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Extremal Viscosity Solutions of Almost Periodic Hamilton-Jacobi Equations

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Abstract

This paper deals with viscosity solutions of Hamilton-Jacobi equations in which the Hamiltonian H is weakly monotone with respect to the zero order term: this leads to non-uniqueness of solutions, even in the class of *periodic* or *almost periodic* (briefly a.p.) functions. The lack of uniqueness of a.p. solutions leads to introduce the notion of minimal (maximal) a.p. solution and to study its properties. The classes of *asymptotically almost periodic* (briefly a.a.p.) and *pseudo almost periodic* (briefly p.a.p.) functions are also considered.

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1 Introduction

Let us consider the Cauchy problem

$$\begin{cases} u_t + H(t,x,u,Du) = 0\,, & (t,x) \in (0,T)\times\mathbb{R}^N, \\ u(0,x) = u_0(x)\,, & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

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where T is a given positive number, H is a continuous real-valued function defined on $[0,T]\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N$ and u_0 is a given real-valued continuous function defined on $\mathbb{R}^N$; u is the real-valued unknown function and u_t and Du denote, respectively, the partial derivative with respect to t and the gradient with respect to x of u.

We are interested to the viscosity solutions of (1.1), introduced by Crandall and Lions [17]. This kind of solution need not be differentiable anywhere: the only regularity required in its definition is continuity. Those authors (see also [15]) studied existence and uniqueness of viscosity solutions, where uniqueness follows from comparison results. There are equivalent definitions of viscosity super(sub)solutions or solutions. One of the principal ones uses the *test functions* (see [2]) and it can be enlighted in the framework of second order PDE as a maximum principle's application. The second definition (see [15, 16]) is based on the notion of super(sub)jets: in the second order PDE framework super(sub)jets play a fundamental role to prove uniqueness. We refer to [16], [2], [25], [32] and [24] and references therein for precise definitions and main results on the subject. We recall also that in [23] the existence of viscosity solutions is proved via Perron's method. In [1] one can find a viscosity solution's approach to optimal control theory. In the case of Carnot groups, the existence of bounded uniformly continuous viscosity solutions is proved in [7], see also [27]; for the Heisenberg group case see [6].

In the following we will deal only with viscosity super(sub)solutions or solutions, so we will omit the term "viscosity" for the sake of simplicity. Moreover we recall that the definition of viscosity supersolution (or subsolution or solution) implies its lower semicontinuity (resp. upper semicontinuity or continuity).

It is well known that a monotonicity hypothesis of the type

$$(H_1) \qquad \begin{cases} \forall R>0,\, \exists \gamma_R>0 : H(t,x,u,p)-H(t,x,v,p)\geq \gamma_R(u-v) \\ \quad \text{for every } t\in[0,T],\, x\in\mathbb{R}^N,\, -R\leq v\leq u\leq R,\, p\in\mathbb{R}^N \end{cases}$$

is essential for uniqueness, whereas a basic assumption for existence is

$$(H_2) \qquad \exists M>0 : H(t,x,-M,0)\leq 0\leq H(t,x,M,0),\ t\in[0,T],\, x\in\mathbb{R}^N.$$

Assumption (H_2) gives a subsolution $-M$ and a supersolution M to the equation in (1.1).

In order to obtain existence and uniqueness results for the solution of (1.1), we add to (H_1) and (H_2) the following assumptions:

$$(H_3) \qquad \begin{cases} \forall R>0,\, \exists m_R : |H(t,x,u,p)-H(t,y,u,p)|\leq m_R(|x-y|(1+|p|)) \\ \text{for every } t\in[0,T],\, x,y\in\mathbb{R}^N,\, -R\leq u\leq R,\, p\in\mathbb{R}^N, \\ \text{where } \lim\limits_{z\to 0} m_R(z)=0, \end{cases}$$

$$(H_4) \qquad \forall R>0,\, H \text{ is uniformly continuous on } [0,T]\times\mathbb{R}^N\times[-R,R]\times\bar{B}_R.$$

When the Hamiltonian H satisfies (H_1), (H_2), (H_3) and (H_4), then for every $u_0\in BUC(\mathbb{R}^N)$ [1] there exists a unique solution $u\in BUC([0,T]\times\mathbb{R}^N)$ of (1.1) (see [12, Theorem 4]).

[1] *Bounded Uniformly Continuous functions*

Let us consider now the equation

$$u_t + H(t,x,u,Du) = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{1.2}$$

and extend to $t \in \mathbb{R}$ the assumptions (H_1), (H_2), (H_3) and (H_4). When H is periodic or almost periodic (briefly a.p.) in t uniformly w.r.t. (x, u, p), the above assumptions can guarantee existence and uniqueness of the periodic (respectively a.p.) solution to the equation (1.2).

Almost periodicity is involved in many mathematical contexts: probability and stochastic processes, see [33], [18], [13] and [5]; homogeneization, see [4] and [30]; stability, see [35], [26] and [34]; abstract spaces, see [19] and [37].

In this new setting we observe that the strict monotonicity of the Hamiltonian w.r.t. u in (H_1) (due to the positivity of γ_R) happens to be crucial, not only for the regularity, but notably for the uniqueness of the solution. Examples of equations with periodic, weakly increasing Hamiltonian, but having different periodic solutions can be found in [29].

In this paper we study the behavior of a.p. solutions of (1.2) when $\gamma_R = 0$, *i.e.* instead of (H_1) we suppose that H, almost periodic in t, satisfies the following weaker condition:

(H_5) $\qquad H(t,x,u,p) - H(t,x,v,p) \geq 0, \quad$ for every $u \geq v$, $(t,x,p) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N$.

This problem was partially treated in [12] for periodic and a.p. solutions and completed in [29] for periodic solutions.

The nonuniqueness of the periodic solutions of (1.2) when the periodic H satisfies (H_5) instead of (H_1), brings the authors of [29] to define a notion of *minimal (maximal) periodic solution* of (1.2). They are able to construct such extremal solution in several ways, studying its possible dependence on M and proposing an appropriate algorithm to compute it numerically: see [29] for details.

We want to gain similar results about a.p. solutions of (1.2).

The plan of the paper is as follows.

In Section 2 we recall existence, uniqueness and regularity results of almost periodic solutions of (1.2) under the conditions (H_1), (H_2), (H_3) and (H_4).

In Section 3, under the assumptions (H_2), (H_3), (H_4) and (H_5), supposing $H(t,x,u,p) = \mathcal{H}(x, u, p) - f(t)$ with $f(t)$ almost periodic, we construct the minimal (maximal) a.p. solution of (1.2) and we study its regularity.

In Section 4 we deal with the a.p. solutions of the one dimensional evolution equation

$$y'(t) + g(y(t)) = f(t), \quad t \in \mathbb{R}, \tag{1.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous weakly increasing function, and f is an a.p. function. In this simpler case we are able to describe the features of the a.p. solutions: see Proposition 4.5.

Because of the monotonicity of the Hamiltonian, condition (H_2) holds even if the constant M is replaced by a greater one $\overline{M} > M$.

In Section 5 we prove that, under additional assumptions, the minimal a.p. solution $\bar{u}$ depends only on the constant M_0, *i.e.* on the smaller value for which (H_2) holds.

In Section 6 we consider asymptotically almost periodic (briefly a.a.p.) solutions of

$$\begin{cases} u_t + \mathcal{H}(x,u,Du) = f(t), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.4}$$

where f is a.a.p., *i.e.* $f = g + \sigma$ where g is a.p. in $\mathbb{R}$ and σ is continuous in $\mathbb{R}^+$ such that $\lim_{t\to+\infty}\sigma(t) = 0$.

The a.a.p. functions were first introduced by M. Fréchet on 1941 [22]. The asymptotic property of the functions is often applied to prove the existence of a.p. solutions, see [36], [31]. Moreover the existence of an a.p or a.a.p. solution has close relation to the applications, as neural networks or epidemiology, see [20].

If $H(t, x, u, p) = \mathcal{H}(x, u, p) - f(t)$ satisfies (H_1), (H_2), (H_3), (H_4) and f is a.a.p., then we obtain existence and uniqueness of the a.a.p. solution u of (1.4), whose a.p. component u_{ap} ($u = u_{ap} + u_\sigma$) is the unique a.p. solution of

$$u_t + \mathcal{H}(x,u,Du) = g(t), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{1.5}$$

When (H_5) holds instead of (H_1), the solution of (1.4) is still unique: see Remark 6.3.

In Section 7 we consider pseudo almost periodic (briefly p.a.p.) solutions of (1.2).

Those functions are a new generalization of a.p. functions, introduced by Zhang [38] on 1992: the author discussed their applications to some differential equations. After that, the research of p.a.p. solutions of various differential equations continued until today: see [19] and references therein.

If $H(t,x,u,p) = \mathcal{H}(x,u,p) - f(t)$ and f is p.a.p., that is $f = g + \varphi$ where g is a.p. and φ is ergodic zero, *i.e.*

$$\frac{1}{2T}\int_{-T}^{T} |\varphi(t)|dt \to 0, \quad \text{as } T \to +\infty,$$

in [38] it is proved that (1.2) admits a unique p.a.p. solution, assuming that (H_1), (H_2), (H_3) and (H_4) hold. When (H_5) holds instead of (H_1) we lose uniqueness, but we can state and prove an existence result for the minimal (maximal) p.a.p. solution also in this case.

2 Preliminaries

In this section we recall some basic properties of viscosity solutions, and we use the following weak form of (H_1):

$$(H_1)_{\mathbb{R}} \qquad \begin{cases} \forall R > 0,\, \exists \gamma_R \in \mathbb{R} : H(t,x,u,p) - H(t,x,v,p) \geq \gamma_R(u-v) \\ \text{for every } t \in [0,T],\, x \in \mathbb{R}^N,\, -R \leq v \leq u \leq R,\, p \in \mathbb{R}^N. \end{cases}$$

Theorem 2.1. *([12, Corollary 1], Decay estimate) Let H satisfy $(H_1)_{\mathbb{R}}$, (H_3), (H_4) and suppose $f^1, f^2 \in BUC\big([0,T]\times\mathbb{R}^N\big)$. Let u be a bounded subsolution of*

$$u_t + H(t,x,u,Du) = f^1(t,x), \;\; (t,x) \in (0,T)\times\mathbb{R}^N, \tag{2.1}$$

and v be a bounded supersolution of

$$v_t + H(t,x,v,Dv) = f^2(t,x), \;\; (t,x) \in (0,T)\times\mathbb{R}^N \tag{2.2}$$

such that

$$\lim_{t\downarrow 0}(u(t,x)-u(0,x))_+ = \lim_{t\downarrow 0}(v(t,x)-v(0,x))_- = 0, \text{ uniformly for } x\in\mathbb{R}^N$$

and

$$u(0,\cdot)\in BUC(\mathbb{R}^N) \quad \text{or} \quad v(0,\cdot)\in BUC(\mathbb{R}^N).$$

Then we have for every $t\in[0,T]$

$$\begin{aligned} e^{\gamma t}\sup_{x\in\mathbf{R}^N}(u(t,x)-v(t,x))_+ &\le \|(u(0,\cdot)-v(0,\cdot))_+\|_{L^\infty(\mathbb{R}^N)}+ \\ &+\int_0^t e^{\gamma s}\|\left(f^1(s,\cdot)-f^2(s,\cdot)\right)_+\|_{L^\infty(\mathbb{R}^N)}\,ds, \end{aligned} \tag{2.3}$$

where $\gamma=\gamma_{R_0}$, $R_0=\max\left\{\sup_{[0,T]\times\mathbb{R}^N}u,\,-\inf_{[0,T]\times\mathbb{R}^N}v\right\}$.

Corollary 2.2. *([28, Corollary 3.6]) Let H satisfy* (H_1), (H_3), (H_4) *and suppose* $f^1,f^2\in BUC(\mathbb{R}\times\mathbb{R}^N)$. *Let u be a bounded subsolution of (2.1) and v be a bounded supersolution of (2.2) on* $\mathbb{R}\times\mathbb{R}^N$. *Then, for every* $t\in\mathbb{R}$

$$\sup_{x\in\mathbb{R}^N}(u(t,x)-v(t,x))\le\sup_{s\le t}\left[\int_s^t\sup_{x\in\mathbb{R}^N}\left(f^1(\sigma,x)-f^2(\sigma,x)\right)d\sigma\right].$$

Theorem 2.3. *([12, Theorem 4],* Existence and uniqueness of solutions to (1.1) *) Assume that H satisfies* $(H_1)_\mathbb{R}$, (H_2), (H_3) *and* (H_4). *Then, for every* $u_0\in BUC(\mathbb{R}^N)$ *there exists a unique solution* $u\in BUC\left([0,T]\times\mathbb{R}^N\right)$ *of (1.1),* $\forall T>0$.

2.1 Almost periodic functions

There are different but equivalent definitions of almost periodicity. The Bochner definition is most convenient for showing the algebraic and topological properties of the set of the real a.p. functions of the real variable. The Bohr definition is better when one wants to show that a solution of a differential equation is a.p. We will use both of them in the paper.

Continuous periodic functions are a.p. but the converse is, in general, false. For example $f(t)=\cos t+\cos(\sqrt{2}t)$ is a.p. and continuous but not periodic.

In this subsection we recall the definition and some fundamental properties of almost periodic functions. For more details (and other equivalent definitions) we refer for example to [14], [21], [36].

Proposition 2.4. *Let* $f:\mathbb{R}\to\mathbb{R}$ *be a continuous function. The following conditions are equivalent*

1) (H.Bohr) $\forall\epsilon>0$, $\exists\ell(\epsilon)>0$ *such that* $\forall a\in\mathbb{R}\ \exists\tau\in[a,a+\ell(\epsilon))$ *satisfying*

$$|f(t+\tau)-f(t)|<\epsilon\ ,\ t\in\mathbb{R}. \tag{2.4}$$

A number τ *verifying* (2.4) *is called* $\epsilon-$*almost period.*

2) (S.Bochner) *For every real sequence $(h_n)_n$, there exists a subsequence $(h_{n_k})_k$ such that $(f(\cdot + h_{n_k}))_k$ converges uniformly on $\mathbb{R}$.*

Definition 2.5. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. We say that f is *almost periodic* (briefly a.p.) if it satisfies one of the previous two conditions. We denote by $AP(\mathbb{R})$ the set of all such functions.

$AP(\mathbb{R})$ is the smallest complete normed space (w.r.t. the uniform norm) containing all continuous periodic functions.

Proposition 2.6. *Assume that $f \in AP(\mathbb{R})$. Then*

i) f is bounded and uniformly continuous in $\mathbb{R}$.

ii) $(1/T)\int_a^{a+T} f(t)\,dt$ converges as $T \to +\infty$ uniformly w.r.t. $a \in \mathbb{R}$. The limit is called the average *of f and denoted*

$$\langle f\rangle := \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)\,dt, \text{ uniformly w.r.t. } a \in \mathbb{R}.$$

iii) F, a primitive of f, is a.p. if and only if F is bounded.

Remark 2.7. If f is periodic, then $\langle f\rangle$ denotes the usual definition of mean of f over one period.

The following definition of almost periodicity is suitable for differential equations [36].

Definition 2.8. We say that $u : \mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ is *almost periodic* in $t \in \mathbb{R}$ uniformly w.r.t. $x \in \mathbb{R}^N$ if u is continuous in t uniformly w.r.t. x and

$$\forall\epsilon > 0\ \exists\ell(\epsilon) > 0 \text{ such that } \forall a \in \mathbb{R}\ \exists\tau \in [a, a+\ell(\epsilon)) \text{ satisfying}$$

$$|u(t+\tau, x) - u(t, x)| < \epsilon,\ (t, x) \in \mathbb{R}\times\mathbb{R}^N.$$

Let us denote by $AP(\mathbb{R}\times\mathbb{R}^N)$ the set of all such functions.

Theorem 2.9. *([12, Proposition 14]) Assume that $H = \mathcal{H}(x, u, p) - f(t)$ satisfies (H_1), (H_2), (H_3) and (H_4) with $\gamma = \gamma_M > 0$, where M is the constant in (H_2), and $f \in BUC(\mathbb{R})$. Then, there exists a unique solution $u \in BUC([a, b]\times\mathbb{R}^N)$ of equation (1.2) for every $a, b \in \mathbb{R}$.*

Moreover, if $f \in AP(\mathbb{R})$, then $u \in BUC(\mathbb{R}\times\mathbb{R}^N) \cap AP(\mathbb{R}\times\mathbb{R}^N)$.

3 Almost periodic minimal solutions

In this section we will suppose $H(t, x, u, p) = \mathcal{H}(x, u, p) - f(t)$, where $f \in AP(\mathbb{R})$. Moreover we will suppose that H satisfies (H_2), (H_3), (H_4) and (H_5) (instead of (H_1)). In this setting we can fall into the lack of uniqueness for the a.p. solutions of (1.2). This brings us to introduce the notion of minimal a.p. solution. Referring to Definition 2.8 we have

Definition 3.1. A bounded solution $u \in AP(\mathbb{R}\times\mathbb{R}^n)$ of (1.2) is called *minimal* if $u \le v$ for every bounded supersolution $v \in AP(\mathbb{R}\times\mathbb{R}^n)$ of (1.2) such that $v \ge -M$, where M is the constant in (H_2).

In the following we will analyze the existence (since uniqueness is obvious) of such a minimal solution, and how to attain it. The definition of maximal solution can be given in the same way, and also the results can be proved analogously, then we will deal in the following only with minimal solutions.

Let us consider the case $f \in BUC(\mathbb{R})$; this assumption guarantees existence and uniqueness of the bounded a.p. solution u_α of the following equation

$$\alpha(u+M)+\partial_t u+H(t,x,u,Du)=0\,,\ (t,x)\in\mathbb{R}\times\mathbb{R}^N, \tag{3.1}$$

for every $\alpha > 0$: see Theorem 2.9. Let us observe that $[\alpha(u+M)+H(t,x,u,p)]$ satisfies (H_1), (H_2), (H_3) and (H_4).

Lemma 3.2. *([29, Lemma 2.3]) Let* $0<\alpha\le\beta$, *then*

i) $-M \le u_\beta(t,x) \le u_\alpha(t,x) \le M\,, \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^N.$

ii) $u_\alpha(t,x) \le u_\beta(t,x) + 2M\left(\dfrac{\beta-\alpha}{\alpha}\right), \qquad (t,x)\in\mathbb{R}\times\mathbb{R}^N.$

It follows from Lemma 3.2 that $(u_\alpha)_{\alpha>0}$ is a monotone decreasing, uniformly bounded sequence of continuous functions. Then, when $\alpha \to 0$, u_α converges towards a function $\bar{u}$, possibly discontinuous, given by

$$\bar{u} = \sup_{\alpha>0} u_\alpha = \lim_{\alpha\to 0} u_\alpha. \tag{3.2}$$

One can verify that $\overline{u}$ is a lower semicontinuous viscosity solution of (1.2): see for example [3]. If $f \in AP(\mathbb{R})$, then $u_\alpha \in AP(\mathbb{R}\times\mathbb{R}^n)$: see Theorem 2.9.

If $\bar{u}$ is a.p., then it is the minimal a.p. solution of (1.2). In fact, if v is an a.p. bounded supersolution of (1.2) such that $v \ge -M$, then in virtue of Corollary 2.2, we have $v \ge u_\alpha$ for every $\alpha > 0$, and letting $\alpha \to 0$ we obtain $v \ge \bar{u}$.

We need to know if $\bar{u}$ is a.p..

Theorem 3.3. *Let*

$$H(t,x,u,p) = \mathcal{H}(x,u,p) - f(t)\,,\ (t,x,u,p)\in\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N\,,$$

where $f \in BUC(\mathbb{R})$ *and* $h(t) := \int_0^t f(\sigma)\,d\sigma$, $t\in\mathbb{R}$, *belongs to* $AP(\mathbb{R})$.

Then we have $\bar{u} \in AP(\mathbb{R}\times\mathbb{R}^N)$.

Moreover $(u_\alpha)_{\alpha>0}$ *and* $\overline{u}$ *have common periods.*

Proof. As h is almost periodic, then, for every $\epsilon > 0$, there exists $\ell(\frac{\epsilon}{2})$ such that every interval of length $\ell(\frac{\epsilon}{2})$ contains an $\frac{\epsilon}{2}$−almost period of h. Take an interval of length $\ell(\frac{\epsilon}{2})$ and let τ an $\frac{\epsilon}{2}$−almost period of h in this interval. By Corollary 2.2 we have, for every $\alpha > 0$, $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, that

$$\begin{aligned} u_\alpha(t+\tau,x) - u_\alpha(t,x) &\le \sup_{s\le t}\left[\int_s^t (f(\sigma+\tau) - f(\sigma))\,d\sigma\right] = \\ &= \sup_{s\le t}\left[\int_{s+\tau}^{t+\tau} f(\sigma)\,d\sigma - \int_s^t f(\sigma)\,d\sigma\right] = \\ &= \sup_{s\le t}\left[(h(t+\tau)-h(t)) - (h(s+\tau)-h(s))\right] \le \epsilon. \end{aligned}$$

When $\alpha \to 0$ one gets, for every $(t,x) \in \mathbb{R}\times\mathbb{R}^N$,

$$\bar{u}(t+\tau,x) - \bar{u}(t,x) \le \epsilon.$$

□

The following hypothesis is used in Proposition 3.4:

$$(H_6) \qquad \begin{cases} \forall R > 0 \ \lim_{|p|\to\infty} H(t,x,u,p) = \infty, \\ \text{uniformly for } (t,x,u) \in \mathbb{R}\times\mathbb{R}^N\times[-R,R], \end{cases}$$

Proposition 3.4. *Let us suppose*

$$H(t,x,u,p) = \mathcal{H}(x,u,p) - f(t),\ (t,x,u,p) \in \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N,$$

where $f \in BUC(\mathbb{R})$. Then $\bar{u}$ is Lipschitz continuous in $t \in \mathbb{R}$ uniformly w.r.t. $x \in \mathbb{R}^N$.

When (H_6) holds true, then $\bar{u}$ is Lipschitz continuous on the whole $\mathbb{R}\times\mathbb{R}^N$.

Proof. Let us fix $k \in \mathbb{R}$ and $\alpha > 0$. The function

$$v_\alpha(t,x) := u_\alpha(t+k,x),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N$$

solves the following equation:

$$\alpha(u+M) + \partial_t u + \mathcal{H}(x,u,Du) = f(t+k),\ (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

By using Corollary 2.2 we have

$$\begin{aligned} u_\alpha(t+k,x) - u_\alpha(t,x) &\le \sup_{s\le t}\int_s^t (f(\sigma+k) - f(\sigma))\,d\sigma \\ &= \sup_{s\le t}\left(\int_t^{t+k} f(\sigma)\,d\sigma - \int_s^{s+k} f(\sigma)\,d\sigma\right) \\ &\le 2|k|\sup_{\mathbb{R}}|f|. \end{aligned}$$

Hence the functions $(u_\alpha)_{\alpha>0}$ are Lipschitz, uniformly w.r.t. $\alpha > 0$, in $t \in \mathbb{R}$, uniformly w.r.t. $x \in \mathbb{R}^N$.

Let us suppose now that (H_6) holds true: the Lipschitz continuity w.r.t. $x \in \mathbb{R}^N$ follows as in the proof of [12, Theorem 5].

Observe that, in virtue of the uniform boundedness of u_α, $\partial_t u_\alpha$, f and by (H_6) together with equation (3.1) (where we can suppose $\alpha \in (0,1)$), we get the uniform boundedness of Du_α. □

4 Almost periodic solutions for one dimensional evolution equations

Let us consider the one dimensional case

$$y'(t)+g(y(t)) = f(t)\ ,\ t\in\mathbb{R}, \tag{4.1}$$

where $g:\mathbb{R}\to\mathbb{R}$ is increasing and Lipschitz continuous and $f\in AP(\mathbb{R})$.

Proposition 4.1. *Assume that g is strictly increasing. Then there exists at most one a.p. solution for* (4.1).

Proof. Let y_1, y_2 be two a.p. solutions for (4.1). By taking the difference between the two equations and multiplying by $y_1(t)-y_2(t)$ we get

$$\frac{1}{2}\frac{d}{dt}|y_1(t)-y_2(t)|^2+[g(y_1(t))-g(y_2(t))][y_1(t)-y_2(t)] = 0\,,\ t\in\mathbb{R}. \tag{4.2}$$

Since g is increasing we have $[g(y_1(t))-g(y_2(t))][y_1(t)-y_2(t)]\geq 0$ for every $t\in\mathbb{R}$; we deduce that the function $\alpha(t):=|y_1(t)-y_2(t)|$ is decreasing.

The function α is a.p.: hence for all positive integers k there exists ℓ_k such that

$$\forall a\in\mathbb{R}\ \exists\tau\in[a,a+\ell_k)\ :\ |\alpha(t+\tau)-\alpha(t)|<\frac{1}{k},\quad \forall t\in\mathbb{R}.$$

If $k=1$ there exists $\tau_1\in[0,\ell_1)$ such that $|\alpha(t+\tau_1)-\alpha(t)|<1$ for all $t\in\mathbb{R}$.

If $k=2$ there exists $\tau_2\in[\ell_1,\ell_1+\ell_2)$ such that $|\alpha(t+\tau_2)-\alpha(t)|<\frac{1}{2}$ for every $t\in\mathbb{R}$, etc.

If $k\geq 3$ there exists $\tau_k\in\left[\sum_{j=1}^{k-1}\ell_j,\sum_{j=1}^{k}\ell_j\right)$ such that $|\alpha(t+\tau_k)-\alpha(t)|<\frac{1}{k}$ for every $t\in\mathbb{R}$.

By Proposition 2.4 and Definition 2.5 there exists a subsequence of $(\tau_k)_k$, denoted again $(\tau_k)_k$ for the sake of simplicity, such that $(\alpha(t+\tau_k))_k$ converges to $\alpha(t)$ uniformly w.r.t. $t\in\mathbb{R}$.

But α is decreasing, then for every $t\in\mathbb{R}$ and every integer $h,k\geq 1$

$$\alpha(t+\tau_{k+h})\leq\alpha(t+\tau_k)\leq\alpha(t).$$

Letting $h\to+\infty$, we get $\alpha(t+\tau_k)=\alpha(t)$ for every $t\in\mathbb{R}$. This implies that α is periodic. But, as α is decreasing, this is a contradiction, unless α is constant.
Hence $\alpha(t)=|y_1(t)-y_2(t)|$ does not depend on t and therefore, from (4.2) we obtain

$$[g(y_1(t))-g(y_2(t))][y_1(t)-y_2(t)] = 0\,,\ t\in\mathbb{R}.$$

The strict monotonicity of g implies that $y_1=y_2$. □

If g is weakly increasing, then (4.1) can have more than one a.p. solution (see [11, Remark 2.2]). An easy consequence of Proposition 4.1 is the following:

Proposition 4.2. *([11, Proposition 2.3]) Let g be an increasing function. Let y_1, y_2 be two a.p. solutions of* (4.1). *Then there exists a constant $C\in\mathbb{R}$ such that*

$$y_1(t)-y_2(t)=C\ ,\ t\in\mathbb{R}.$$

Let us define the function

$$h(t) := \int_0^t (f(s) - \langle f \rangle)\, ds, \quad t \in \mathbb{R}, \tag{4.3}$$

and the set

$$O\langle f \rangle := \{ y \in \mathbb{R} : g(y + h(t)) = \langle f \rangle\,, \forall\, t \in \mathbb{R} \}.$$

We have obviously

$$O\langle f \rangle \subset g^{-1}(\langle f \rangle),$$

and the following proposition holds.

Proposition 4.3. *Let g be an increasing function. Let us suppose that the function h defined in* (4.3) *is a.p. Then the equation* (4.1) *has different a.p. solutions iff*

$$\mathrm{Int}(O\langle f \rangle) \neq \emptyset \tag{4.4}$$

Proof. Assume that (4.1) has two different a.p. solutions $y_1 \neq y_2$. By Proposition 4.2 there exists $C \neq 0$ such that $y_2(t) = y_1(t) + C$ for every $t \in \mathbb{R}$. We obtain from (4.1) that $g(y_1) = g(y_2)$, *i.e.* $g(y_1) = g(y_1 + C)$. As g is increasing, we deduce that g is constant, *i.e.* $g = C_0(t)$ on each interval $[y_1(t), y_1(t) + C]$, for every $t \in \mathbb{R}$. To evaluate $C_0(t)$ we integrate between 0 and t the equation (4.1) letting $y = y_1$: we obtain

$$\int_0^t g(y_1(s))\, ds + y_1(t) - y_1(0) = \int_0^t f(s)\, ds,$$

and then

$$C_0(t) + \frac{y_1(t) - y_1(0)}{t} = \frac{1}{t} \int_0^t f(s)\, ds.$$

As y_1 is bounded, letting $t \to +\infty$ we obtain that $C_0(t)$ is a constant, say $C_0(t) = \langle f \rangle$, and then

$$g(y_1(t)) = g(y_2(t)) = \langle f \rangle\,, \; t \in \mathbb{R}. \tag{4.5}$$

One immediately gets from (4.5) and (4.1) that

$$y_i'(t) = f(t) - \langle f \rangle\,, \; t \in \mathbb{R}, \quad i = 1, 2,$$

whence for (4.3) we get

$$y_i(t) = h(t) + C_i\,, \; t \in \mathbb{R}, \quad i = 1, 2.$$

Then we have that $C_i \in O\langle f \rangle$, $i = 1, 2$. For every $y \in (C_1, C_2)$ we have that $\langle f \rangle = g(y_1) \leq g(y + h) \leq g(y_2) = \langle f \rangle$, hence $g(y + h) = \langle f \rangle$ and $y \in O\langle f \rangle$. We conclude that $[C_1, C_2] \subseteq O\langle f \rangle$ and (4.4) is proved.

Conversely, suppose that there exist y and $C > 0$ such that $y, y + C \in O\langle f \rangle$. It is easy to check that the functions

$$y_1(t) = y + h(t)\,, \; y_2(t) = y + h(t) + C\,, \; t \in \mathbb{R},$$

are two different a.p. solutions of (4.1). □

Let g be increasing and h be a.p. Suppose that (4.1) has more than one a.p. solution. By Proposition 4.3 we have Int $(O\langle f\rangle) \neq \emptyset$. Let us define

$$I\langle f\rangle := \{y \in \mathbb{R} \,|\, g(y) = \langle f\rangle\}.$$

Let us observe that $O\langle f\rangle \subseteq I\langle f\rangle$ and that both $O\langle f\rangle$ and $I\langle f\rangle$ are intervals. We argue by contradiction that, for example, $I\langle f\rangle$ is not an interval. Then there exist y_0, y_1, $y_2 \in \mathbb{R}$ such that $y_1 < y_0 < y_2$ and $g(y_1) = g(y_2) = \langle f\rangle$ whereas $g(y_0) \neq \langle f\rangle$. The monotonicity of g produces a contradiction.

Let us assume that

$$\mathrm{Int}\,(I\langle f\rangle) = (a,b), \tag{4.6}$$

eventually unbounded. As the function h is a.p., then it is bounded (unless $f \equiv 0$, but this is the trivial case). Let us define

$$h_m = \inf_{t\in\mathbb{R}} h(t)\,,\;\; h_M = \sup_{t\in\mathbb{R}} h(t). \tag{4.7}$$

We have the following Lemma.

Lemma 4.4. *In the above notation, the following formula holds:*

$$\mathrm{Int}\,(\mathrm{O}\langle f\rangle) = (a-h_m, b-h_M) =: (\underline{C}, \overline{C}) \subseteq (a,b). \tag{4.8}$$

Proof. If $y \in \mathrm{Int}\,(O\langle f\rangle)$, then $g(y+h(t)) = \langle f\rangle$ for every $t \in \mathbb{R}$, and then, because of the monotonicity and the continuity of g (taking inf and sup on $\mathbb{R}$), we get

$$y+h_m\,,\; y+h_M \in \mathrm{Int}\,(I\langle f\rangle)\,,$$

and then

$$a < y+h_m < y+h_M < b.$$

So we have

$$\forall y \in \mathrm{Int}\,(O\langle f\rangle)\,,\; a-h_m < y < b-h_M,$$

and then

$$\mathrm{Int}\,(O\langle f\rangle) \subseteq (a-h_m,\, b-h_M)\,.$$

On the contrary

$$y \in (a-h_m,\, b-h_M) \implies a < y+h_m < y+h_M < b.$$

So $g(y+h_m) = \langle f\rangle = g(y+h_M)$, and then we have

$$\langle f\rangle = g(y+h_m) \leq g(y+h(t)) \leq g(y+h_M) = \langle f\rangle\,,\; t \in \mathbb{R}.$$

We conclude that $y \in \mathrm{Int}\,(O\langle f\rangle)$. □

Note that if (4.4) holds and h is a.p., then every a.p. solution of (4.1) will necessarily be of the form

$$y(t) = C + h(t), \quad t \in \mathbb{R}, \tag{4.9}$$

with $C \in \mathrm{Int}\,(O\langle f\rangle)$. In particular

- if $-\infty < \underline{C}$, then $y(t) = \underline{C} + h(t)$ is the minimal a.p. solution of (4.1);
- if $\overline{C} < +\infty$, then $y(t) = \overline{C} + h(t)$ is the maximal a.p. solution of (4.1).

Indeed if $C \in \mathrm{Int}(O\langle f\rangle)$, then $g(C+h(t)) = \langle f\rangle$ for every $t \in \mathbb{R}$ and then $y(t) = C + h(t)$, $t \in \mathbb{R}$, solves (4.1) and is a.p.. Conversely, if $C \notin O\langle f\rangle$, then $y(t) = C + h(t)$ cannot solve (4.1).

We have thus proved the following Proposition:

Proposition 4.5. *Let g be an increasing function. Let us suppose that the function h defined by* (4.3) *is a.p. and suppose that the equation* (4.1) *has different a.p. solutions. Then every a.p. solution of* (4.1) *is of the form*

$$y(t) = C + h(t)\,, \ t \in \mathbb{R},$$

with $C \in \mathrm{Int}(\mathrm{O}\langle f\rangle)$, where $\mathrm{Int}(\mathrm{O}\langle f\rangle)$ *is described by* (4.8).

5 The Hamilton-Jacobi case. The dependence on M.

The a.p. solution u_α of (3.1) depends, a priori, on the parameters α and M, where M is the constant in (H_2); it follows that $\overline{u}$, the limit of u_α when α goes to zero, depends, a priori, on M. Let us denote hereafter by $u_\alpha(M)$ the a.p. solution of (3.1) and by $\overline{u}(M)$ its limit when α goes to zero.

Our aim in this section is to study the dependence on M of the minimal solution $\overline{u}(M)$ of the equation

$$\partial_t u + \mathcal{H}(x,u,Du) = f(t)\,, \ (t,x) \in \mathbb{R}\times\mathbb{R}^N \tag{5.1}$$

where $\mathcal{H}(x,u,p) - f(t)$ satisfies (H_2), (H_3), (H_4) and (H_5), $f \in AP((R))$ and h, defined by (4.3), is a.p.. The monotonicity of H implies that, if (H_2) is true for some M_0, it will be true for every $M \geq M_0$. Let us define M_0 as

$$M_0 := \inf\{M > 0 | (H_2)\,\text{is true}\}.$$

Theorem 5.1. *Let $H : \mathbb{R}\times\mathbb{R}^N\times\mathbb{R}\times\mathbb{R}^N \to \mathbb{R}$ satisfy (H_2) (with constant M_0), (H_3), (H_4), (H_5) and*

$$(H_7) \qquad \sup_{x\in\mathbb{R}^n} \mathcal{H}(x,u,0) \ \text{is Lipschitz.}$$

Let $f \in AP(\mathbb{R})$ and let h, defined by (4.3)*, be in $AP(\mathbb{R})$.*
Then, for every $M \geq M_0$, we have $\bar{u}(M) = \bar{u}(M_0)$.

Proof. Using the comparison result Corollary 2.2, we have that

$$u_\alpha(M) \leq u_\alpha(M_0)\,, \text{for every } M \geq M_0.$$

This implies that, letting $\alpha \to 0$,

$$\overline{u}(M) \leq \overline{u}(M_0)\,, \text{ for every } M \geq M_0.$$

In order to prove $\overline{u}(M) = \overline{u}(M_0)$, for every $M \geq M_0$, we define

$$g(u) := \sup_{x \in \mathbb{R}^n} \mathcal{H}(x,u,0).$$

The function g is Lipschitz, increasing and satisfies

$$g(-M_0) \leq f(t) \leq g(M_0)\,,\ \forall\, t \in \mathbb{R}.$$

So it satisfies (H_2), (H_4) and (H_5). Let us consider, for $M \geq M_0$ and $\alpha > 0$, the ode

$$\alpha(y(t)+M) + y'(t) + g(y(t)) = f(t)\ ,\ t \in \mathbb{R}. \tag{5.2}$$

We know from Theorem 2.9 that (5.2) admits a unique a.p. solution, which we denote $y_\alpha(M)$.

We can easily verify, because of the definition of g, that $y_\alpha(M)$ is a subsolution of the following equation

$$\alpha(u+M) + \partial_t u + \mathcal{H}(x,u,Du) = f(t),\ (t,x) \in \mathbb{R} \times \mathbb{R}^N. \tag{5.3}$$

Then, by the comparison principle Corollary 2.2, we have

$$u_\alpha(M) \geq y_\alpha(M).$$

Suppose that (4.1) has more than one a.p. solution. As $g(-M_0) \leq \langle f \rangle$, then $-M_0 \leq a \leq \underline{C}$ (a and $\underline{C}$ are defined in (4.6) and (4.8)) and then $\underline{C} > -\infty$ and $\underline{C} \in O\langle f \rangle$, that is $\underline{C} + h$ and $\overline{y}(M) := \lim_{\alpha\to 0} y_\alpha(M)$ are both minimal solutions of (4.1), so $\underline{C} + h = \overline{y}(M)$.

Then

$$\underline{C} + h = \lim_{\alpha \to 0} y_\alpha(M) \leq \lim_{\alpha\to 0} u_\alpha(M) = \overline{u}(M).$$

Referring to (4.7), (4.8), (4.9), we have $\underline{C} = a - \inf h \geq -M_0 - \inf h$, then $\underline{C} + h \geq -M_0$. It follows that $\overline{u}(M) \geq -M_0$, then we obtain $\overline{u}(M) \geq \overline{u}(M_0)$.

The proof can be concluded in a similar but simpler way when (4.1) admits a unique a.p. solution. $\square$

6 Asymptotically almost periodic solutions

6.1 Asymptotically almost periodic solutions in $\mathbb{R}^+$

In this subsection we prove existence and uniqueness of asymptotically almost periodic (briefly a.a.p.) viscosity solutions to the equation

$$\begin{cases} u_t + \mathcal{H}(x,u,Du) = f(t), & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N\,, \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{6.1}$$

where f is a.a.p. and $u_0 \in BUC(\mathbb{R}^N)$. We recall the definition of a.a.p. function, referring the interested reader to [36] for more details. Set

$$\begin{aligned} \psi_0(\mathbb{R}^+) &:= \left\{ \sigma \in C^0(\mathbb{R}^+) \mid \lim_{t\to+\infty} \sigma(t) = 0 \right\} \\ \psi_0\left(\mathbb{R}^+ \times \mathbb{R}^N\right) &:= \left\{ \sigma \in C^0\left(\mathbb{R}^+ \times \mathbb{R}^N\right) \mid \lim_{t\to+\infty} \sigma(t,x) = 0 \text{ uniformly w.r.t. } x \in \mathbb{R}^N \right\}. \end{aligned}$$

Definition 6.1. A function $f \in C^0(\mathbb{R}^+)$ $\left(u \in C^0\left(\mathbb{R}^+ \times \mathbb{R}^N\right)\right)$ is called *asymptotically almost periodic* (briefly a.a.p.) in $t \in \mathbb{R}^+$ (uniformly w.r.t. $x \in \mathbb{R}^N$) if
$f(t) = g(t) + \sigma(t)$, $t \in \mathbb{R}^+$, where $g \in AP(\mathbb{R})$ and $\sigma \in \psi_0(\mathbb{R}^+)$
$(u(t, x) = g(t, x) + \sigma(t, x)$, $(t, x) \in \mathbb{R}^+ \times \mathbb{R}^N$, where $g \in AP(\mathbb{R} \times \mathbb{R}^N)$ and $\sigma \in \psi_0\left(\mathbb{R}^+ \times \mathbb{R}^N\right))$.

We denote by $AAP(\mathbb{R}^+)$ $(AAP(\mathbb{R}^+ \times \mathbb{R}^N))$ the set of all such functions.

Theorem 6.2. *Let $f \in AAP(\mathbb{R}^+)$ and suppose that $\mathcal{H}(x, u, p)$ satisfies*

$$\begin{cases} \mathcal{H}(x, -M, p) \le f(t) \le \mathcal{H}(x, M, p), & (t, x) \in \mathbb{R}^+ \times \mathbb{R}^N \\ \mathcal{H}(x, -M, p) \le g(t) \le \mathcal{H}(x, M, p), & (t, x) \in \mathbb{R} \times \mathbb{R}^N \end{cases} \tag{6.2}$$

for a suitable constant $M > 0$. Assume that the following assumptions hold: (H_1) with $\gamma = \gamma_M > 0$, (H_3) and (H_4) on $\mathbb{R} \times \mathbb{R}^N$.

Then the problem (6.1) *admits a unique solution*

$$V \in BUC([0,T) \times \mathbb{R}^N) \cap AAP(\mathbb{R}^+ \times \mathbb{R}^N)$$

for every $T > 0$.

Proof. Let $f = g + \sigma$ where $g \in AP(\mathbb{R})$ and $\sigma \in C(\mathbb{R}^+)$ satisfies

$$\lim_{t \to +\infty} \sigma(t) = 0. \tag{6.3}$$

Let us consider the equation

$$u_t + \mathcal{H}(x, u, Du) = g(t), \ (t, x) \in \mathbb{R} \times \mathbb{R}^N. \tag{6.4}$$

In virtue of Theorem 2.9 there exists a unique solution $U \in BUC(\mathbb{R} \times \mathbb{R}^N) \cap AP(\mathbb{R} \times \mathbb{R}^N)$ of equation (6.4).
In virtue of Theorem 2.3 there exists a unique solution $V \in BUC([0,T) \times \mathbb{R}^N)$ of (6.1) for every $T > 0$.

By (6.3), for every $\epsilon > 0$, there exists $t^1_\epsilon > 0$ such that $|\sigma(t)| < \gamma\epsilon$ for every $t \ge t^1_\epsilon$.

In virtue of Theorem 2.1 we have that, $\forall\, t \ge t^1_\epsilon$,

$$\begin{aligned} \sup_{x \in \mathbb{R}^N} |U(t,x) - V(t,x)| \quad &\le e^{-\gamma(t - t^1_\epsilon)} \sup_{x \in \mathbb{R}^N} |U(t^1_\epsilon, x) - V(t^1_\epsilon, x)| + \\ &+ e^{-\gamma t} \int_{t^1_\epsilon}^t e^{\gamma s} |\sigma(s)|\, ds. \end{aligned} \tag{6.5}$$

Let us take $t^2_\epsilon \ge t^1_\epsilon$ such that

$$2M e^{-\gamma(t^2_\epsilon - t^1_\epsilon)} < \epsilon.$$

Then from (6.5) we have

$$\forall\, \epsilon > 0\ \exists t^2_\epsilon > 0\ :\ \sup_{x \in \mathbb{R}^N} |U(t,x) - V(t,x)| < 2\epsilon,\ \forall\, t \ge t^2_\epsilon$$

Let us define

$$-W(t,x) := U(t,x) - V(t,x), \quad (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N.$$

Then $\lim\limits_{t \to +\infty} W(t,x) = 0$ uniformly w.r.t. $x \in \mathbb{R}^N$, and $V := U + W$ is (the unique) solution of (6.1) which is a.a.p. in $t \in \mathbb{R}^+$ uniformly w.r.t. $x \in \mathbb{R}^N$. $\square$

Remark 6.3. If (H_5) holds instead of (H_1), then possibly there exist more than one a.p. solution of (6.4), but Theorem 2.3 assures that there exists only one solution of (6.1) which is still a.a.p.

The following example illustrates this situation.

Example 6.4. Let us consider the following problem (partially borrowed from [29])

$$\begin{cases} \partial_t u + |\partial_x u| + g(u) - \frac{1}{4}|\cos x| + \frac{1}{4}\cos t + e^{-t} = 0, & (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N, \\ u(0,x) = \frac{1}{4}\sin x + \frac{1}{2}, & x \in \mathbb{R}^N. \end{cases} \tag{6.6}$$

where

$$g(u) = \begin{cases} u+1, & u < -1, \\ 0\ , & u \in [-1,1] \\ u-1, & 1 < u. \end{cases}$$

The assumption (H_2) is satisfied for $M \geq M_0 = \frac{9}{4}$. The assumptions (H_3), (H_4) and (H_5) also hold. One can check that the functions

$$\begin{aligned} u(t,x) &= -\tfrac{1}{2} + \tfrac{1}{4}(\sin x - \sin t) + e^{-t} \\ h(t,x) &= -\tfrac{1}{2} + \tfrac{1}{4}(-|\sin x| - \sin t) + e^{-t} \\ v(t,x) &= -\tfrac{1}{2} + \tfrac{1}{4}(|\sin x| - \sin t) + e^{-t} \end{aligned}$$

are all a.a.p. solutions of the equation in (6.6), but they result respectively the solution, a subsolution and a supersolution of the problem (6.6).

6.2 Asymptotically almost periodic solution in $\mathbb{R}$

Some authors consider a.a.p. functions defined on the whole $\mathbb{R}$ instead of $\mathbb{R}^+$, as we have done in the above subsection. The present subsection is devoted to summarize the results concerning functions defined on $\mathbb{R}$. Set

$$\begin{aligned} \psi_0(\mathbb{R}) &:= \left\{\sigma \in C^0(\mathbb{R}) \mid \lim_{t\to+\infty} \sigma(t) = 0\right\} \\ \psi_0\left(\mathbb{R}\times\mathbb{R}^N\right) &:= \left\{\sigma \in C^0\left(\mathbb{R}\times\mathbb{R}^N\right) \mid \lim_{t\to+\infty} \sigma(t,x) = 0 \text{ uniformly w.r.t. } x \in \mathbb{R}^N\right\}. \end{aligned}$$

Definition 6.5. A function $f \in C(\mathbb{R})$ $\left(u \in C\left(\mathbb{R}\times\mathbb{R}^N\right)\right)$ is called *asymptotically almost periodic* (briefly a.a.p.) (uniformly w.r.t. $x \in \mathbb{R}^N$) if
$f(t) = g(t) + \sigma(t)$, $t \in \mathbb{R}$, where $g \in AP(\mathbb{R})$ and $\sigma \in \psi_0(\mathbb{R})$
$(u(t,x) = g(t,x) + \sigma(t,x)$, $(t,x) \in \mathbb{R}\times\mathbb{R}^N$, where $g \in AP(\mathbb{R}\times\mathbb{R}^N)$ and $\sigma \in \psi_0\left(\mathbb{R}\times\mathbb{R}^N\right))$.

We denote by $AAP(\mathbb{R})$ ($AAP(\mathbb{R}\times\mathbb{R}^N)$) the set of all such functions.

Theorem 6.6. *Let f and $\mathcal{H}$ be as in Theorem 6.2 in $\mathbb{R}\times\mathbb{R}^N$. Then the equation* (5.1) *admits a unique solution $V \in BUC([a,b]\times\mathbb{R}^N) \cap AAP(\mathbb{R}\times\mathbb{R}^N)$ for every $a, b \in \mathbb{R}$, $a < b$. Moreover $V = U + W$, where $U \in BUC(\mathbb{R}\times\mathbb{R}^N) \cap AP(\mathbb{R}\times\mathbb{R}^N)$ solves* (6.4) *and $W \in C^0(\mathbb{R}\times\mathbb{R}^N)$, $W(t,x) \to 0$ for $t \to \infty$, uniformly w.r.t. $x \in \mathbb{R}^N$.*

Proof. The outline of the proof is similar to that of Theorem 6.1. The condition $W(t,x) \to 0$ for $t \to -\infty$, uniformly for $x \in \mathbb{R}^N$, follows from Corollary 2.1. □

Let us suppose now that (H_5) holds instead of (H_1). Let us suppose $f = g + \sigma$ with

$f, g \in BUC(\mathbb{R})$. In virtue of Theorem 2.9 we can define V_α, U_α and $W_\alpha = V_\alpha - U_\alpha$ for every $\alpha > 0$ as before, and prove that they converge for $\alpha \to 0$: see section 3. If $\overline{V}$, $\overline{U}$ and $\overline{W}$ are their respective limits, then $\overline{V}$ solves $u_t + \mathcal{H} = f$, while $\overline{U}$ solves $u_t + \mathcal{H} = g$ and $\overline{W}$ is such that $\overline{W} = \overline{V} - \overline{U}$.

Theorem 6.7. *Let $f \in AAP(\mathbb{R})$, and suppose that* (6.4), (H_3), (H_4), (H_5) *hold on* $\mathbb{R} \times \mathbb{R}^n$ *together with $h \in AAP(\mathbb{R})$, where $h(t) = \int_0^t g(\sigma)\, d\sigma$. If*

$$W_\alpha \to \overline{W}, \quad \text{uniformly on } [0, +\infty) \times \mathbb{R}^N, \text{ for } \alpha \to 0, \tag{6.7}$$

then $\overline{U} \in AP(\mathbb{R} \times \mathbb{R}^N)$, $\overline{W} \in C(\mathbb{R} \times \mathbb{R}^N)$, $\overline{W}(t, x) \to 0$ for $t \to \infty$ and $\overline{V} \in AAP(\mathbb{R} \times \mathbb{R}^N)$ is the minimal a.a.p. solution of the equation (5.3) *in* $\mathbb{R} \times \mathbb{R}^n$.

Proof. We obtain from Corollary 2.2 that $\overline{W}(t, x) \to 0$ for $t \to -\infty$ uniformly w.r.t. $x \in \mathbb{R}^n$.
Moreover it follows from (6.7) that $\overline{W}(t, x) \to 0$ for $t \to +\infty$ uniformly w.r.t. $x \in \mathbb{R}^n$.
Finally, from Theorem 3.3 we get $\overline{U} \in AP(\mathbb{R} \times \mathbb{R}^N)$, then the conclusion follows. □

7 Pseudo almost periodic solutions

On 2011, generalizing [12], the authors of [39] proved, under usual hypothesis on the Hamiltonian, existence and uniqueness of time pseudo almost periodic (briefly p.a.p.) viscosity solutions of Hamilton-Jacobi second order parabolic equations (see [39, Theorem 3.11]), when $H(t,x,u,p,X) = \mathcal{H}(x,u,p,X) - f(t)$ and f is p.a.p. As the authors of [39] used the method of [12], followed also in Theorem 2.9 of the present paper, then we can argue that existence and uniqueness of time p.a.p. viscosity solutions of the first order Hamilton-Jacobi equation (1.2) can analogously be established.

Let us denote by $\Psi(\mathbb{R})$ ($\Psi(\mathbb{R} \times \mathbb{R}^N)$) the space of all real bounded and continuous functions on $\mathbb{R}$ ($\mathbb{R} \times \mathbb{R}^N$) with the supremum norm. Set

$$PAP_0(\mathbb{R}) := \left\{ \varphi \in \Psi(\mathbb{R}) \mid \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t)|\, dt = 0 \right\}$$

$$PAP_0(\mathbb{R} \times \mathbb{R}^N) := \left\{ \varphi \in \Psi(\mathbb{R} \times \mathbb{R}^N) \mid \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\varphi(t,x)|\, dt = 0 \right.$$
$$\left. \text{uniformly w.r.t. } x \in \mathbb{R}^N \right\}$$

Definition 7.1. A function $f \in \Psi(\mathbb{R})$ ($f \in \Psi(\mathbb{R} \times \mathbb{R}^N)$) is called *pseudo almost periodic* in t (uniformly w.r.t. $x \in \mathbb{R}^N$) if $f = g + \varphi$ where $g \in AP(\mathbb{R})$ ($g \in AP(\mathbb{R} \times \mathbb{R}^N)$) and $\varphi \in PAP_0(\mathbb{R})$ ($\varphi \in PAP_0(\mathbb{R} \times \mathbb{R}^N)$). The functions g and φ are called respectively the a.p. component and the ergodic perturbation of the function f. Denote by $PAP(\mathbb{R})$ ($PAP(\mathbb{R} \times \mathbb{R}^N)$) the set of all such functions.

Theorem 7.2. *Assume that* $H(t,x,u,p) = \mathcal{H}(x,u,p) - f(t)$ *satisfies* (H_1), (H_2), (H_3) *and* (H_4) *with* $\gamma = \gamma_M > 0$, *where* $M > 0$ *is the constant in* (H_2), *and suppose* $f \in PAP(\mathbb{R})$.

Then there exists a unique function $u \in BUC(\mathbb{R}\times\mathbb{R}^N) \cap PAP(\mathbb{R}\times\mathbb{R}^N)$ *which is viscosity solution of* (1.2).

Proof. Follow the outline of [39, Theorem 3.11]. □

If (H_5) holds instead of (H_1) we lose uniqueness of the p.a.p. solution.

Example 7.3. Let the even function $\sigma \in C^1(\mathbb{R})$ be defined, for $t \geq 0$, as follows:

$$\sigma(t) = \begin{cases} \dfrac{1}{2e}(3-t^2), & \text{if } 0 \leq t < 1, \\ e^{-t}\,, & \text{if } 1 \leq t. \end{cases}$$

We observe that both σ and σ' are ergodic zero. Let us consider the equation

$$\partial_t u + |\partial_x u| + g(u) - \frac{1}{4}|\cos x| + \frac{1}{4}\cos t + \sigma'(t) = 0\,,\ (t,x) \in \mathbb{R}^+ \times \mathbb{R}^N \tag{7.1}$$

where $g(u)$ is defined as in the Example 6.4. The functions

$$\begin{cases} h(t,x) = -\frac{1}{2} + \frac{1}{4}(-|\sin x| - \sin t) - \sigma(t) \\ v(t,x) = -\frac{1}{2} + \frac{1}{4}(|\sin x| - \sin t) - \sigma(t) \end{cases}$$

are both p.a.p. solutions of the equation (7.1).

Minimal (maximal) p.a.p. viscosity solutions of (1.2) can be defined as in Section 3. In the following Theorem we get that, under suitable assumptions, the minimal solution is in $PAP(\mathbb{R}\times\mathbb{R}^N)$.

Theorem 7.4. *Assume that* $H(t,x,u,p) = \mathcal{H}(x,u,p) - f(t)$ *satisfies* (H_2), (H_3), (H_4), (H_5) *and let* $f \in PAP(\mathbb{R})$. *For every* $\alpha > 0$ *let* $u_\alpha = g_\alpha + \varphi_\alpha$ *be the p.a.p. solution of*

$$\alpha(u+M) + u_t + \mathcal{H}(x,u,Du) = f(t)\,,\ (t,x) \in \mathbb{R}\times\mathbb{R}^N, \tag{7.2}$$

where g_α *and* φ_α *are respectively the a.p. component and the ergodic perturbation of* u_α, *according to Theorem 7.1. Let* $\bar{u}$ *be the limit of* u_α *when* $\alpha \to 0$. *Suppose that*

$$\begin{cases} \lim\limits_{\alpha\to 0} g_\alpha =: \bar{g} \\ \lim\limits_{\alpha\to 0} \varphi_\alpha =: \bar{\varphi} \end{cases} \quad \text{uniformly w.r.t. } (t,x) \in \mathbb{R}\times\mathbb{R}^N.$$

Then $\bar{g} \in AP(\mathbb{R}\times\mathbb{R}^N)$, $\bar{\varphi} \in PAP_0(\mathbb{R}\times\mathbb{R}^N)$, *and* $\bar{u} = (\bar{g}+\bar{\varphi}) \in PAP(\mathbb{R}\times\mathbb{R}^N)$ *is the minimal p.a.p. solution of* (1.2).

Proof. It is easy to prove that $\bar{g} \in AP(\mathbb{R}\times\mathbb{R}^N)$ (see [36, Theorem 2.5]). If

$$\lim_{\alpha\to 0}\sup_{t\in\mathbb{R}}|\varphi_\alpha(t,x)-\bar{\varphi}(t,x)| = 0, \text{ uniformly w.r.t. } x\in\mathbb{R}^N,$$

then, for $\alpha \to 0$

$$\sup_{T>0}\frac{1}{2T}\int_{-T}^{T}|\varphi_\alpha-\bar{\varphi}|\,dt \to 0, \text{ uniformly w.r.t. } x\in\mathbb{R}^N,$$

which implies, when $\alpha \to 0$, that

$$\sup_{T>0}\left|\frac{1}{2T}\int_{-T}^{T}|\varphi_\alpha|\,dt - \int_{-T}^{T}|\bar{\varphi}|\,dt\right| \to 0, \text{ uniformly w.r.t. } x\in\mathbb{R}^N.$$

Let us define

$$f(\alpha,T,x) := \frac{1}{2T}\int_{-T}^{T}|\varphi_\alpha|\,dt,\ f(T,x) := \frac{1}{2T}\int_{-T}^{T}|\bar{\varphi}|\,dt.$$

Then $\lim_{\alpha\to 0} f(\alpha,T,x) = f(T,x)$ uniformly w.r.t. $T>0$ and $x\in\mathbb{R}^N$. As consequence we have

$$\lim_{T\to+\infty} f(T,x) = \lim_{\alpha\to 0}\lim_{T\to+\infty} f(\alpha,T,x) = 0, \text{ uniformly w.r.t. } x\in\mathbb{R}^N.$$

Hence we have $\bar{\varphi} \in PAP_0(\mathbb{R}\times\mathbb{R}^N)$. So we have $\bar{u} = (\bar{g}+\bar{\varphi}) \in PAP(\mathbb{R}\times\mathbb{R}^N)$; following the outline of Section 3, we easily prove that it is the minimal p.a.p. solution of (1.2). □

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