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New Developments on Nirenberg's Problem for Compact Perturbations of Quasimonotone Expansive Mappings in Reflexive Banach Spaces

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Abstract

Let X be a real locally uniformly convex reflexive Banach space with locally uniformly convex dual space X^*. Let $T : X \to X^*$ be demicontinuous, quasimonotone and α-expansive, and $C : X \to X^*$ be compact such that either (i) $\langle Tx + Cx, x\rangle \geq -d\|x\|$ for all $x \in X$ or (ii) $\langle Tx + Cx, x\rangle \geq -d\|x\|^2$ for all $x \in X$ and some suitable positive constants α and d. New surjectivity results are given for the operator $T + C$. The results are new even for $C = \{0\}$, which gives a partial positive answer for Nirenberg's problem for demicontinuous, quasimonotone and α-expansive mapping. Existence result on the surjectivity of quasimonotone perturbations of multivalued maximal monotone operator is included. The theory is applied to prove existence of generalized solution in $H_0^1(\Omega)$ of nonlinear elliptic equation of the type

$$\begin{cases} -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x))) + G_\lambda(x, u(x)) = f(x) & \text{in } \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases}$$

where $f \in L^2(\Omega)$, Ω is a nonempty, bounded and open subset of $\mathbb{R}^N$ with smooth boundary, $\lambda > 0$, $G_\lambda(x,u) = -div(\beta(\nabla u(x))) + \lambda u(x) + a_0(x, u(x), \nabla u(x)) + g(x, u(x))$, $\beta : \mathbb{R}^N \to \mathbb{R}^N$, $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ $(i = 0, 1, 2, ..., N)$ and $g : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfy certain conditions.

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1 Introduction

In what follows, X denotes a real locally uniformly convex Banach space with locally uniformly convex dual space X^*. If X is a real Hilbert space, it is denoted by H. A mapping $T : X \to X^*$ is called α-expansive if there exists $\alpha > 0$ such that

$$\|Tx - Ty\| \geq \alpha\|x - y\| \text{ for all } x \in X \text{ and } y \in X.$$

It is called expansive if $\alpha = 1$. An operator $C : X \to X^*$ is called compact if it is continuous and maps every bounded subset B of X into relatively compact subset of X^*. Nirenberg [34] stated a problem as to whether a continuous expansive mapping from a real Hilbert space H into itself whose range contains an open set is surjective or not. It is not hard to observe that every continuous expansive mapping is injective and $R(T)$ is a closed subset of H. The problem can be solved positively if one can show that $R(T)$ is an open subset of H. Moreover, it is well-known that $R(T)$ is open if H is finite dimensional space as a result of invariance of domain theorem. For the solvability of the problem for $T = I - B$ with B compact or contraction or k-set contraction, the reader is referred to Nussbaum [35]. The reader is referred to the results of Browder [13] and Minty [31] for a positive answer if T is strongly monotone, i.e., there exists $c > 0$ such that

$$\langle Tx - Ty, x - y\rangle \geq c\|x - y\|^2 \text{ for all } x \in H \text{ and } y \in H\ .$$

Chang and Li [19] gave a positive answer for the problem if H is replaced by a Banach space X with an additional assumption that T is Fréchet differentiable at each $x \in X$, i.e., for each $x \in X$, the Frećhet derivative of T at x, $T'(x)$ exists and

$$\limsup_{x \to x_0} \|T'(x) - T'(x_0)\| < 1.$$

A counterexample was given by Morel and Steinlein [32] demonstrating that the problem is not solvable if $T : \ell^1(N) \to \ell^1(N)$ is continuous expansive. Even in a Hilbert space $H = \ell^2(N)$, Szczepanski [36, 37] gave a partial negative answer for the problem. Furthermore, Kartsatos [25] proved that any continuous expansive mapping $T : H \to H$ is surjective provided that, there exists $\alpha \in (0, 1)$ such that

$$\langle Tx - Ty, x - y\rangle \geq -\alpha\|x - y\|^2 \text{ for all } x \in H \text{ and } y \in H \tag{1.1}$$

instead of using the condition $\overset{\circ}{R(T)} \neq \emptyset$. Recently, Xiang [38] gave surjectivity result for an $h-$ expansive mapping with $h > 0$ and such that there exists $c \in (0, \frac{\sqrt{2}}{2}h)$ satisfying

$$\langle Tx - Ty, x - y\rangle \geq -c\|x - y\|^2 \text{ for all } x \in H \text{ and } y \in H \tag{1.2}$$

instead of using the condition $\overset{\circ}{R(T)} \neq \emptyset$. It is worth mentioning that the result of Karstatos [25] is more general than that of Xiang [38] if $h = 1$. The author believes that the result of Kartsatos [25] should have been cited by Xiang [38]. Existence results for single continuous quasimonotone expansive operator defined from a real separable Hilbert space into itself can be found in the paper by Berkovits [7]. For results on Nirenberg-type problems in

Banach lattices, the reader is referred to the paper by A. Duma and I. Duma [20].

In this paper, we give surjectivity results for the operator $T+C$, where $T : X \to X^*$ is demicontinuous, quasimonotone and α-expansive and $C : X \to X^*$ is a compact operator provided that $T+C$ is weakly coercive satisfying one of the following conditions.

(i) $\alpha > 0$ and there exists $d \geq 0$ such that

$$\langle Tx + Cx, x\rangle \geq -d\|x\| \text{ for all } x \in X;$$

It is not difficult to see that any monotone operator $A : X \supseteq D(A) \to X^*$ with $0 \in D(A)$ satisfies (i). Indeed, we see that

$$\langle Ax, x\rangle = \langle Ax - A(0), x\rangle + \langle A(0), x\rangle \geq -\|A(0)\|\|x\|$$

for all $x \in D(A)$. Furthermore, Browder and Hess [14] used (i) as a sufficient condition for regularity and surjectivity of maximal monotone perturbations of regular generalized pseudomonotone operators. For definitions, properties and existence results involving pseudomonotone, generalized pseudomonotone and regular generalized pseudomonotone operators, the reader is referred to the paper of Browder and Hess [14]. One can observe that (i) implies

$$\begin{aligned}\langle Tx + Cx, x\rangle &\geq -d\|x\| \geq -d\Big(\frac{\|x\|^2}{2} + \frac{1}{2}\Big)\\ &\geq -d\|x\|^2 - d \text{ for all } x \in X,\end{aligned}$$

which implies that (i) is some what stronger than condition (ii) below. However, (ii) does not give an ontoness result if $\alpha > 0$ and $d > 0$ are arbitrary as in (i). The shift operator $S : \ell^2(N) \to \ell^2(N)$ given by

$$S(x_1, x_2, ...,) = (0, x_1, x_2, ...)$$

fails to be surjective eventhough it is continuous $\alpha-$expansive mapping with $\alpha = 1$ and satisfies

$$\langle Sx, x\rangle \geq -\|x\|^2 \text{ for all } x = (x_i) \in \ell^2(N).$$

The advantage of condition (i) is that, it provides surjectivity result for compact perturbations of any demicontinuous, quasimonotone and α-expansive mapping without any restriction on the positive constant α. In addition, there are a number differential operators which satisfies condition (ii) below (cf. Theorem 3.1 of this paper). Therefore, the paper addresses, all relevant cases (i), (ii) and (iii).

(ii) $\alpha \geq 1$ and there exists $d \in (0,1)$ such that

$$\langle Tx + Cx, x\rangle \geq -d\|x\|^2 \text{ for all } x \in X;$$

(iii) $\alpha > 0$ and there exists $d \in (0,\alpha)$ such that

$$\langle Tx + Cx, x\rangle \geq -d\|x\|^2 \text{ for all } x \in X.$$

We observe that (i) is stronger than (ii) provided that $\alpha \geq 1$ and $d \in (0,1)$, and (i) is stronger than (iii) if $\alpha > 0$ and $d \in (0,\alpha)$. In a real Hilbert space H, for $C = \{0\}$, it can be easily seen that condition (ii) is weaker than condition (1.1), which was used by Kartsatos [25], and (iii) is weaker than (1.2), used by Xiang [38]. We like to mention here that the restrictions on the expansive constant α and $d > 0$ in conditions (i), (ii) and (iii) are essential.

In Theorem 2.1, we have used condition (i) to give surjectivity result for the operator $T+C$, where $T : X \to X^*$ is demicontinuous, quasimonotone and α-expansive with $\alpha > 0$ arbitrary, and $C : X \to X^*$ is compact such that $T+C$ is weakly coercive, i.e., $\|Tx+Cx\| \to \infty$ as $\|x\| \to \infty$. It is not hard to see that weak coercivity condition is satisfied if $C = \{0\}$ and T is α-expansive. In particular, it is shown that the above inner product condition is sufficient for surjectivity of a demicontinuous, quasimonotone and α-expansive mapping with arbitrary $\alpha > 0$. To the best of the author's knowledge, this result is new even for $C = \{0\}$. Theorem 2.3 provides surjectivity result for weakly coercive operator $T+C$ along with conditions (ii) and (iii). Theorem 2.1 and Theorem 2.3 provide new surjectivity results for compact perturbations of expansive mapping with suitable positive constants α and d. Finally, a surjectivity result is given for operators of the type $\lambda I + N + A + C$, where $N : H \to H$ is Lipschitz quasimonotone, $A : H \supseteq D(A) \to 2^H$ is maximal monotone and $C : H \to H$ is compact under suitable inner product and norm conditions. In particular, existence of solution u_λ in $D(A)$ of an eigenvalue problem of the type

$$\lambda u + Nu + Au + Cu \ni 0, u \in D(A),$$

is given with positive constant λ satisfying suitable condition. As a result, Theorem 2.5 is applied to prove existence of generalized solution for nonlinear elliptic differential equation in appropriate real Hilbert spaces.

In what follows, the following definitions are useful.

Definition 1.1. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be

(i) "monotone" if

$$\langle u^* - v^*, x - y\rangle \geq 0 \text{ for every } (x,y) \in D(T)\times D(T),\ u^* \in Tx \text{ and } v^* \in Ty.$$

(ii) "maximal monotone" if $R(T + \lambda J) = X^*$ for every $\lambda > 0$, where $J : X \to 2^{X^*}$ is the normalized duality mapping given by

$$Jx = \{x^* \in X^* : \langle x^*, x\rangle = \|x^*\|^2 = \|x\|^2\}.$$

This is equivalent to saying that T is maximal monotone if and only if T is monotone and $\langle u^* - u_0^*, x - x_0\rangle \geq 0$ for every $(x,u^*) \in G(T)$ imply $x_0 \in D(T)$ and $u_0^* \in Tx_0$.

(iii) "coercive" if either $D(T)$ is bounded or there exists a function $\psi : [0,\infty) \to (-\infty,\infty)$ such that $\psi(t) \to \infty$ as $t \to \infty$ and

$$\langle y^*, x\rangle \geq \psi(\|x\|)\|x\| \text{ for all } x \in D(T) \text{ and } y^* \in Tx.$$

(iv) " weakly coercive" if either $D(T)$ is bounded or $|Tx| \to \infty$ as $\|x\| \to \infty$, where

$$|Tx| = \inf\{\|v^*\| : x \in D(T), v^* \in Tx\}.$$

Browder and Hess [14] introduced the following definitions of multivalued pseudomonotone and generalized pseudomonotone operators. The original definition of single valued pseudomonotone operators is due to Brézis [10].

Definition 1.2. An operator $T : X \supset D(T) \to 2^{X^*}$ is said to be

(a) "pseudomonotone" if the following conditions are satisfied.

(i) For every $x \in D(T)$, Tx is nonempty, closed, convex and bounded subset of X^*.

(ii) T is finitely continuous, i.e., T is "weakly upper semicontinuous" on each finite-dimensional subspace F of X, i.e., for every $x_0 \in D(T) \cap F$ and every weak neighborhood V of Tx_0 in X^*, there exists a neighborhood U of x_0 in F such that $TU \subset V$.

(iii) For every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$ and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \le 0,$$

we have that for every $x \in D(T)$, there exists $y^*(x) \in Tx_0$ such that

$$\langle y^*(x), x_0 - x\rangle \le \liminf_{n\to\infty} \langle y_n^*, x_n - x\rangle.$$

In particular, replacing x_0 in place of x in the above inequality, pseudomonotonicity of T implies

$$\liminf_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \ge 0.$$

(b) "generalized pseudomonotone" if for every sequence $\{x_n\} \subset D(T)$ and every sequence $\{y_n^*\}$ with $y_n^* \in Tx_n$ such that $x_n \rightharpoonup x_0 \in D(T)$, $y_n^* \rightharpoonup y_0^*$ in X^* and

$$\limsup_{n\to\infty} \langle y_n^*, x_n - x_0\rangle \le 0,$$

we have $y_0^* \in Tx_0$ and $\langle y_n^*, x_n\rangle \to \langle y_0^*, x_0\rangle$ as $n \to \infty$.

(c) "quasimonotone" if (i) and (ii) of (a) hold and for any sequence $\{x_n\}$ in $D(T)$ such that $x_n \rightharpoonup x_0$ in X as $n \to \infty$ and every sequence $\{w_n^*\}$ with $w_n^* \in Sx_n$, we have

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \ge 0.$$

(d) "of type (S_+)" if (i) and (ii) of (a) hold and for any sequence $\{x_n\}$ in $D(T)$ such that $x_n \rightharpoonup x_0$ in X as $n \to \infty$ and every $w_n^* \in Sx_n$ with

$$\limsup_{n\to\infty} \langle w_n^*, x_n - x_0\rangle \le 0,$$

we have $x_n \to x_0 \in D(T)$ and there exists a subsequence denoted again by $\{w_n^*\}$ such that $w_n^* \rightharpoonup w_0^* \in Tx_0$ as $n \to \infty$.

Since X and X^* are locally uniformly convex, it is well-known that J is single valued, bounded, bicontinuous, coercive, maximal monotone and of type (S_+). Furthermore, it is well-known that the class of quasimonotone operators includes the class of pseudomonotone operators. For basic and further properties of monotone, pseudomonotone and generalized pseudomonotone operators, the reader is referred to the books due to Barbu [4, 5], Zeidler [39] and the references therein.

The following basic Lemma is due to Browder and Hess [14, Proposition 15, p. 289].

Lemma 1.3. *Let K be a compact convex subset of X and $T : K \to 2^{X^*}$ an operator such that for every $x \in K$, Tx is a nonempty, closed, convex and bounded subset of X^*. Assume that T is upper semicontinuous with X^* being given its weak topology. Let $f^* \in X^*$. Then there exist elements $x_0 \in K$ and $y_0^* \in Tx_0$ such that*

$$\langle y_0^* - f^*, x - x_0\rangle \le 0 \text{ for all } x \in K.$$

For every $f^* \in X^*$, $-T$ is upper semicontinuous. Thus, Lemma 1.3 implies that there exist $x_0^* \in K$ and $v_0^* \in -Tx_0$ (i.e., $v_0^* = -w_0^*$ for some $w_0^* \in Tx_0$) such that

$$\langle -w_0^* - (-f^*), x - x_0\rangle \le 0 \text{ for all } x \in K,$$

i.e., $\langle w_0^* - f^*, x - x_0\rangle \ge 0$ for all $x \in K$.

The following lemma can be found in Browder [12, Proposition 7.2, p. 81].

Lemma 1.4. *Let X be a reflexive Banach space, A be a nonempty bounded subset of X and $x_0 \in \overline{A^w}$, where $\overline{A^w}$ is the weak closure of A in X. Then there exists a sequence $\{x_n\}$ in A such that $x_n \rightharpoonup x_0$ in X as $n \to \infty$.*

Brézis, Crandall and Pazy [9] gave the following important result for maximal monotone operators from a reflexive Banach space X into its dual space X^*.

Lemma 1.5. *Let B be a maximal monotone set in $X^* \times X$. If $(u_n^*, u_n) \in B$ for all positive integer n such that $u_n \rightharpoonup u$ and $u_n^* \rightharpoonup u^*$ as $n \to \infty$ and either*

$$\limsup_{n,m\to\infty} \langle u_n^* - u_m^*, u_n - u_m\rangle \le 0$$

or

$$\limsup_{n\to\infty} \langle u_n^* - u^*, u_n - u\rangle \le 0,$$

then $(u^*, u) \in B$ and $\langle u_n^*, u_n\rangle \to \langle u^*, u\rangle$ as $n \to \infty$.

2 Main Results

This section addresses the main contributions of the work. Theorem 2.1 gives a new surjectivity result for compact perturbations of demicontinuous, quasimonotone and α-expansive mappings. For $C = \{0\}$, this result provides a partial positive answer for Nirenberg's problem [34]. The importance of Theorem 2.1 over the results of Kartsatos [25] and Xiang [38] is that, it gives an ontoness result for any demicontinuous, quasimonotone and α-expansive mapping with arbitrary $\alpha > 0$ while the result of Kartsatos is for $\alpha = 1$ and $d \in (0,1)$ satisfying (1.1) and result of Xiang [38] is for $\alpha > 0$ and $c \in (0, \frac{\sqrt{2}}{2}\alpha)$ satisfying (1.2). Furthermore, the results due to Kartsatos [25] and Xiang [38] are for single continuous expansive mapping defined from a real Hilbert space H into itself. However, the results in this paper are for compact perturbations of demicontinuous, quasimonotone and α-expansive mapping defined from a real reflexive Banach space into its dual space X^*.

Theorem 2.1. *Let $T : X \to X^*$ be a demicontinuous, quasimonotone and α-expansive mapping with $\alpha > 0$ and $C : X \to X^*$ be a compact operator. Assume, further, that $T + C$ is weakly coercive and there exists $d \geq 0$ such that*

$$\langle Tx + Cx, x\rangle \geq -d\|x\| \text{ for all } x \in X.$$

Then $T + C$ is surjective.

Proof. Let $\varepsilon > 0$ and Λ be the collection of all finite dimensional subspaces of X. For each $F \in \Lambda$, let $j_F : F \to X$ be the inclusion mapping and $j_F^* : X^* \to F$ be the dual projection onto F. For each fixed $\varepsilon > 0$ and each $y^* \in X^*$, we see that

$$\langle Tx + Cx + \varepsilon Jx - y^*, x\rangle \geq \|x\|^2\Big(\varepsilon - \frac{d + \|y^*\|}{\|x\|}\Big)$$

for all $x \in X\backslash\{0\}$. As a result, there exists $R_\varepsilon = R(\varepsilon) > 0$ such that

$$\langle Tx + Cx + \varepsilon Jx - y^*, x\rangle > 0 \tag{2.1}$$

for all $x \in \partial B_{R_\varepsilon}(0)$. Let $K_\varepsilon = \overline{B}_{R_\varepsilon}(0)$ and $K_F^\varepsilon = K_\varepsilon \cap F$. Since K_ε is bounded, K_F^ε is a compact subset of F. The continuity of T implies that $j_F^*(T + C + \varepsilon I)j_F : F \to F$ is continuous. Since $j_F^* y^* \in F$ for any $y^* \in X^*$, using Lemma 1.3, there exists $x_F \in K_F^\varepsilon$ such that

$$\langle j_F^*(Tx_F + Cx_F + \varepsilon Jx_F - y^*), x - x_F\rangle \geq 0$$

for all $x \in K_F^\varepsilon$, which is equivalent to say that

$$\langle Tx_F + Cx_F + \varepsilon Jx_F - y^*, x - x_F\rangle \geq 0$$

for all $x \in K_F^\varepsilon$. Since K_ε is closed, convex and bounded, the family $\{x_F\}_{F\in\Lambda}$ is uniformly bounded and K_ε is weakly compact subset of X. For each $F \in \Lambda$, we define

$$V_F := \bigcup_{F'\in\Lambda,\ F\subset F'} \{x_{F'}\},$$

where $x_{F'} \in K_{F'}^\varepsilon$ satisfies

$$\langle Tx_{F'} + Cx_{F'} + \varepsilon Jx_{F'} - y^*, x - x_{F'}\rangle \geq 0 \text{ for all } x \in K_{K_{F'}}^\varepsilon.$$

We observe that, for every F, $\overline{V_F}^w$ is a weakly closed subset of the weakly compact subset K_ε and the family $\{\overline{V_F}^w\}$ satisfies the finite intersection property. Therefore, we have

$$V := \bigcap_{F\in\Lambda} \overline{V_F}^w \neq \emptyset.$$

Let $x_0^\varepsilon \in V \subseteq K_\varepsilon$ and $x \in K_\varepsilon$. Choose a finite dimensional subspace F_0 containing x and x_0^ε. By using Lemma 1.4, we choose a sequence $\{x_n^\varepsilon\}$ in V_{F_0} such that $x_n^\varepsilon \rightharpoonup x_0^\varepsilon$ as $n \to \infty$. By the definition of V_{F_0}, for each n, we choose F_n such that $F_0 \subseteq F_n$ and $x_n^\varepsilon \in K_{F_n}^\varepsilon$ for all n. Moreover, using the definition of x_n^ε, we have

$$\langle T x_n^\varepsilon + C x_n^\varepsilon + \varepsilon J x_n^\varepsilon, y - x_n^\varepsilon \rangle \geq \langle y^*, y - x_n^\varepsilon \rangle \tag{2.2}$$

for all $y \in K_{F_0}^\varepsilon$ and n. Since $x_0^\varepsilon \in K_{F_0}^\varepsilon$, it follows that

$$\langle T x_n^\varepsilon + C x_n^\varepsilon + \varepsilon J x_n^\varepsilon, x_0^\varepsilon - x_n^\varepsilon \rangle \geq \langle y^*, x_0^\varepsilon - x_n^\varepsilon \rangle$$

for all n. Since C is compact (i.e., quasimonotone) and T is quasimonotone, it follows that

$$\liminf_{n\to\infty} \langle T x_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon \rangle \geq 0 \text{ and } \liminf_{n\to\infty} \langle C x_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon \rangle \geq 0.$$

Consequently, we obtain that

$$\begin{aligned}
\varepsilon \limsup_{n\to\infty} \langle J x_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon \rangle &\leq \limsup_{n\to\infty} \Big(-\langle T x_n^\varepsilon + C x_n^\varepsilon - y^*, x_n^\varepsilon - x_0^\varepsilon \rangle \Big) \\
&= -\liminf_{n\to\infty} \langle T x_n^\varepsilon + C x_n^\varepsilon - y^*, x_n^\varepsilon - x_0^\varepsilon \rangle \\
&\leq -\liminf_{n\to\infty} \langle T x_n^\varepsilon - y^*, x_n^\varepsilon - x_0^\varepsilon \rangle - \liminf_{n\to\infty} \langle C x_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon \rangle \\
&\leq 0
\end{aligned}$$

Since J is of type (S_+), we conclude that $x_n^\varepsilon \to x_0^\varepsilon$ as $n \to \infty$. Since J and C are continuous, and T is demicontinuous, we get $J x_n^\varepsilon \to J x_0^\varepsilon$, $C x_n^\varepsilon \to C x_0^\varepsilon$ and $T x_n^\varepsilon \rightharpoonup T x_0^\varepsilon$ as $n \to \infty$. Letting $n \to \infty$ in (2.2), we arrive at

$$\langle T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon, y - x_0^\varepsilon \rangle \geq \langle y^*, y - x_0^\varepsilon \rangle$$

for all $y \in K_{F_0}^\varepsilon$. Since $x \in K_{F_0}^\varepsilon$, we conclude that

$$\langle T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon, x - x_0^\varepsilon \rangle \geq \langle y^*, x - x_0^\varepsilon \rangle.$$

Since $x \in K_\varepsilon$ is arbitrary, it follows that

$$\langle T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon, x - x_0^\varepsilon \rangle \geq \langle y^*, x - x_0^\varepsilon \rangle \text{ for all } x \in K_\varepsilon. \tag{2.3}$$

By definition of the subdifferential $\partial I_{K_\varepsilon}$ of the indicator function on K_ε, it follows that

$$y^* - (T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon) \in \partial I_{K_\varepsilon}(x_0^\varepsilon)$$

i.e., there exists $u_0^* \in \partial I_{K_\varepsilon}(x_0^\varepsilon)$ such that

$$u_0^* + T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon = y^*.$$

Since $0 \in K_\varepsilon$, using 0 in place of x in (2.3) , the definition of $\partial I_{K_\varepsilon}$ yields $\langle u_0^*, x_0^\varepsilon \rangle \geq 0$, which implies that

$$\langle T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon - y^*, x_0^\varepsilon \rangle \leq 0.$$

If $x_0^\varepsilon \in \partial K_\varepsilon$, from (2.1), we obtain that

$$0 \geq \langle T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon - y^*, x_0^\varepsilon \rangle > 0,$$

i.e., $0 < 0$. However, this is absurd. Therefore, we conclude that $x_0^\varepsilon \in \overset{\circ}{K}_\varepsilon$. Since $\partial I_{K_\varepsilon}(x) = \{0\}$ for all $x \in \overset{\circ}{K}_\varepsilon$, we conclude that $u_0^* = 0$. As a result, the equation

$$T x_0^\varepsilon + C x_0^\varepsilon + \varepsilon J x_0^\varepsilon = y^* \text{ holds.}$$

Thus, for each sequence $\{\varepsilon_n\}$ such that $\varepsilon_n \downarrow 0^+$ as $n \to \infty$, setting $y_n := x_0^{\varepsilon_n}$, we see that

$$T y_n + C y_n + \varepsilon_n J y_n = y^* \text{ for all } n. \tag{2.4}$$

Next we show that $\{y_n\}$ is bounded. Suppose not, i.e., there exists a subsequence, denoted again by $\{y_n\}$, such that $\|y_n\| \to \infty$ as $n \to \infty$. The hypothesis of the theorem implies that

$$\begin{aligned} \varepsilon_n \|y_n\|^2 &= -\langle T y_n + C y_n - y^*, y_n \rangle \\ &\leq (d + \|y^*\|)\|y_n\| \text{ for all } n, \end{aligned}$$

i.e., dividing this inequality by $\|y_n\|$ for all large n, we arrive at $\varepsilon_n \|y_n\| \leq d + \|y^*\|$ for all large n. As a result, we obtain

$$\begin{aligned} \|T y_n + C y_n\| &\leq \varepsilon_n \|y_n\| + \|y^*\| \\ &\leq d + 2\|y^*\| \text{ for all large } n \end{aligned}$$

Since $\|T y_n + C y_n\| \to \infty$ whenever $\|y_n\| \to \infty$, we arrive at $\infty \leq d + 2\|y^*\|$, which is impossible. Therefore, $\{y_n\}$ is bounded. Using the compactness of C, we assume that $C y_n \to y_0^*$ as $n \to \infty$, which implies that $T y_n \to y^* - y_0^*$ as $n \to \infty$. For all positive integers n and m, the α-expansiveness of T implies

$$\alpha \|y_n - y_m\| \leq \|T y_n - T y_m\| \to 0$$

as $n, m \to \infty$, i.e., $\|y_n - y_m\| \to 0$ as $n, m \to \infty$, i.e., $y_n \to y_0 \in X$ as $n \to \infty$. The demicontinuity of T and continuity of C imply that $T y_n \rightharpoonup T y_0$ and $C y_n \to C y_0$ as $n \to \infty$. Finally, letting $n \to \infty$ in (2.4), we obtain that

$$T y_0 + C y_0 = y^*.$$

Since $y^* \in X^*$ is arbitrary, the surjectivity of $T + C$ follows. The proof is complete. □

The following corollary is immediate consequence of Theorem 2.1.

Corollary 2.2. *Let $T : X \to X^*$ be a demicontinuous, quasimonotone and α-expansive mapping with $\alpha > 0$. Assume, further, that there exists $d \geq 0$ such that*

$$\langle T x, x \rangle \geq -d\|x\| \text{ for all } x \in X.$$

Then T is surjective.

Proof. The proof follows by setting $C = \{0\}$ in Theorem 2.1 because $\|T x\| \to \infty$ as $\|x\| \to \infty$ as a consequence of expansiveness of T. □

The author would like to mention here that Corollary 2.2 gives a partial positive answer for Nirenberg's problem for arbitrary demicontinuous, quasimonotone and α-expansive mapping under the given inner product condition in any reflexive Banach space X.

In Theorem 2.3 below, we give a new result for compact perturbations of demicontinuous, quasimonotone and α-expansive mapping $T : X \to X^*$ under a more general inner product condition, which is weaker than the condition used by Kartsatos [25] and Xiang [38].

Theorem 2.3. *Let $T : X \to X^*$ be a demicontinuous, quasimonotone and α-expansive mapping and $C : X \to X^*$ be a compact operator. Assume, further, that*

(i) (a) $\alpha \geq 1$ and there exists $d \in (0,1)$ or (b) $\alpha > 0$ and there exists $d \in (0,\alpha)$ such that

$$\langle Tx + Cx, x\rangle \geq -d\|x\|^2 \quad \text{for all } x \in X; \tag{2.5}$$

(ii) there exists $\mu \geq 0$ such that $\|Tx + Cx\| \geq \alpha\|x\| - \mu$ for all $x \in X$.

Then $T + C$ is surjective.

Proof. Suppose (i) (a) (i.e., $\alpha \geq 1$ and $d \in (0,1)$) and (ii) hold. Let $\varepsilon > 0$ and $\tilde{J}x = \|x\|Jx$, $x \in X$, where $J : X \to X^*$ is the normalized duality mapping. Then $\tilde{J}$ is continuous from X into X^*. Indeed, if for any sequence $\{x_n\}$ in X such that $x_n \to x_0$ as $n \to \infty$, it follows that $\|x_n\| \to \|x_0\|$ as $n \to \infty$. By the continuity of J, we obtain that

$$\begin{aligned} \|\tilde{J}x_n - \tilde{J}x_0\| &= \|\|x_n\|Jx_n - \|x_0\|Jx_0\| \\ &= \|(\|x_n\| - \|x_0\|)Jx_n + \|x_0\|(Jx_n - Jx_0)\| \\ &\leq \|x_n\|\|\|x_n\| - \|x_0\|\| + \|x_0\|\|Jx_n - Jx_0\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, the operator $T + C + \varepsilon\tilde{J}$ is demicontinuous from X into X^*. Furthermore, using $\tilde{J}$ and (2.5), we see that

$$\begin{aligned} \langle Tx + Cx + \varepsilon\tilde{J}x, x\rangle &= \langle Tx + Cx + \varepsilon\|x\|Jx, x\rangle \\ &\geq \varepsilon\|x\|^3 - d\|x\|^2 \\ &= \|x\|^3\left(\varepsilon - \frac{d}{\|x\|}\right) \text{ for all } x \in X\backslash\{0\}. \end{aligned}$$

Thus, there exists $R_\varepsilon = R(\varepsilon) > 0$ such that

$$\langle Tx + Cx + \varepsilon\tilde{J}x, x\rangle > 0$$

for all $x \in \partial B_{R_\varepsilon}(0)$. For each $y^* \in X^*$, using the finite dimensional argument used in the proof of Theorem 2.1, we conclude that there exists a sequence $\{x_n^\varepsilon\}$ in X such that $x_n^\varepsilon \rightharpoonup x_0^\varepsilon$ as $n \to \infty$ and

$$\langle Tx_n^\varepsilon + Cx_n^\varepsilon + \varepsilon\|x_n^\varepsilon\|Jx_n^\varepsilon, x_0^\varepsilon - x_n^\varepsilon\rangle \geq \langle y^*, x_0^\varepsilon - x_n^\varepsilon\rangle \text{ for all } n. \tag{2.6}$$

Since T and C are quasimonotone and $x_n^\varepsilon \rightharpoonup x_0^\varepsilon$ as $n \to \infty$, we see that

$$\begin{aligned}\liminf_{n\to\infty}\langle Tx_n^\varepsilon + Cx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle &\geq \liminf_{n\to\infty}\langle Tx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle + \liminf_{n\to\infty}\langle Cx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle \\ &\geq 0\end{aligned}$$

As a result, using this inequality along with (2.6), we obtain that

$$\limsup_{n\to\infty}\|x_n^\varepsilon\|\langle Jx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle \leq 0.$$

If $x_0^\varepsilon = 0$, then we have

$$\limsup_{n\to\infty}\|x_n^\varepsilon\|\langle Jx_n^\varepsilon, x_n^\varepsilon\rangle = \limsup_{n\to\infty}\|x_n^\varepsilon\|^3 \leq 0,$$

$x_n^\varepsilon \to 0$ as $n \to \infty$. Assume $x_0^\varepsilon \neq 0$, i.e., there exists a subsequence, denoted again by $\{x_n^\varepsilon\}$, such that $\|x_n^\varepsilon\| \geq Q$ for all n and some $Q > 0$. Assume without loss of generality that $\|x_n^\varepsilon\| \to a_\varepsilon > 0$ as $n \to \infty$. This implies that

$$\begin{aligned}\limsup_{n\to\infty}\langle Jx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle &= \limsup_{n\to\infty}\frac{1}{\|x_n^\varepsilon\|}\langle\|x_n^\varepsilon\|Jx_n^\varepsilon, x_n - x_0\rangle \\ &= \frac{1}{a_\varepsilon}\limsup_{n\to\infty}\langle\|x_n^\varepsilon\|Jx_n^\varepsilon, x_n^\varepsilon - x_0^\varepsilon\rangle \\ &\leq 0\end{aligned}$$

Since J is of type (S_+), we conclude that $x_n^\varepsilon \to x_0^\varepsilon$ as $n \to \infty$. Since T is demicontinuous and, $\tilde{J}$ and C are continuous, it follows that $Tx_n^\varepsilon \rightharpoonup Tx_0^\varepsilon$, $\tilde{J}x_n^\varepsilon \to \tilde{J}x_0^\varepsilon$ and $Cx_n^\varepsilon \to Cx_0^\varepsilon$ as $n \to \infty$, respectively. In addition, we get $\langle Tx_n^\varepsilon, x_n^\varepsilon\rangle \to \langle Tx_0^\varepsilon, x_0^\varepsilon\rangle$, $\langle\tilde{J}x_n^\varepsilon, x_n^\varepsilon\rangle \to \langle\tilde{J}x_0^\varepsilon, x_0^\varepsilon\rangle$ and $\langle Cx_n^\varepsilon, x_n^\varepsilon\rangle \to \langle Cx_0^\varepsilon, x_0^\varepsilon\rangle$ as $n \to \infty$, i.e., it follows that

$$\langle Tx_n^\varepsilon + Cx_n^\varepsilon + \tilde{J}x_n^\varepsilon, x_n^\varepsilon\rangle \to \langle Tx_0^\varepsilon + Cx_0^\varepsilon + \tilde{J}x_0^\varepsilon, x_0^\varepsilon\rangle \text{ as } n \to \infty.$$

Following the arguments used in the proof of Theorem 2.1, using $K_\varepsilon = \overline{B}_\varepsilon(0)$, for each $y^* \in X^*$ and $\varepsilon > 0$, we arrive at

$$Tx_0^\varepsilon + Cx_0^\varepsilon + \varepsilon\tilde{J}x_0^\varepsilon = y^*.$$

Then for each $\varepsilon_n \downarrow 0^+$ as $n \to \infty$, there exists $y_n := x_0^{\varepsilon_n}$ in X such that

$$Ty_n + Cy_n + \varepsilon_n\|y_n\|y_n = y^* \text{ for all } n, \text{ where } y_n := x_0^{\varepsilon_n}. \tag{2.7}$$

Next we show that $\{y_n\}$ is bounded. To this end, we assume, there exists a subsequence, denoted again by $\{y_n\}$, such that $\|y_n\| \to \infty$ as $n \to \infty$. Thus, using the inner product condition in the hypothesis of the theorem together with (2.7), we arrive at

$$\begin{aligned}\varepsilon_n\|y_n\|^3 &= -\langle Ty_n + Cy_n, y_n\rangle + \langle y^*, y_n\rangle \\ &\leq d\|y_n\|^2 + \|y^*\|\|y_n\| \text{ for all } n.\end{aligned}$$

Dividing both sides of this inequality by $\|y_n\|$ for all large n yields

$$\varepsilon_n\|y_n\| \le d + \frac{\|y^*\|}{\|y_n\|},$$

i.e., $\limsup_{n\to\infty} \varepsilon_n\|y_n\| \le d < 1$. Furthermore, condition (ii) together with (2.7) imply

$$\begin{aligned} -\mu + \alpha\|y_n\| &\le \|Ty_n + Cy_n\| \\ &\le \varepsilon_n\|y_n\|^2 + \|y^*\| \text{ for all } n. \end{aligned}$$

Dividing this inequality by $\|y_n\|$ for all large n, we arrive at

$$1 \le \alpha \le \limsup_{n\to\infty}\Big(\varepsilon_n\|y_n\| + \frac{\|y^*\|}{\|y_n\|}\Big) \le d < 1,$$

which is impossible. Therefore, the sequence $\{y_n\}$ is bounded. Since C is compact, there exists a subsequence, denoted again by $\{y_n\}$, such that $Cy_n \to h_0^*$ as $n \to \infty$. As a result from (2.7), we obtain that

$$Ty_n \to y^* - h_0^* \text{ as } n \to \infty.$$

Since T is α-expansive, it follows that

$$\alpha\|y_n - y_m\| \le \|Ty_n - Ty_m\| \to 0 \text{ as } n, m \to \infty,$$

i.e., $\{y_n\}$ is a Cauchy sequence in X. This gives $y_n \to y_0$ in X as $n \to \infty$. Since T is demicontinuous and C is continuous, we conclude that $Ty_n \rightharpoonup Ty_0$ and $Cy_n \to Cy_0$ as $n \to \infty$. Finally, letting $n \to \infty$ in (2.7), we arrive at

$$Ty_0 + Cy_0 = y^*.$$

Since $y^* \in X^*$ is arbitrary, we conclude that $T + C$ is surjective. The proof of the theorem using conditions (i) (b) and (ii) follows a similar argument. The details are omitted here. □

The following corollary is a consequence of Theorem 2.3.

Corollary 2.4. *Let $T : X \to X^*$ be a demicontinuous, quasimonotone and α-expansive mapping such that (i) of Theorem 2.3 holds with positive constants α and d satisfying $\langle Tx, x\rangle \ge -d\|x\|^2$ for all $x \in X$. Then T is surjective.*

Proof. We notice that condition (ii) of Theorem 2.3 is not required because of the expansiveness of T. Indeed, for some fixed $u_0 \in X$, by the hypothesis that T is α-expansive, we see that

$$\begin{aligned} \|Tx\| &= \|Tx - Tu_0 + Tu_0\| \\ &\ge \|Tx - Tu_0\| - \|Tu_0\| \\ &\ge \alpha\|x - u_0\| - \|Tu_0\| \\ &\ge \alpha\|x\| - (\alpha\|u_0\| + \|Tu_0\|) \\ &= \alpha\|x\| - \mu, \text{ where } \mu = \alpha\|u_0\| + \|Tu_0\| \text{ for all } x \in X, \end{aligned} \tag{2.8}$$

i.e., condition (ii) of Theorem 2.3 is satisfied. Thus, the proof follows by setting $C = 0$ in Theorem 2.3. □

Corollary 2.4 gives a positive answer for Nirenberg's problem under the given inner product condition. The stronger inner product conditions used by Kartsatos [25] and Xiang [38] are weakened for demicontinuous, quasimonotone and α-expansive mappings defined from a reflexive Banach space X into its dual space X^*.

If $X = H$, a real Hilbert space, the following theorem gives surjectivity result for operators of the type $\lambda I + N + A + C$, where $N : H \to H$ is Lipschitz quasimonotone, $A : H \supseteq D(H) \to 2^H$ is maximal monotone and $C : H \to H$ is compact, possibly multivalued and $\lambda > 0$ is a positive constant. For $f = 0$, Theorem 2.5 gives solvability of an eigenvalue problem

$$\lambda u + Nu + Au + Cu \ni 0 \text{ in } D(A)$$

for an appropriate eigenvalue $\lambda > 0$.

Theorem 2.5. *Let $A : H \supseteq D(A) \to 2^H$ be maximal monotone with $0 \in A(0)$, $N : H \to H$ be Lipschitiz quasimonotone with Lipschitz constant $\ell > 0$. Let $C : H \to H$ be compact such that there exists $k > 0$ satisfying $\|Cx\| \le k\|x\|$ for all $x \in H$. Assume, further, that there exists $\tilde{d} > 0$ such that*

$$\langle Nx + Cx, x\rangle \ge -\tilde{d}\|x\|^2 \text{ for all } x \in H. \tag{2.9}$$

If $0 < \tau^* < \tilde{d}$, where $\tau^* = max\{\ell + k, \frac{\tilde{d}+\ell}{2}\}$, then for any $\lambda \in (\tau^*, \tilde{d})$, the operator $\lambda I + N + A + C$ is surjective.

Proof. For each $\varepsilon > 0$, let A_ε be the Yosida approximant of A. It is well-known that $A_\varepsilon : H \to H$ is bounded, continuous and monotone. Since $0 \in A(0)$, it follows that $A_\varepsilon(0) = 0$ for all $\varepsilon > 0$. For each $\lambda \in (\tau^*, \tilde{d})$, the monotonicity of A_ε implies that

$$\langle \lambda(x - y) + A_\varepsilon x - A_\varepsilon y, x - y\rangle \ge \lambda\|x - y\|^2$$

for all x, y in H, i.e.,

$$\|(\lambda x + A_\varepsilon x) - (\lambda y + A_\varepsilon y)\| \ge \lambda\|x - y\|$$

for all $x \in H$ and $y \in H$, i.e., $\lambda I + A_\varepsilon$ is continuous, monotone and λ-expansive. On the other hand,

$$\begin{aligned}\|(\lambda x + A_\varepsilon x + Nx) - (\lambda y + A_\varepsilon y + Ny)\| &\ge \|(\lambda x + A_\varepsilon x) - (\lambda y + A_\varepsilon y)\| \\ &\quad - \|Nx - Ny\| \\ &\ge (\lambda - \ell)\|x - y\|\end{aligned}$$

for all $x \in H$ and $y \in H$. Since $\lambda > \ell + k$, it follows that, for each $\varepsilon > 0$, the operator $\lambda I + N + A_\varepsilon$ is continuous, quasimonotone and α-expansive with $\alpha = \lambda - \ell > 0$. The inner product condition on $N + C$ implies

$$\begin{aligned}\langle \lambda x + Nx + A_\varepsilon x + Cx, x\rangle &\ge \lambda\|x\|^2 - \tilde{d}\|x\|^2 \\ &= -(\tilde{d} - \lambda)\|x\|^2 \text{ for all } x \in H.\end{aligned}$$

We observe that $\tilde{d} - \lambda > 0$ because $\lambda < \tilde{d}$. Furthermore, we have

$$\begin{aligned}\|\lambda x + Nx + A_\varepsilon x + Cx\| &\ge \|\lambda x + A_\varepsilon x + Nx\| - \|Cx\| \\ &\ge (\lambda - \ell)\|x\| - \|Cx\| \\ &\ge (\lambda - (\ell + k))\|x\| \text{ for all } x \in H.\end{aligned}$$

Since $\lambda > \frac{\tilde{d}+\ell}{2}$, i.e., $0 < \tilde{d} - \lambda < \lambda - \ell$, using $\lambda I + N + A_\varepsilon$ in place of T, $\lambda - \ell$ in place of α and $\tilde{d} - \lambda$ in place of d and using conditions (i) (b) and (ii) of Theorem 2.3, we conclude that, for each λ as in the hypothesis and $\varepsilon > 0$, the operator $\lambda I + A_\varepsilon + N + C$ is surjective, i.e., for each $f \in H$ and $\varepsilon_n \downarrow 0^+$, there exists $x_n \in H$ such that

$$\lambda x_n + N x_n + A_{\varepsilon_n} x_n + C x_n = f \tag{2.10}$$

for all n. Using (2.10), we get that

$$(\lambda - (\ell + k))\|x_n\| \leq \|\lambda x_n + N x_n + A_{\varepsilon_n} x_n + C x_n\| \leq \|f\|$$

for all n, which implies the boundedness of $\{x_n\}$. Since N is Lipschitz mapping, we see that

$$\|N x_n\| \leq \|N x_n - N0\| + \|N0\| \leq \ell \|x_n\| + \|N0\|$$

for all n, which implies the boundedness of $\{N x_n\}$. Since C is compact, there exists a subsequence, denoted again by $\{x_n\}$, such that $C x_n \to y_0$ as $n \to \infty$. Thus the sequence $\{A_{\varepsilon_n} x_n\}$ is bounded. Let $v_n^* = A_{\varepsilon_n} x_n$ for all n. Assume by passing into subsequences that $x_n \rightharpoonup x_0$ and $v_n^* \rightharpoonup v_0^*$ as $n \to \infty$. Let J_{ε_n} be the Yosida resolvent of A. Using the properties of the Yosida approximant and resolvent of A, it is well-known that $J_{\varepsilon_n} x_n \in D(A)$, $A_{\varepsilon_n} x_n \in A(J_{\varepsilon_n} x_n)$ and $J_{\varepsilon_n} x_n = x_n - \varepsilon_n A_{\varepsilon_n} x_n$ for all n. By (2.10), we have

$$\begin{aligned}
\limsup_{n\to\infty} \langle A_{\varepsilon_n} x_n, x_n - x_0 \rangle &\leq \limsup_{n\to\infty} \Big(-\langle \lambda x_n + N x_n + C x_n, x_n - x_0 \rangle \Big) \\
&= -\liminf_{n\to\infty} \langle \lambda x_n + N x_n + C x_n, x_n - x_0 \rangle \\
&\leq -\liminf_{n\to\infty} \langle \lambda x_n, x_n - x_0 \rangle - \liminf_{n\to\infty} \langle N x_n, x_n - x_0 \rangle \\
&\quad - \liminf_{n\to\infty} \langle C x_n, x_n - x_0 \rangle.
\end{aligned}$$

Since N is quasimonotone and C is compact, we have

$$\liminf_{n\to\infty} \langle N x_n, x_n - x_0 \rangle \geq 0 \text{ and} \liminf_{n\to\infty} \langle C x_n, x_n - x_0 \rangle \geq 0.$$

As a result, we get

$$\limsup_{n\to\infty} \langle A_{\varepsilon_n} x_n, x_n - x_0 \rangle \leq 0.$$

Moreover, it follows that

$$\begin{aligned}
\limsup_{n\to\infty} \langle A_{\varepsilon_n} x_n, J_{\varepsilon_n} x_n - x_0 \rangle &= \limsup_{n\to\infty} \langle A_{\varepsilon_n} x_n, (J_{\varepsilon_n} x_n - x_n) + (x_n - x_0) \rangle \\
&\leq \limsup_{n\to\infty} (-\varepsilon_n \|A_{\varepsilon_n} x_n\|^2) + \limsup_{n\to\infty} \langle A_{\varepsilon_n} x_n, x_n - x_0 \rangle \\
&\leq 0.
\end{aligned}$$

Consequently, using the maximal monotonicity of A along with Lemma 1.5, we conclude that $x_0 \in D(A)$ and $v_0^* \in A x_0$ and $\langle A_{\varepsilon_n} x_n, x_n \rangle \to \langle v_0^*, x_0 \rangle$ as $n \to \infty$. Thus, using the quasimonotonicity of N and C, we obtain that

$$\limsup_{n\to\infty} \langle x_n, x_n - x_0 \rangle \leq 0,$$

which implies

$$\limsup_{n\to\infty} \|x_n\| \le \|x_0\|.$$

As a result, the uniform convexity of H implies $x_n \to x_0$ as $n \to \infty$, which again gives $Nx_n \to Nx_0$ and $Cx_n \to Cx_0$ as $n \to \infty$. Finally, letting $n \to \infty$ in (2.10), we conclude that

$$\lambda x_0 + Nx_0 + Cx_0 + v_0^* = f.$$

Thus, for every $f \in H$, the inclusion problem

$$\lambda u + Nu + Cu + Au \ni f,\ u \in D(A)$$

is solvable, i.e., for each λ as in the hypothesis, $R(\lambda I + N + C + A) = H$. The proof is complete. □

For more references on surjectivity of perturbations of multivalued maximal monotone operators under a certain type of coercivity assumptions, we advise the reader to refer Browder and Hess [14], Brézis [10], Brézis and Nirenberg [11], Kenmochi [26, 27, 28], Le [30], Guan, Kartsatos and Skrypnik [22], Guan and Kartsatos [23] and the references therein. Recent results on topological degree and variational inequality theories for multivalued pseudomonotone perturbations of maximal monotone operators can be found in the papers due to Asfaw and Kartsatos [1, 2, 3] and the references therein. Various examples of single valued and/or multivalued operators of pseudomonotone type can be found in the paper by Kenmochi [28], Carl, Le and Motreanu [16], Carl [17] and Carl and Motreanu [18].

3 Application to elliptic equations

Let Ω be a nonempty, bounded and open subset of $\mathbb{R}^N$ with smooth boundary. Let $g : \Omega\times\mathbb{R} \to \mathbb{R}$ be Carathèodory function, i.e., for each fixed $u \in \mathbb{R}$, the function $x \longmapsto g(x,u)$ is measurable and for almost every $x \in \Omega$, the function $u \longmapsto g(x,u)$ is continuous. Assume, further, that

(C1) there exists $k_1 > 0$ such that

$$|g(x,s)| \le k_1|s| \text{ for all } s \in \mathbb{R} \text{ and almost all } x \in \Omega;$$

(C2) there exists $\tau \ge 0$ such that

$$g(x,s)s \ge -\tau|s|^2 \text{ for all } s \in \mathbb{R} \text{ and almost all } x \in \Omega.$$

(C 3) there exist a continuous monotone function $\beta : \mathbb{R}^N \to \mathbb{R}^N$ and $C_1 > 0$ such that

$$|\beta(r)| \le C_1(1 + |r|) \text{ for all } r \in \mathbb{R}^N.$$

Let $H = H_0^1(\Omega)$. Define the mapping $C : H \to H$ by

$$\langle Cu, \phi \rangle := \int_\Omega g(x, u(x))\phi(x)dx, \; u \in H, \phi \in H. \tag{3.1}$$

Using the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$, it follows that C is a compact operator. For examples of single valued as well as multivalued differential operators, the reader is referred to the paper of Browder [15], Hu and Papageorgiou [24], Berkovits [6], Kobayash and Otani [29], Fitzpatrick and Petryshyn [21] and the references therein.

Suppose that, for each $i = 0, 1, 2, ..., N$, the function $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ satisfies the following conditions.

(A1) $a_i(x, s, \xi)$ satisfies the Carathéodory conditions, i.e., for each $(x, \xi) \in \mathbb{R} \times \mathbb{R}^N$, the function $x \longmapsto a_i(x, s, \xi)$ is measurable and for almost all $x \in \Omega$, the function $(s, \xi) \longmapsto a_i(x, s, \xi)$ is continuous. For each $i = 0, 1, 2, ..., N$, there exist constants $c_i > 0$ such that

$$|a_i(x, \eta, \xi) - a_i(x, \eta', \xi')| \le c_i(|\eta - \eta'| + |\xi - \xi'|)$$

a.e. for $x \in \Omega$, and for all (η, ξ) and (η', ξ') in $\mathbb{R} \times \mathbb{R}^N$, where $|\xi - \xi'|$ denotes the norm of $\xi - \xi'$ in $\mathbb{R}^N$.

(A2) The functions a_i $(i = 0, 1, 2, 3, ., ., ., N)$ satisfy a monotonicity condition with respect to ξ in the form

$$\sum_{i=1}^{N} (a_i(x, s, \xi) - a_i(x, s, \xi'))(\xi - \xi') > 0$$

for a.e. $x \in \Omega$, and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

(A3) There exist $\nu \ge 0$, $\mu \ge 0$ such that

$$\sum_{i=1}^{N} a_i(x, s, \xi)\xi_i \ge -\nu(|\xi|^2 + |s|^2) \text{ and } a_0(x, s, \xi)s \ge -\mu(|\xi|^2 + |s|^2)$$

for a.e. $x \in \Omega$ and all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$.

We consider a second-order elliptic differential operator of the form

$$Bu(x) = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u, \nabla u(x)) + a_0(x, u(x), \nabla u(x)), \; x \in \Omega, \; u \in H,$$

where $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_N}\right)$. The operator B generates an operator $\tilde{N} : H \to H$ given by

$$\langle \tilde{N}u, \varphi \rangle = \int_\Omega \left(\sum_{i=1}^{N} a_i(x, u, \nabla u) \frac{\partial \varphi}{\partial x_i} + a_0(x, u(x), \nabla u(x))\varphi(x) \right) dx \tag{3.2}$$

for all $u \in H$ and $\varphi \in H$. It is well known that under conditions (A1) and (A2) the operator $\tilde{N}$ is pseudomonotone. For further details and more examples , the reader is referred to the recent paper by Mustonen [33] and the paper and handbook of Kenmochi [27, 28]. We demonstrate the applicability of the theory for existence of weak solution of an elliptic differential equation given by

$$\begin{cases} -\sum\limits_{i=1}^{N} \frac{\partial}{\partial x_i} a_i(x, u(x), \nabla u(x))) + G_\lambda(x, u(x)) = f(x) & \text{in } \Omega \\ u(x) = 0 & x \in \partial\Omega, \end{cases} \tag{3.3}$$

where $f \in L^2(\Omega)$, $G_\lambda(x,u) = -div(\beta(\nabla u(x))) + \lambda u(x) + a_0(x, u(x), \nabla u(x)) + g(x, u(x))$ and β satisfies condition (C_3).

It is well-known that the divergence operator $-div(\beta(\nabla u(x)))$ generates a continuous maximal monotone operator $A : H \to H$ given by

$$\langle Au, \varphi \rangle = \int_\Omega \beta(\nabla u(x)) \nabla \varphi(x) dx, u \in H, \varphi \in H. \tag{3.4}$$

For further details and more examples, the reader is referred to Barbu [4, 5], Kenmoch [27, 28], Browder [13, 12, 15], Zeidler [39] and the references therein. Existence of weak solution for (3.3) is to mean finding $u \in H$ such that

$$\int_\Omega (G_\lambda(x, u(x)) - f)\varphi(x)dx + \sum_{i=1}^{n} \int_\Omega a_i(x, u(x), \nabla u(x)) \frac{\partial}{\partial x_i} \varphi(x)dx = 0 \tag{3.5}$$

for all $\varphi \in H$.

The following theorem demonstrates the applicability of the results to nonlinear elliptic boundary value problems of the type (3.3).

Theorem 3.1. *Let Ω be a nonempty, bounded and open subset of $\mathbb{R}^N$ with smooth boundary. Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be Carathéodory function satisfying conditions* (C1) *and* (C2). *Assume, further, that* (C3) *holds and the functions $a_i : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ ($i = 0, 1, 2, 3, ..., N$) are Carathéodory functions satisfying conditions* (A1) *through* (A3). *Then for each $f \in L^2(\Omega)$, equation* (3.3) admits a weak solution in $H_0^1(\Omega)$.

Proof. We consider the Hilbert space $H = H_0^1(\Omega)$, which is a closed subspace of the Sobolev space $H^1(\Omega)$, where $H^1(\Omega) = \{u \in L^2(\Omega) : \frac{\partial u}{\partial x_i} \in L^2(\Omega)$ for all $i = 1, 2, ..., N\}$, $\frac{\partial u}{\partial x_i}$ is the distributional derivative of u. The norm of $u \in H$ is given by

$$\|u\| = \|u\|_{L^2(\Omega)} + \sum_{i=1}^{N} \left\| \frac{\partial u}{\partial x_i} \right\|_{L^2(\Omega)}.$$

Let C, $\tilde{N}$ and A be as defined in (3.1), (3.2) and (3.4), respectively. It is well-known that $A : H \to H$ is maximal monotone (cf. Barbu [4, 5]). The operator $C : H \to H$ is compact(cf. Fitzpatrick and Petryshyn [21]). Furthermore, in the recent paper of Mustonnen [33], it is shown that $\tilde{N} : H \to H$ is bounded, continuous, nonmonotone and pseudomonotone, i.e., continuous quasimonotone. For all $u \in H$ and $\varphi \in H$, using Hólder's inequality, we see that

$$\begin{aligned}
\|Cu\| &= \sup_{\|\varphi\|=1} |\langle Cu, \varphi\rangle| \\
&= \sup_{\|\varphi\|=1} \Big| \int_\Omega k_1 u(x)\varphi(x)dx \Big| \\
&\le \sup_{\|\varphi\|=1} \int_\Omega |k_1 u(x)||\varphi(x)|dx \\
&\le \sup_{\|\varphi\|=1} (k_1 \|u\|_{L^2(\Omega)} \|\varphi\|_{L^2(\Omega)} \\
&\le \sup_{\|\varphi\|=1} k_1 \|u\| \|\varphi\| = k_1 \|u\|,
\end{aligned}$$

i.e., $\|Cu\| \le k_1\|u\|$ for all $u \in H$. Next we show that the inner product condition in Theorem 2.5 is satisfied. To this end, by conditions (A3) and (C2), we obtain that

$$\begin{aligned}
\langle \tilde{N}u + Cu, u\rangle &= \sum_{i=1}^{N} \int_\Omega a_i(x, u(x), \nabla u(x)) \frac{\partial u(x)}{\partial x_i} dx + \int_\Omega a_0(x, u(x), \nabla u(x)) u(x) dx \\
&+ \int_\Omega g(x, u(x)) u(x) dx \\
&\ge -(\nu + \mu) \int_\Omega (|u|^2 + |\nabla u|^2) dx - \tau \int_\Omega |u|^2 dx \\
&\ge -(\nu + \tau + \mu) \int_\Omega (|u|^2 + |\nabla u|^2) dx \\
&\ge -\tilde{d}\|u\|^2 \text{ for all } u \in H,
\end{aligned}$$

where $\tilde{d} = \nu + \mu + \tau$, i.e., $\langle \tilde{N}u + Cu, u\rangle \ge -\tilde{d}\|u\|^2$ for all $u \in H$. Finally, we show that $\tilde{N}$ is Lipschitz continuous. To this end, for all $u \in H$, $v \in H$ and $\varphi \in H$, we get the estimate

$$\begin{aligned}
|\langle \tilde{N}u - \tilde{N}v, \varphi\rangle| &\le \int_\Omega \sum_{i=1}^{N} |a_i(x, u(x), \nabla u(x)) - a_i(x, v(x), \nabla v(x))| \Big| \frac{\partial \varphi(x)}{\partial x_i} \Big| dx \\
&+ \int_\Omega |a_0(x, u(x), \nabla u(x)) - a_0(x, v(x), \nabla v(x))||\varphi(x)| dx \\
&\le \sum_{i=1}^{N} c_i \int_\Omega \Big[|u(x) - v(x)| + |\nabla u(x) - \nabla v(x)| \Big] \Big| \frac{\partial \varphi(x)}{\partial x_i} \Big| dx \\
&+ \int_\Omega \Big[|u(x) - v(x)| + |\nabla u(x) - \nabla v(x)| \Big] |\varphi(x)| dx \\
&= \sum_{i=1}^{N} c_i \Big(\int_\Omega |u(x) - v(x)| \Big| \frac{\partial \varphi(x)}{\partial x_i} \Big| dx + \int_\Omega |\nabla u(x) - \nabla v(x)| \Big| \frac{\partial \varphi(x)}{\partial x_i} dx \Big) \\
&+ \int_\Omega |u(x) - v(x)||\varphi(x)| dx + \int_\Omega |\nabla u(x) - \nabla v(x)||\varphi(x)| dx.
\end{aligned}$$

Next employing Hólder's and Minkowski's inequalities, and the definition of $\|u-v\|$ and $\|\varphi\|$ in H, we obtain

$$\begin{aligned}\sum_{i=1}^{N} c_i \int_\Omega |(\nabla u)(x)-(\nabla v)(x)|\Big|\frac{\partial\varphi}{\partial x_i}\Big|dx &= \sum_{i=1}^{N} c_i \int_\Omega \bigg(\sum_{j=1}^{N}\Big|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\Big|^2\bigg)^{\frac12}\Big|\frac{\partial\varphi}{\partial x_i}\Big|dx\\
&\le \sum_{i=1}^{N} c_i \int_\Omega \sum_{j=1}^{N}\Big|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\Big|\Big|\frac{\partial\varphi}{\partial x_i}\Big|dx\\
&\le \sum_{i=1}^{N}\bigg[c_i\bigg(\int_\Omega\bigg(\sum_{j=1}^{N}\Big|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\Big|\bigg)^2dx\bigg)^{\frac12}\\
&\quad\times\bigg(\int_\Omega\Big|\frac{\partial\varphi}{\partial x_i}\Big|^2dx\bigg)^{\frac12}\bigg]\\
&= \sum_{i=1}^{N} c_i\bigg\|\sum_{j=1}^{N}\Big|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\Big|\bigg\|_{L^2(\Omega)}\Big\|\frac{\partial\varphi}{\partial x_i}\Big\|_{L^2(\Omega)}\\
&\le \sum_{i=1}^{N} c_i\sum_{j=1}^{N}\Big\|\frac{\partial u}{\partial x_j}-\frac{\partial v}{\partial x_j}\Big\|_{L^2(\Omega)}\Big\|\frac{\partial\varphi}{\partial x_i}\Big\|_{L^2(\Omega)}\\
&\le \bigg(\sum_{i=1}^{N} c_i\bigg)\|u-v\|\|\varphi\|\end{aligned}$$

for all $u\in H$, $v\in H$ and $\varphi\in H$. Similarly, by analogous argument, we obtain

$$\sum_{i=1}^{N} c_i\int_\Omega |u(x)-v(x)|\Big|\frac{\partial\varphi}{\partial x_i}\Big|dx \le \bigg(\sum_{i=1}^{N} c_i\bigg)\|u-v\|\|\varphi\|$$

and

$$\int_\Omega\Big(|u(x)-v(x)|+|\nabla u(x)-\nabla v(x)|\Big)|\varphi(x)|dx \le 2\|u-v\|\|\varphi\|$$

for all $u\in H$, $v\in H$ and $\varphi\in H$. Consequently, combining these estimates, we arrive at

$$|\langle\tilde N u-\tilde N v,\varphi\rangle| \le 2\bigg(1+\sum_{i=1}^{N} c_i\bigg)\|u-v\|\|\varphi\|$$

for all $u\in H$, $v\in H$ and φ in H, which implies

$$\|\tilde N u-\tilde N v\| = \sup_{\|\varphi\|=1}|\langle\tilde N u-\tilde N v,\varphi\rangle| \le \ell\|u-v\|$$

for all $u\in H$ and $v\in H$, where $\ell = 2\Big(1+\sum_{i=1}^{N} c_i\Big)$. This shows that $\tilde N$ is Lipschiz continuous and pseudomonotone. For all such $\tilde d>0$ and $k>0$ such that $\sum_{i=1}^{N} c_i < \frac{\tilde d-k-2}{2}$, $\tilde d$ satisfies the hypothesis of Theorem 2.5. Thus, for all $\lambda\in(\tau^*,\tilde d)$, (3.5) is solvable in $H_0^1(\Omega)$, where $\tau^* = \max\{\ell+k, \frac{\tilde d+\ell}{2}\}$. The proof is complete. □

We like to mention here that the conclusions of Theorem 2.1 through Theorem 2.5 hold if the compact operator C is multivalued. The argument of the proof follows similarly and the details are omitted. Examples on multivalued compact operators and applications on elliptic as well as parabolic equations can be found, for example, in the paper by Hu and Papageorgiou [24], Berkovits and Tienari [8], Mustonen [33], and the references therein.

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