(T, K_r, K) = 1 \text{ and } i(T, K_R, K) = 0,$$

in such a case, the additivity and the solution properties of fixed point index imply that

$$i(T, K_R \setminus \bar{K}_r, K) = i(T, K_R, K) - i(T, K_r, K) = -1,$$

and T admits a positive fixed point u, with $r < \|u\| < R$.

Consider the function $H_1 : [0,1] \times K \to K$ defined by $H_1(t,u) = tTu + (1-t)N_2(u)$ and let us prove existence of $R > 0$ large enough such that for all $t \in [0,1]$ equation $H_1(t,u) = u$ has no solution in ∂K_R. To the contrary, suppose that for all integers $n \ge 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that

$$u_n = t_n T(u_n) + (1 - t_n) N_2(u_n).$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n = t_n \frac{T(u_n)}{\|u_n\|} + (1-t_n)\frac{N_2(u_n)}{\|u_n\|}$$

and

$$\tilde{N}_2(v_n) = \tilde{N}_2\left(t_n \frac{T(u_n)}{\|u_n\|} + (1-t_n)\frac{N_2(u_n)}{\|u_n\|}\right). \tag{2.4}$$

Because $\tilde{N}_2$ is completely continuous, N_2 is uniformly continuous on $\bar{B}(0_E, 2)$ and by the homogeneity condition imposed on N_2 in Hypothesis (2.2), there is a subsequence of integers (n_l), $\hat{t} \in [0,1]$ and $w \in P_2$ such that $\lim t_{n_l} = \hat{t}$, $w = \lim \tilde{N}_2(v_{n_l}) = \lim N_2(u_{n_l}) / \|u_{n_l}\| = \lim N_2(u_{n_l} - G_2(u_{n_l})) / \|u_{n_l}\| = \lim N_2(u_{n_l} + G_3(u_{n_l})) / \|u_{n_l}\| = w$. Note that $w > 0_E$. Indeed, if $\lim N_2(u_{n_l}) / \|u_{n_l}\| = 0_E$, then we have from

$$\frac{T(u_{n_l})}{\|u_{n_l}\|} \leq \alpha \frac{N_2(u_{n_l} + G_3(u_{n_l}))}{\|u_{n_l}\|}$$

and the normality of the cone K that $\lim T(u_{n_l}) / \|u_{n_l}\| = 0_E$, leading to

$$\lim v_{n_l} = \lim\left(t_n \frac{T(u_{n_l})}{\|u_{n_l}\|} + (1-t_{n_l})\frac{N_2(u_{n_l})}{\|u_{n_l}\|}\right) = 0_E,$$

contradicting $\|v_{n_l}\| = 1$.

At this stage, letting $l \to \infty$ in

$$\begin{aligned}
\tilde{N}_2(v_{n_l}) &= \tilde{N}_2\left(t_{n_l} \frac{T(u_{n_l})}{\|u_{n_l}\|} + (1-t_{n_l})\frac{N_2(u_{n_l})}{\|u_{n_l}\|}\right) \\
&\geq \tilde{N}_2\left(t_{n_l} \frac{N_2(u_{n_l} - G_2(u_{n_l}))}{\|u_{n_l}\|} + (1-t_{n_l})\frac{N_2(u_{n_l})}{\|u_{n_l}\|}\right)
\end{aligned}$$

we obtain $w \geq \tilde{N}_2(w)$ and $1 \in \Lambda_{P_2}^{\tilde{N}_2}$ contradicting $\lambda_{P_2}^{\tilde{N}_2} > 1$.Thus, our claim is proved, and for such a real number $R > 0$, we deduce from the homotopy property of the fixed point index and Lemma 1.3 that

$$i(T, K_R, P) = i(H_1(1,\cdot), K_R, K) = i(H_1(0,\cdot), K_R, K) = i(N_2, K_R, K) = 0.$$

In similar way, consider the function $H_2 : [0,1] \times K \to K$ defined by $H_2(t,u) = tTu + (1-t)N_1 u$ and let us prove existence of $r > 0$ small enough such that for all $t \in [0,1]$, equation $H_2(t,u) = u$ has no solution in ∂K_r. To the contrary, suppose that for all integers $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial K_{1/n}$ such that

$$u_n = t_n T u_n + (1-t_n) N_1(u_n).$$

Note that $v_n = u_n / \|u_n\| \in \partial K_1$ and satisfies

$$v_n = t_n \frac{T u_n}{\|u_n\|} + (1-t_n) \frac{N_1(u_n)}{\|u_n\|}.$$

Then

$$\tilde{N}_1(v_n) = \tilde{N}_1 \left(t_n \frac{T u_n}{\|u_n\|} + (1-t_n) \frac{N_1(u_n)}{\|u_n\|} \right).$$

Because $\tilde{N}_1$ is completely continuous, N_1 is uniformly continuous on $\bar{B}(0_E, 2)$ and the homogeneity condition imposed on N_1 in Hypothesis (2.2), there is a subsequence of integers (n_l), $\hat{t} \in [0,1]$ and $\omega \in P_1$ such that $\lim t_{n_l} = \hat{t}$, $\lim \tilde{N}_1(v_{n_l}) = \lim N_1(u_{n_l}) / \left\| u_{n_l} \right\| = \lim N_2 \left(u_{n_l} + G_1 \left(u_{n_l} \right) \right) / \left\| u_{n_l} \right\| = \omega$.

Note that $\omega \succ 0_E$. Indeed, if $\lim N_1(v_{n_l}) = 0_E$, then we have from

$$\frac{T\left(u_{n_l}\right)}{\left\|u_{n_l}\right\|} \preceq \frac{N_1\left(u_{n_l} + G_1\left(u_{n_l}\right)\right)}{\left\|u_{n_l}\right\|}$$

and the normality of the cone K that $\lim T\left(u_{n_l}\right) / \left\|u_{n_l}\right\| = 0_E$, leading to

$$\lim v_{n_l} = \lim \left(t_n \frac{T\left(u_{n_l}\right)}{\left\|u_{n_l}\right\|} + (1-t_{n_l}) \frac{N_2(u_{n_l})}{\left\|u_{n_l}\right\|} \right) = 0_E,$$

contradicting $\left\| v_{n_l} \right\| = 1$.

At this stage, letting $l \to \infty$ in

$$\begin{aligned} \tilde{N}_1(v_{n_l}) &= \tilde{N}_1 \left(t_{n_l} \frac{T\left(u_{n_l}\right)}{\left\|u_{n_l}\right\|} + (1-t_{n_l}) \frac{N_1(u_{n_l})}{\left\|u_{n_l}\right\|} \right) \\ &\preceq \tilde{N}_1 \left(t_{n_l} \frac{N_1\left(u_{n_l} + G_1\left(u_{n_l}\right)\right)}{\left\|u_{n_l}\right\|} + (1-t_{n_l}) \frac{N_1(u_{n_l})}{\left\|u_{n_l}\right\|} \right), \end{aligned}$$

we obtain $\omega \preceq \tilde{N}_1(\omega)$ and $1 \in \Theta_{P_1}^{\tilde{N}_1}$, contradicting $\theta_{P_1}^{\tilde{N}_1} < 1$. Thus, our claim is proved and for such a real number $r > 0$, we deduce from the homotopy property of the fixed point index and Lemma 1.2 that

$$i(T, K_r, K) = i(H_2(1, \cdot), K_r, K) = i(H_2(0, \cdot), K_r, K) = i(N_1, K_r, K) = 1.$$

This completes the proof. □

Theorem 2.12. *Assume that the cone K is normal and there exist operators $N_1, N_2 \in Q_K^K(E)$, $\tilde{N}_1 \in Q_K^{P_1}(E)$, $\tilde{N}_2 \in Q_K^{P_2}(E)$, where P_1, P_2 are two cones contained in K and functions $G_1, G_2 : K \to K$ such that N_1, N_2 are uniformely continuous on $\bar{B}(0_E, 2)$, $\tilde{N}_2$ is lower bounded, $\theta_P^{\tilde{N}_1} < 1 < \lambda_P^{\tilde{N}_2}$, $\tilde{N}_1, \tilde{N}_2$ are increasing and for all $u \in K$,*

$$N_2\left(u - G_2(u)\right) \preceq T(u) \preceq N_1\left(u + G_1(u)\right).$$

If either

$$\begin{cases} G_1(u) = \circ(\|u\|) \text{ near } 0 \text{ and } G_i(u) = \circ(\|u\|) \text{ near } \infty \text{ for } i = 2,3, \\ \lim_{t\to 0} \dfrac{N_1(tu)}{t} = \tilde{N}_1(u) \text{ and } \lim_{t\to+\infty} \dfrac{N_2(tu)}{t} = \tilde{N}_2(u) \text{ uniformly in } \partial B(0_E,1) \end{cases} \tag{2.5}$$

or

$$\begin{cases} G_1(u) = \circ(\|u\|) \text{ near } \infty \text{ and } G_i(u) = \circ(\|u\|) \text{ near } 0 \text{ for } i = 2,3, \\ \lim_{t\to+\infty} \dfrac{N_1(tu)}{t} = \tilde{N}_1(u) \text{ and } \lim_{t\to 0} \dfrac{N_2(tu)}{t} = \tilde{N}_2(u) \text{ uniformly in } \partial B(0_E,1), \end{cases}$$

then T admits at least one fixed point.

Proof. We present the proof in the case where (2.5) holds. The other case is checked similarly. As in the proof of Theorem 2.11, we have to prove existence of $0 < r < R$ such that

$$i(T,K_r,K) = 1 \text{ and } i(T,K_R,K) = 0.$$

Consider the function $H_3 : [0,1]\times K \to K$ defined by $H_3(t,u) = (1-t)Tu + tN_2(u)$ and let us prove existence of $R > 0$ large enough such that for all $t \in [0,1]$, equation $H_3(t,u) = u$ has no solution in ∂K_R. To the contrary, suppose that for all integers $n \geq 1$, there exist $t_n \in [0,1]$ and $u_n \in \partial K_n$ such that

$$u_n = H_3(t_n,u_n) = (1-t_n)Tu_n + t_nN_2(u_n).$$

Note that $v_n = u_n/\|u_n\| \in \partial K_1$ and satisfies

$$v_n = (1-t_n)\frac{T(u_n)}{\|u_n\|} + t_n\frac{N_2(u_n)}{\|u_n\|}$$

and

$$\tilde{N}_2(v_n) = \tilde{N}_2\left((1-t_n)\frac{T(u_n)}{\|u_n\|} + t_n\frac{N_2(u_n)}{\|u_n\|}\right). \tag{2.6}$$

Because $\tilde{N}_2$ is completely continuous and by the homogeneity condition imposed on N_2 in Hypothesis (2.5), there is a subsequence of integers (n_l), $\hat{t} \in [0,1]$ and $w \in P_2$ such that $\lim t_{n_l} = \hat{t}$, $\lim N_2(u_{n_l})/\left\|u_{n_l}\right\| = \lim N_2\left(u_{n_l} - G_2\left(u_{n_l}\right)\right)/\left\|u_{n_l}\right\| = w$.

Note that the lower boundeness of $\tilde{N}_2$ on the cone P, leads to $\|w\| \geq \tilde{N}^-_{2,P} > 0$. Thus, letting $l \to \infty$ in

$$\begin{aligned} \tilde{N}_2(v_{n_l}) &= \tilde{N}_2\left((1-t_{n_l})\frac{T(u_{n_l})}{\left\|u_{n_l}\right\|} + t_{n_l}\frac{N_2(u_{n_l})}{\left\|u_{n_l}\right\|}\right) \\ &\geq \tilde{N}_2\left((1-t_{n_l})\frac{N_2\left(u_{n_l} - G_2\left(u_{n_l}\right)\right)}{\left\|u_{n_l}\right\|} + t_{n_l}\frac{N_2(u_{n_l})}{\left\|u_{n_l}\right\|}\right), \end{aligned}$$

we get $w \geq \tilde{N}_2(w)$ and $1 \in \Lambda_{P_2}^{\tilde{N}_2}$ contradicting $\lambda_{P_2}^{\tilde{N}_2} > 1$.Thus, our claim is proved and for such a real number $R > 0$, we deduce from homotopy property of the fixed point index and Lemma 1.3 that

$$i(T,K_R,K) = i(H_3(1,\cdot),K_R,K) = i(H_3(0,\cdot),K_R,K) = i(N_2,K_R,K) = 0.$$

Arguing as in the proof of Theorem 2.11, we prove existence of $r > 0$ small enough such that $i(T, K_r, K) = 1$. This completes the proof. □

Remark 2.13. Theorem 2.12 holds true if we replace the condition K normal by the condition that N_1 is lower bounded on P_1.

3 Application to bvps

We are concerned in this section with existence of positive solutions to the bvp

$$\begin{cases} -(a(t)\phi(t,u'(t)))' = f(t,u(t)), \text{ a.e. } t \in (0,1) \\ u'(0) = u(1) = 0 \end{cases} \tag{3.1}$$

where $a, b : [0,1] \to [0,+\infty)$ are measurable functions such that $a(t) > 0$ a.e. $t \in [0,1]$, $meas\{b(t) > 0\} > 0$, $\phi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is continuous, $\phi(t,\cdot) : \mathbb{R} \to \mathbb{R}$ is an odd increasing homeomorphism and $f \in C([0,1] \times [0,+\infty), [0,+\infty))$.

In all of this section, $\psi : [0,1] \times \mathbb{R} \to \mathbb{R}$ is the continuous function such that $\psi(t,\cdot)$ is the inverse function of $\phi(t,\cdot)$.

Throughout we assume that

$$\begin{gathered} \exists \alpha, \beta \in (0,+\infty) \text{ such that for all } t, x \geq 0 \text{ and } s \in (0,1) \\ s^{\beta}\phi(t,x) \leq \phi(t,sx) \leq s^{\alpha}\phi(t,x), \end{gathered} \tag{3.2}$$

leading to

$$s^{\frac{1}{\alpha}}\psi(t,x) \leq \psi(t,sx) \leq s^{\frac{1}{\beta}}\psi(t,x) \text{ for all } t, x \geq 0 \text{ and } s \in (0,1). \tag{3.3}$$

Let $\phi^+, \phi^-, \psi^+, \psi^-$ be the odd functions defined on $[0,+\infty)$ by

$$\phi^+(x) = \begin{cases} x^{\alpha} \text{ if } x \leq 1 \\ x^{\beta} \text{ if } x \geq 1 \end{cases} \quad \phi^-(x) = \begin{cases} x^{\beta} \text{ if } x \leq 1 \\ x^{\alpha} \text{ if } x \geq 1 \end{cases}$$

$$\psi^+(x) = \begin{cases} x^{\frac{1}{\beta}} \text{ if } x \leq 1 \\ x^{\frac{1}{\alpha}} \text{ if } x \geq 1 \end{cases} \quad \psi^-(x) = \begin{cases} x^{\frac{1}{\alpha}} \text{ if } x \leq 1 \\ x^{\frac{1}{\beta}} \text{ if } x \geq 1 \end{cases}$$

and note that ψ^-, ψ^+ are, respectively, the inverse functions of ϕ^+, ϕ^-.

Typical examples of a functions ϕ satisfying (3.2) are: $\phi(t,u) = \sum_{i=1}^{i=k} \zeta_i |u|^{p_i(t)-1} u$, where for $i = 1, ...k$, $\zeta_i > 0$ and $p_i \in C([0,1], (0,+\infty))$ and $\phi(t,u) = \sum_{i=1}^{i=k} \omega_i(t) |u|^{q_i-1} u$, where for $i = 1, ...k$, $q_i > 0$ and $\omega_i \in C([0,1], (0,+\infty))$. This shows that the differential operator considered in this section is more general that in [3], [4] and [9].

The main results of this section will be obtained under the following conditions on a and b.

$$\psi^+\left(\frac{1}{a}\right) \in L^1_{loc}(0,1],\ b \in L^1_{loc}[0,1) \text{ and } \Gamma = \sup_{t\in(0,1)} \left(\frac{1}{a(t)} \int_0^t b(s)ds\right) dt < \infty. \tag{3.4}$$

$$\lim_{t\to 0}\frac{1}{a(t)}\int_0^t b(s)ds=0. \tag{3.5}$$

Also, in all of this section, E is the Banach space of all continuous functions defined on $[0,1]$ equipped with its sup-norm denoted $\|\cdot\|$, and K is the normal cone of nonnegative functions in E.

A typical example of weights a,b satisfying (3.4) and (3.5) is

$$a(t)=\frac{t^m}{(1-t)^n}\ b(t)=\frac{(1-t)^n}{t^m},\ m,n\in\mathbb{N}.$$

Because of Hypothesis (3.4), the operator $N:E\to E$, defined for $h\in E$ and $t\in[0,1]$, by $Nh(t)=\int_t^1\psi\left(s,\frac{1}{a(s)}\int_0^s b(\tau)h(\tau)d\tau\right)ds$ is well defined and we have that $N(K)\subset K$, and N is a completely continuous operator.

Let $F:K\to K$ be the mapping defined, for $u\in K$, by $Fu(t)=f(t,u(t))$ and note that F is continuous and bounded (maps bounded sets into bounded sets). Set $T=NF$ and observe that $T(K)\subset K$, T is a completely continuous operator and because of Hypothesis (3.5), u is a positive solution to (3.1) if and only if u is a fixed point of the operator T.

Now, let $\lambda>0$ and $N_\lambda^-,\ N_\lambda^+,N_\alpha,N_\beta:E\to E$ be the operators defined by

$$N_\lambda^- u(t)=\int_t^1\psi^-\left(\int_0^s\frac{b(\tau)\phi^+(\lambda u(\tau))}{a(s)}d\tau\right)ds,\ N_\lambda^+ u(t)=\int_t^1\psi^+\left(\int_0^s\frac{b(\tau)\phi^-(\lambda u(\tau))}{a(s)}d\tau\right)ds,$$
$$N_\alpha u(t)=\int_t^1\varphi_{1/\alpha}\left(\int_0^s\frac{b(\tau)\varphi_\alpha(u(\tau))}{a(s)}d\tau\right)ds \text{ and } N_\beta u(t)=\int_t^1\varphi_{1/\beta}\left(\int_0^s\frac{b(\tau)\varphi_\beta(u(\tau))}{a(s)}d\tau\right)ds,$$

where for $\theta>0$, $\varphi_\theta(x)=|x|^{\theta-1}x$.

It is easy to prove the following lemma.

Lemma 3.1. *Assume that Hypotheses (3.2), (3.4) hold. Then*

i) *for $N=N_\lambda^-,\ N_\lambda^+,N_\alpha,N_\beta$, N is completely continuous, $N(K)\subset K$ and N is upper bounded on K,*

ii) *If $\bar\rho=\int_0^1\psi^+(1/a(s))ds<\infty$, then $N_\alpha(K)\subset P_\alpha$, $N_\beta(K)\subset P_\beta$ and N_α,N_β are respectively lower bounded on P_α and P_β where for $\nu=\alpha,\beta$, P_ν is the cone defined by*

$$P_\nu=\{u\in K,\ u(t)\geq\rho_\nu(t)\|u\|\ \textit{for all}\ t\in[0,1]\}$$

with

$$\rho_\nu(t)=\frac{1}{\bar\rho_\nu}\int_t^1\varphi_{\frac{1}{\nu}}\left(\frac{1}{a(s)}\right)ds \textit{ and } \bar\rho_\nu=\int_0^1\varphi_{\frac{1}{\nu}}\left(\frac{1}{a(s)}\right)ds.$$

Lemma 3.2. *Assume that Hypotheses (3.2), (3.4) hold. Then*

$$\lim_{\gamma\to 0}\frac{N_\lambda^+(\gamma u)}{\gamma}=\lambda N_\beta(u),\quad \lim_{\gamma\to+\infty}\frac{N_\lambda^+(\gamma u)}{\gamma}=\lambda N_\alpha(u),$$
$$\lim_{\gamma\to 0}\frac{N_\lambda^-(\gamma u)}{\gamma}=\lambda N_\alpha(u),\quad \lim_{\gamma\to+\infty}\frac{N_\lambda^-(\gamma u)}{\gamma}=\lambda N_\beta(u),$$

uniformly in $\partial B(0_E,1)$.

Proof. It easy to see that

$$\lim_{\delta\to 0}\frac{\phi^{+}(\delta u)}{\delta^{\alpha}}=\varphi_{\alpha}(u),\quad \lim_{\delta\to +\infty}\frac{\phi^{+}(\delta u)}{\delta^{\beta}}=\varphi_{\beta}(u),$$
$$\lim_{\delta\to 0}\frac{\phi^{-}(\delta u)}{\delta^{\beta}}=\varphi_{\beta}(u),\quad \lim_{\delta\to +\infty}\frac{\phi^{-}(\delta u)}{\delta^{\alpha}}=\varphi_{\alpha}(u),$$

uniformly for u in bounded intervals, and then

$$\lim_{\delta\to 0}\frac{\psi^{-}(\delta u)}{\delta^{\frac{1}{\alpha}}}=\varphi_{\frac{1}{\alpha}}(u),\quad \lim_{\delta\to +\infty}\frac{\psi^{-}(\delta u)}{\delta^{\frac{1}{\beta}}}=\varphi_{\frac{1}{\beta}}(u),$$
$$\lim_{\delta\to 0}\frac{\psi^{+}(\delta u)}{\delta^{\frac{1}{\beta}}}=\varphi_{\frac{1}{\beta}}(u),\quad \lim_{\delta\to 0}\frac{\psi^{+}(\delta u)}{\delta^{\frac{1}{\alpha}}}=\varphi_{\frac{1}{\alpha}}(u),$$

uniformly for u in bounded intervals.

Now, let $\varepsilon>0$ be small enough, that there exists $\delta_0>0$ such that, for $\delta\le\delta_0$,

$$\delta^{\beta}\left(\varphi_{\beta}(u)-\varepsilon\right)\le\phi^{-}(\delta u)\le\delta^{\beta}\left(\varphi_{\beta}(u)+\varepsilon\right)\text{ for }u\in[-1,1]$$

and

$$\delta^{\frac{1}{\beta}}\left(\varphi_{\frac{1}{\beta}}(u)-\varepsilon\right)\le\psi^{+}(\delta u)\le\delta^{\frac{1}{\beta}}\left(\varphi_{\frac{1}{\beta}}(u)+\varepsilon\right)\text{ for }u\in[-\Gamma(\lambda+1)/\lambda,\Gamma(\lambda+1)/\lambda].$$

These inequalities lead to

$$\begin{aligned}\frac{N_{\lambda}^{+}(\delta u)(t)}{\delta}&\le\int_t^1\varphi_{\frac{1}{\beta}}\left(\frac{1}{a(s)}\int_0^s b(\tau)\left(\varphi_{\beta}(\lambda u(\tau))+\varepsilon\right)d\tau\right)ds+\varepsilon\\&\le\lambda\int_t^1\varphi_{\frac{1}{\beta}}\left(\frac{1}{a(s)}\int_0^s b(\tau)\left(\varphi_{\beta}(u(\tau))+(\varepsilon/\lambda)\right)d\tau\right)ds+\varepsilon\end{aligned}\tag{3.6}$$

and

$$\begin{aligned}\frac{N_{\lambda}^{+}(\delta u)(t)}{\delta}&\ge\int_t^1\varphi_{\frac{1}{\beta}}\left(\frac{1}{a(s)}\int_0^s b(\tau)\left(\varphi_{\beta}(\lambda u(\tau))-\varepsilon\right)d\tau\right)ds-\varepsilon\\&\ge\lambda\int_t^1\varphi_{\frac{1}{\beta}}\left(\frac{1}{a(s)}\int_0^s b(\tau)\left(\varphi_{\beta}(u(\tau))-(\varepsilon/\lambda)\right)d\tau\right)ds-\varepsilon\end{aligned}\tag{3.7}$$

for all $u\in\partial B(0_E,1)$.

Finally, we have from (3.6) and (3.7)

$$-\varepsilon-\lambda c(\beta,\varepsilon/\lambda)\le\frac{N_{\lambda}^{+}(\delta u)(t)}{\delta}-\lambda N_{\beta}u(t)\le\varepsilon+\lambda c(\beta,\varepsilon/\lambda)$$

where

$$C(\epsilon,\beta)=\max_{x\in[-1,1]}\left(\varphi_{\frac{1}{\beta}}(x+\epsilon)-\varphi_{\frac{1}{\beta}}(x)\right)=\begin{cases}\epsilon^{\frac{1}{\beta}}\text{ if }\beta\ge 1\\ \left(\frac{\epsilon}{\beta}\right)(1+\epsilon)^{\frac{1}{\beta}-1}\text{ if }\beta\le 1.\end{cases}$$

Thus, we have proved that $\lim_{\delta\to 0}N_{\lambda}^{+}(\delta u)/\delta=\lambda N_{\beta}u$ uniformly for $u\in\partial B(0_E,1)$. The other limits are checked similarly. □

Proposition 3.3. *Assume that Hypotheses (3.2) and (3.4) hold. Then for all $p>0$ the operator N_p has a unique positive eigenvalue $\mu(p)$.*

Proof. Let (ξ_n) and (η_n) be two sequences of real numbers such that $0 < \xi_n < \eta_n < 1$, (ξ_n) is decreasing and (η_n) is increasing. Let

$$a_n(t) = \begin{cases} a(t) \text{ if } t \in (\xi_n, 1) \\ \sup(a(t), a(\xi_n)) \text{ if } t \in (0, \xi_n) \end{cases} \qquad b_n(t) = \begin{cases} b(t) \text{ if } t \in (0, \eta_n) \\ 0 \text{ if } t \in (\eta_n, 1) \end{cases}$$

and note that $1/a_n, b_n \in L^1[0,1]$. Let us define for all $n \in \mathbb{N}$, $N_{p,n} : E \to E$ by $N_{p,n}u(t) = \int_t^1 \varphi_{1/p}\left(\frac{1}{a_n(s)} \int_0^s b_n(\tau)\varphi_p(u(\tau))\,d\tau\right) ds$. Clearly $N_{p,n}$ is completely continuous and $N_{p,n}(K) \subset K$. By considering the restriction of $N_{p,n}$ to the space $E_\#$, we have that $N_{p,n}(K_\# \setminus \{0_E\}) \subset O$ where

$$E_\# = \left\{u \in C^1[0,1] : u'(0) = u(1) = 0\right\}$$
$$O = \{u \in E_\# : u(t) > 0 \text{ for all } t \in [0,1) \text{ and } u'(1) < 0\}.$$

Thus, similar arguments to that used in the proof of Theorem 4.10 in [8] lead to N_θ has a unique positive eingenvalue $\mu_n(p)$ with $\mu_n(p) = \theta_K^{N_{p,n}} = \lambda_K^{N_{p,n}}$.

Since $(\mu_n(p))$ is nondecreaing and bounded by Γ, it converges to some $\mu(p)$. Also, we have that

$$\lim\left\|N_p - N_{p,n}\right\| = \lim \sup_{u \in \bar{B}(0_E,1)} \left\|N_p(u) - N_{p,n}(u)\right\| = 0$$

and

$$i\left(\gamma N_p, K_1, K\right) = \lim i\left(\gamma N_{p,n}, K_1, K\right) = 1 \text{ for all } \gamma < \mu(p) \text{ and}$$
$$i\left(\gamma N_p, K_1, K\right) = \lim i\left(\gamma N_{p,n}, K_1, K\right) = 0 \text{ for all } \gamma > \mu(p).$$

This shows that $\mu(p)$ is the unique positive eigenvalue of N_p. This ends the proof. □

Set for $\nu = 0, \infty$

$$f_\nu = \liminf_{u \to \nu}\left(\min_{t \in [0,1]} \frac{\psi^-(f(t,u))}{u}\right) \quad f^\nu = \limsup_{u \to \nu}\left(\max_{t \in [0,1]} \frac{\psi^+(f(t,u))}{u}\right)$$

$$f^{\nu,+} = \limsup_{u \to \nu}\left(\max_{t \in [0,1]} \frac{\psi^+(f(t,u))}{u}\right)$$

and let $\mu(p)$ for $p = \alpha, \beta$, be the unique positive eigenvalue of N_p given by Proposition 3.3.

Theorem 3.4. *Assume that Hypotheses (3.2), (3.4) and (3.5) hold and $\psi^+(1/a) \in L^1[0,1]$. If either*

$$f^0 \mu(\beta) < 1 < \mu(\beta) f_\infty \tag{3.8}$$

or

$$f^\infty \mu(\alpha) < 1 < \mu(\alpha) f_0, \tag{3.9}$$

then bvp (3.1) admits a positive solution.

Proof. We present the proof in the case where (3.8) holds (the other case is checked similarly). Let $\epsilon > 0$ be such that $\left(f^0 + \epsilon\right)\mu(\beta) < 1 < (f_\infty - \epsilon)\mu(\beta)$. We have then

$$N_2(u - G_2 u) \le Tu \le N_1(u + G_1 u)$$

where

$$\begin{array}{ll} N_1u = N^{+}_{(f^0+\epsilon)}(u) & G_1(u) = \max\left\{f(t,u(t) - \left(f^0+\epsilon\right)u(t), 0\right\} / \left(f^0+\epsilon\right) \\ N_2u = N^{-}_{(f_\infty-\epsilon)}(u) & G_2(u) = \dfrac{c}{(f_\infty-\epsilon)} \end{array}$$

and c is a positive constant large enough.

Furthermore, Lemma 3.2 guarantees that

$$\begin{array}{l} \lim_{\gamma\to+\infty} \dfrac{N_2(\gamma u)}{\gamma} = (f_\infty-\epsilon) N_\beta(u) = \tilde{N}_2u \text{ and} \\ \lim_{\gamma\to 0} \dfrac{N_1(\gamma u)}{\gamma} = \left(f^0+\epsilon\right) N_\alpha(u) = \tilde{N}_1u \text{ uniformly in } \partial B(0_E,1). \end{array}$$

Since Lemma 3.1 guarantees that $\tilde{N}_1, \tilde{N}_2$ are, respectively, lower bounded on P_α and P_β, and upper bounded on K and we have

$$\theta^{\tilde{N}_1}_{P_\alpha} = \left(f^0+\epsilon\right)\mu(\beta) < 1 < (f_\infty-\epsilon)\mu(\beta) = \lambda^{\tilde{N}_2}_{P_\beta}.$$

Therefore, existence of a positive solution to bvp (3.1) follows from Theorem 2.12. □

Theorem 3.5. *Assume that Hypotheses (3.2), (3.4) and (3.5) hold and either*

$$f^0\mu(\beta) < 1 < \mu(\beta) f_\infty,\ f^{\infty,+} < \infty \tag{3.10}$$

or

$$f^\infty\mu(\alpha) < 1 < \mu(\alpha) f_0,\ f^{0,+} < \infty. \tag{3.11}$$

Then bvp (3.1) admits a positive solution.

Proof. We present the proof in the case where (3.10) holds (the other case is checked similarly). Let $\epsilon > 0$ be such that $\left(f^0+\epsilon\right)\mu(\beta) < 1 < (f_\infty-\epsilon)\mu(\beta)$. We have then

$$\begin{array}{l} Tu \leq Tu \leq N_1(u + G_1u) \\ N_2(u - G_2u) \leq Tu \leq \alpha N_2(u + G_3u) \end{array}$$

where

$$\begin{array}{ll} N_1u = N^{+}_{(f^0+\epsilon)}(u) & G_1(u) = \max\left\{f(t,u(t) - \left(f^0+\epsilon\right)u(t), 0\right\} / \left(f^0+\epsilon\right) \\ N_2u = N^{-}_{(f_\infty-\epsilon)}(u) & G_2(u) = \dfrac{c}{(f_\infty-\epsilon)}, \\ \alpha = \psi^{+}\left(\phi^{-}\left(\dfrac{f^{\infty,+}+\epsilon}{f_\infty-\epsilon}\right)\right) & G_3(u) = \dfrac{C}{(f^{\infty,+}+\epsilon)} \end{array}$$

and c, C are positive constant large enough.

We have also

$$\begin{array}{l} \lim_{\gamma\to+\infty} \dfrac{N_2(\gamma u)}{\gamma} = (f_\infty-\epsilon) N_\beta(u) = \tilde{N}_2u \text{ and} \\ \lim_{\gamma\to 0} \dfrac{N_1(\gamma u)}{\gamma} = \left(f^0+\epsilon\right) N_\alpha(u) = \tilde{N}_1u \text{ uniformly in } \partial B(0_E,1). \end{array}$$

Since Lemma 3.1 guarantees that

$$\theta_P^{\tilde{N}_2} = (f_\infty - \epsilon)\mu(\beta) < 1 < \left(f^0 + \epsilon\right)\mu(\beta) = \lambda_P^{\tilde{N}_1}.$$

Therefore, existence of a positive solution to bvp (3.1) follows from Theorem 2.11. □

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