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# DEGENERATE ABSTRACT PARABOLIC EQUATIONS AND APPLICATIONS

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**Abstract**

Linear and nonlinear degenerate abstract parabolic equations with variable coefficients are studied. Here the equations and boundary conditions are degenerated on all boundary and contain some parameters. The linear problem is considered on the moving domain. The separability properties of elliptic and parabolic problems and Strichartz type estimates in mixed $L_{\mathbf{p}}$ spaces are obtained. Moreover, the existence and uniqueness of optimal regular solution of mixed problem for nonlinear parabolic equation is established. Note that, these problems arise in fluid mechanics and environmental engineering.

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## 1 Introduction

In this work, the boundary value problems (BVPs) for parameter dependent degenerate differential-operator equations (DOEs) are considered. Namely, equations and boundary conditions contain small parameters. These problems have numerous applications in PDE,

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pseudo DE, mechanics and environmental engineering. The BVP for DOEs have been studied extensively by many researchers (see e.g. [1-23] and the references therein). A comprehensive introduction to the DOEs and historical references may be found in [1-6]. The maximal regularity properties for DOEs have been studied e.g. in [2, 8-9,16-22, 25]. DOEs in Banach space valued function class are investigated e.g. in [2, 4, 9,15, 20-23, 24, 29]. Nonlinear DOEs studied e.g. in [1,16, 20, 21]. The Fredholm property of BVP for elliptic equations are studied e.g. in [1, 10, 24 ].

The main objective of the present paper is to discuses the initial and BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+B\left(\left(t,x,u,D^{[1]}u\right)\right)u=F\left(t,x,u,D^{[1]}u\right), \tag{0.1}$$

where $a_k$ are complex valued functions, $B$ and $F$ are nonlinear operators in a Banach space $E$ and

$$D^{[1]}u=\left(\frac{\partial^{[1]}u}{\partial x_1},\frac{\partial^{[1]}u}{\partial x_2},...,\frac{\partial^{[1]}u}{\partial x_n}\right),\ x=(x_1,x_2,...,x_n)\in G=\prod_{k=1}^{n}(0,b_k),$$

$$D_k^{[i]}u=u_k^{(i)}=\frac{\partial^{[i]}u}{\partial x_k^i}=\left[x_k^{\alpha_{1k}}(b_k-x_k)^{\alpha_{2k}}\frac{\partial}{\partial x_k}\right]^i u(x),\ 0\leq\alpha_{1k},\alpha_{2k}<1.$$

First, we consider the BVP for the degenerate elliptic DOE with small parameters

$$\sum_{k=1}^{n}\varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+\lambda u+\sum_{k=1}^{n}\varepsilon_k^{\frac{1}{2}}A_k(x)\frac{\partial^{[1]}u}{\partial x_k}=f(x), \tag{0.2}$$

where $a_k$ are complex-valued functions, $\varepsilon_k$ are small parameters, $A(x)$ and $A_k(x)$ are linear operators, $\lambda$ is a complex parameter.

Namely we prove that, for $f\in L_{\mathbf{p}}(G;E)$, $\left|\arg\lambda\right|\leq\varphi$, $0<\varphi\leq\pi$ and sufficiently large $|\lambda|$, problem (0.2) has a unique solution $u\in W_{\mathbf{p}}^{[2]}(G;E(A),E)$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{k}{2}}\varepsilon_k^{\frac{i}{2}}\left\|\frac{\partial^{[i]}u}{\partial x_k^i}\right\|_{L_{\mathbf{p}}(G;E)}+\|Au\|_{L_{\mathbf{p}}(G;E)}\leq C\|f\|_{L_{\mathbf{p}}(G;E)}.$$

Especially, it is shown that the corresponding differential operator is positive and also is a generator of an analytic semigroup. Then by using this result, we prove the well-posedeness of initial and BVP and uniform $L_{\mathbf{p}}-$Strichartz type estimate for the following degenerate abstract parabolic equation with parameters

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}\varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u=f(x,t),\ t\in(0,T),\ x\in G. \tag{0.3}$$

Finally, via maximal regularity properties of (0.3) and contraction mapping argument we derive the existence and uniqueness of solution of the problem (0.1).

Note that, the equation and boundary conditions are degenerated on all edges of boundary $G$. Moreover, it happened with the different rate at different boundary edges, in general.

In application, the system of degenerate nonlinear parabolic equations is presented. Particularly, we consider the system that serves as a model of systems used to describe photochemical generation and atmospheric dispersion of ozone and other pollutants. The model of the process is given by initial and BVP for the atmospheric reaction-advection-diffusion system having the form

$$\frac{\partial u_i}{\partial t} = \sum_{k=1}^{3}\left[a_{ki}(x_k)\frac{\partial^{[2]}u_i}{\partial x_k^2} + b_{ki}(x)\frac{\partial^{[1]}}{\partial x_k}(u_i\omega_k)\right] + \sum_{k=1}^{3} d_k u_k + f_i(u) + g_i, \tag{0.4}$$

where

$$x \in G_3 = \{x = (x_1, x_2, x_3),\ 0 < x_k < b_k,\},$$

$$u_i = u_i(x,t),\ i,\ k = 1,2,3,\ u = u(x,t) = (u_1, u_2, u_3),\ t \in (0,T)$$

and the state variables $u_i$ represent concentration densities of the chemical species involved in the photochemical reaction. The relevant chemistry of the chemical species involved in the photochemical reaction and appears in the nonlinear functions $f_i(u)$, with the terms $g_i$, representing elevated point sources, $a_{ki}(x), b_{ki}(x)$ are real-valued functions. The advection terms $\omega = \omega(x) = (\omega_1(x), \omega_2(x), \omega_3(x))$, describe transport from the velocity vector field of atmospheric currents or wind. In this direction the work [25] and references there can be mentioned. The existence and uniqueness of solution of the problem (0.4) is established by the theoretic-operator method, i.e., this problem reduced to degenerate differential-operator equation.

Modern analysis methods, particularly abstract harmonic analysis, the operator theory, interpolation of Banach Spaces, semigroups of linear operators, microlocal analysis, embedding and trace theorems in vector-valued Sobolev-Lions spaces are the main tools implemented to carry out the analysis.

## 2 Notations, definitions and background

Let $\gamma = \gamma(x)$ be a positive measurable function on $\Omega \subset R^n$ and $E$ be a Banach space. Let $L_{p,\gamma}(\Omega; E)$ denote the space of strongly measurable $E$-valued functions defined on $\Omega$ with the norm

$$\|f\|_{L_{p,\gamma}} = \|f\|_{L_{p,\gamma}(\Omega;E)} = \left(\int \|f(x)\|_E^p \gamma(x)\, dx\right)^{\frac{1}{p}}, 1 \le p < \infty.$$

Let $\mathbf{p} = (p_1, p_2, ..., p_n)$. $L_{\mathbf{p},\gamma}(G; E)$, $G = \prod_{k=1}^{n}(0, b_k)$ will denote the space of all $E$-valued $\mathbf{p}$-summable functions with mixed norm, i.e., the space of all measurable functions $f$ defined on $G$ equipped with norm

$$\|f\|_{L_{\mathbf{p}}(G;E)} = \left(\left(\int_0^{b_n}\left(\ldots\int_0^{b_2}\left(\int_0^{b_1}\|f(x)\|_E^{p_1}\, dx_1\right)^{\frac{p_2}{p_1}} dx_2\right)^{\frac{p_3}{p_2}}\ldots\right)^{\frac{p_n}{p_{n-1}}}\gamma(x)\, dx_n\right)^{\frac{1}{p_n}}.$$

For $\gamma(x) \equiv 1$ we will denote these spaces by $L_p(\Omega; E)$ and $L_{\mathbf{p}}(G; E)$, respectively ( see e.g. [26] for $E = \mathbb{C}$ ).

The Banach space $E$ is called an $UMD$-space if the Hilbert operator

$$(Hf)(x) = \lim_{\varepsilon \to 0} \int\limits_{|x-y|>\varepsilon} \frac{f(y)}{x-y} dy$$

is bounded in $L_p(R, E)$, $p \in (1, \infty)$ ( see. e.g. [27] ). $UMD$ spaces include e.g. $L_p$, $l_p$ spaces and Lorentz spaces $L_{pq}$, $p$, $q \in (1, \infty)$.

Let $\mathbb{C}$ be the set of the complex numbers and

$$S_\varphi = \left\{\lambda;\ \ \lambda \in \mathbb{C},\ \left|\arg \lambda\right| \leq \varphi\right\} \cup \{0\},\ 0 \leq \varphi < \pi.$$

A linear operator $A$ is said to be $\varphi$-positive in a Banach space $E$ with bound $M > 0$ if $D(A)$ is dense on $E$ and $\left\|(A + \lambda I)^{-1}\right\|_{B(E)} \leq M(1 + |\lambda|)^{-1}$ for any $\lambda \in S_\varphi$, $0 \leq \varphi < \pi$, where $I$ is the identity operator in $E$, $B(E)$ is the space of bounded linear operators in $E$. Sometimes $A + \lambda I$ will be written as $A + \lambda$ and denoted by $A_\lambda$. It is known [28, §1.15.1] that the positive operator $A$ has well-defined fractional powers $A^\theta$. Let $E\left(A^\theta\right)$ denote the space $D\left(A^\theta\right)$ with norm

$$\|u\|_{E(A^\theta)} = \left(\|u\|^p + \left\|A^\theta u\right\|^p\right)^{\frac{1}{p}}, 1 \leq p < \infty,\ 0 < \theta < \infty.$$

Let $E_1$ and $E_2$ be two Banach spaces continuously embedding in a locally convex space. By $(E_1, E_2)_{\theta,p}$, $0 < \theta < 1, 1 \leq p \leq \infty$ we will denote the interpolation spaces obtained from $\{E_1, E_2\}$ by the $K$-method [28, §1.3.2].

Weight function $\gamma$ satisfies $A_p$ condition (i.e. $\gamma \in A_p$, $1 < p < \infty$) if there is a constant $C$ such that

$$\left(\frac{1}{|Q|}\int\limits_Q \gamma(x)\,dx\right)\left(\frac{1}{|Q|}\int\limits_Q \gamma^{-\frac{1}{p-1}}(x)\,dx\right)^{p-1} \leq C$$

for all cubes $Q \subset R^n$.

Let $S(R^n; E)$ denote the Schwartz class, i.e., the space of all $E$-valued rapidly decreasing smooth functions on $R^n$. Let $F$ denote the Fourier transformation. A function $\Psi \in C(R^n; B(E))$ is called Fourier multiplier in $L_{p,\gamma}(R^n; E)$ if the map

$$u \to \Phi u = F^{-1}\Psi(\xi) Fu,\ u \in S(R^n; E)$$

is well defined and extends to a bounded linear operator in $L_{p,\gamma}(R^n; E)$. The set of all multipliers in $L_{p,\gamma}(R^n; E)$ will denoted by $M_{p,\gamma}(E)$.

Let $W_h = \left\{\Psi_h \in M_{p,\gamma}(E),\ h \in Q \subset \mathbb{C}\right\}$ be a collection of multipliers in $M_{p,\gamma}(E)$. We say $W_h$ is a uniform collection of multipliers if there exists a positive constant $M$ independent of $h$ such that

$$\left\|F^{-1}\Psi_h F u\right\|_{L_{p,\gamma}(R^n;E)} \leq M\|u\|_{L_{p,\gamma}(R^n;E)}$$

for all $h \in Q$ and $u \in S(R^n; E)$.

Let $\mathbb{N}$ denote the set of natural numbers and $\left\{r_j\right\}$ is an arbitrary sequence of independent symmetric $\{-1, 1\}$-valued random variables on $[0, 1]$. A set $K \subset B(E_1, E_2)$ is called

$R$-bounded ( see e.g. [10] ) if there is a positive constant $C$ such that for all $T_1, T_2, ..., T_m \in K$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy.$$

The smallest $C$ for which the above estimate holds is called a $R$-bound of the collection $K$ and denoted by $R(K)$.

A set $W_h \subset L(E_1, E_2)$ is called uniform $R$-bounded with respect to $h \in Q$ if there is a constant $C$ independent of $h$ such that for all $T_1(h), T_2(h), ..., T_m(h) \in W_h$ and $u_1, u_2, ..., u_m \in E_1$, $m \in \mathbb{N}$

$$\int_0^1 \left\| \sum_{j=1}^m r_j(y) T_j(h) u_j \right\|_{E_2} dy \leq C \int_0^1 \left\| \sum_{j=1}^m r_j(y) u_j \right\|_{E_1} dy.$$

**Definition 1**. A Banach space $E$ is said to be a space satisfying multiplier condition, if for any $\Psi \in C^{(1)}(R; L(E))$ the $R$-boundedness of the set

$$\left\{ \xi^j \Psi^{(j)}(\xi) : \xi \in \mathbb{R} \setminus \{0\},\ j = 0, 1 \right\}$$

implies $\Psi \in M_{p,\gamma}(E)$.

A $\varphi$-positive operator $A$ is said to be $R$-positive in a Banach space $E$ if the set $L_A = \left\{ \xi (A + \xi I)^{-1} : \xi \in S_\varphi \right\}, 0 \leq \varphi < \pi$ is $R$-bounded. Note that, in Hilbert spaces all norm bounded sets are $R$-bounded. Therefore, all positive operators in Hilbert spaces are $R$-positive. If $A$ is a generator of contraction semigroup on $L_q, 1 \leq q \leq \infty$, or $A$ has a bounded imaginary powers with $\left\| A^{it} \right\|_{B(E)} \leq Ce^{\nu |t|}$, $\nu < \frac{\pi}{2}$ in $E \in UMD$, then $A$ is $R$-positive ( e.g. see [10] ).

An operator $A(t)$ is said to be uniformly $\varphi$-positive if $D(A(t))$ is independent of $t$ and dense in $E$ and $\left\| (A(t) + \lambda)^{-1} \right\| \leq \frac{M}{1+|\lambda|}$ for all $\lambda \in S(\varphi)$, $0 \leq \varphi < \pi$, where $M$ is independent of $t$.

$\sigma_\infty(E)$ will denote the space of all compact operators in $E$.

Let $E_0$ and $E$ be two Banach spaces and $E_0$ is continuously and densely embeds into $E$. Let us consider the Sobolev-Lions type space $W^m_{p,\gamma}(a, b; E_0, E)$, consisting of all functions $u \in L_{p,\gamma}(a, b; E_0)$ that have generalized derivatives $u^{(m)} \in L_{p,\gamma}(a, b; E)$ with the norm

$$\|u\|_{W^m_{p,\gamma}} = \|u\|_{W^m_{p,\gamma}(a,b;E_0,E)} = \|u\|_{L_{p,\gamma}(a,b;E_0)} + \left\| u^{(m)} \right\|_{L_{p,\gamma}(a,b;E)} < \infty.$$

Let $\gamma = \gamma(x)$ be a positive measurable function on $(0, 1)$ and

$$W^{[m]}_{p,\gamma} = W^{[m]}_{p,\gamma}(0, 1; E_0, E) = \Big\{ u : u \in L_p(0, 1; E_0),$$

$$u^{[m]} \in L_p(0, 1; E), \|u\|_{W^{[m]}_{p,\gamma}} = \|u\|_{L_p(0,1;E_0)} + \left\| u^{[m]} \right\|_{L_p(0,1;E)} < \infty \Big\},$$

where

$$u^{[i]} = \left( \gamma(x) \frac{d}{dx} \right)^i u(x).$$

Now, let we define $E$-valued Sobolev-Lions type spaces with mixed $L_{\mathbf{p}}$ and $L_{\mathbf{p},\gamma}$ norms. Let

$$\alpha_k(x) = x_k^{\alpha_{1k}}(b_k - x_k)^{\alpha_{2k}},\ \alpha = (\alpha_1, \alpha_2, ..., \alpha_n).$$

Consider $E$-valued weighted space defined by

$$W^{[m]}_{\mathbf{p},\alpha}(G, E(A), E) = \Big\{u; u \in L_{\mathbf{p}}(G; E_0),\ \frac{\partial^{[m]}u}{\partial x_k^m} \in L_{\mathbf{p}}(G; E),$$

$$\|u\|_{W^{[m]}_{\mathbf{p},\alpha}} = \|u\|_{L_{\mathbf{p}}(G;E_0)} + \sum_{k=1}^{n}\left\|\frac{\partial^{[m]}u}{\partial x_k^m}\right\|_{L_{\mathbf{p}}(G;E)} < \infty\Big\}.$$

Let $\varepsilon_k$ be small parameters and $\varepsilon = (\varepsilon_1, \varepsilon_2, ..., \varepsilon_n)$. We denote by $W^m_{\mathbf{p},\gamma}(\Omega; E_0, E)$ the space of all functions $u \in L_{\mathbf{p},\gamma}(\Omega; E_0)$ possessing generalized derivatives $\frac{\partial^m u}{\partial x_k^m} \in L_{\mathbf{p},\gamma}(\Omega; E)$ with the parameterized norm

$$\|u\|_{W^m_{\mathbf{p},\gamma,\varepsilon}(\Omega;E_0,E)} = \|u\|_{L_{\mathbf{p},\gamma}(\Omega;E_0)} + \sum_{k=1}^{n}\varepsilon_k\left\|\frac{\partial^m u}{\partial x_k^m}\right\|_{L_{\mathbf{p},\gamma}(\Omega;E)} < \infty.$$

In a similar way as in [18, Theorems 2.3, 2.4] we have the following result:

**Theorem A$_1$**. Assume the following conditions be satisfied:

(1) $\gamma = \gamma(x)$ is a weight function defined on domain $\Omega \subset R^n$ satisfying $A_p$ condition;

(2) $E$ is a Banach space satisfying the multiplier condition with respect to $\mathbf{p} = (p_1, p_2, ..., p_n)$ and $\gamma$;

(3) $A$ is a $R$-positive operator in $E$ and $0 < \varepsilon_k < T < \infty$, $p_k \in (1, \infty)$; $\beta = (\beta_1, \beta_2, ..., \beta_n)$;

(4) there exists a bounded linear extension operator from $W^m_{\mathbf{p},\gamma}(\Omega; E(A), E)$ to $W^m_{\mathbf{p},\gamma}(R^n; E(A), E)$.

Then, the embedding

$$D^\beta W^m_{\mathbf{p},\gamma}(\Omega; E(A), E) \subset L_{\mathbf{p},\gamma}\left(\Omega; E\left(A^{1-\frac{|\beta|}{m}-\mu}\right)\right)$$

is continuous and for $0 \le \mu \le 1 - \frac{|\beta|}{m}$, $0 < h \le h_0 < \infty$ the following uniform estimate holds

$$\prod_{k=1}^{n}\varepsilon_k^{\frac{\beta_k}{m}}\left\|D^\alpha u\right\|_{L_{\mathbf{p},\gamma}\left(\Omega;E\left(A^{1-\varkappa-\mu}\right)\right)} \le h^\mu \|u\|_{W^m_{\mathbf{p},\gamma,\varepsilon}(\Omega;E(A),E)} + h^{-(1-\mu)}\|u\|_{L_{\mathbf{p},\gamma}(\Omega;E)}$$

for $u \in W^m_{\mathbf{p},\gamma}(\Omega; E(A), E)$.

By reasoning as in [18, Theorems 2.3] and [15, Theorem 3.7] we obtain

**Theorem A$_2$.** Let all conditions of Theorem A$_1$ hold, $\Omega$ be a bounded domain and $A^{-1} \in \sigma_\infty(E)$. Then, for $0 < \mu < 1 - \frac{j}{m}$ the embedding

$$D^j W^m_{\mathbf{p},\gamma}(\Omega; E(A), E) \subset L_{\mathbf{p},\gamma}\left(\Omega; E\left(A^{1-\frac{|\beta|}{m}-\mu}\right)\right)$$

is compact.

Consider the BVP for the degenerate ordinary DOE with parameter

$$Lu = \varepsilon a(x) u^{[2]}(x) + (A(x) + \lambda) u(x) = f, \tag{1.1}$$

$$L_1 u = \sum_{i=0}^{m_1} \varepsilon^{\sigma_i} \delta_i u^{[i]}(0) = 0,\ L_2 u = \sum_{i=0}^{m_2} \varepsilon^{\sigma_i} \beta_i u^{[i]}(1) = 0,\ x \in (0,1), \tag{1.2}$$

where $u^{[i]} = \left[x^{\gamma_1}(1-x)^{\gamma_2} \frac{d}{dx}\right]^i u(x)$, $0 \le \gamma_k < 1$, $\sigma_i = \frac{i}{2} + \frac{1}{2p(1-\gamma_0)}$, $\gamma_0 = \min\{\gamma_1, \gamma_2\}$, $m_k \in \{0,1\}$, $\delta_i, \beta_i$ are complex numbers; $A(x)$ is a linear operator in a Banach space $E$ for $x \in (0,1)$, $\varepsilon$ is a small positive and $\lambda$ is a complex parameter.

We suppose $\delta_{m_1} \ne 0, \beta_{m_1} \ne 0$ and

$$\int_0^x z^{-\gamma_1}(1-z)^{-\gamma_2}\, dz < \infty.$$

Consider the operator $B_\varepsilon$ generated by problem $(1.1)-(1.2)$ for $\lambda = 0$, i.e.,

$$D(B_\varepsilon) = W_{p,\gamma}^{[2]}(0,1;E(A),E,\ L_k) =$$

$$\left\{u:\ u \in W_{p,\gamma}^{[2]}(0,1;E(A),E),\ L_k u = 0,\ k = 1,2\right\},$$

$$B_\varepsilon u = -\varepsilon a(x) u^{[2]} + A(x) u.$$

**Condition 1.1.** Assume the following conditions are satisfied:

(1) $E$ is a Banach space satisfying the multiplier condition with respect to $p$ and the weight function $\gamma(x) = x^{\gamma_1}(1-x)^{\gamma_2}$, $0 \le \gamma_k < 1 - \frac{1}{p}, 1 < p < \infty$, $a \in C([0,1])$ and $a(x) < 0$ for $x \in (0,1)$;

(2) $A$ is a $R$ positive operator in $E$ and $A(x)A^{-1}(x_0) \in C([0,1];B(E))$ for $x, x_0 \in (0,1)$.

By reasoning as in [18, Theorem 5.1] and by using the method used in [17, Theorem 1 ] we get the following:

**Theorem A$_3$.** Let the Condition 1.1 hold. Then, problem (1.1) has a unique solution $u \in W_{p,\gamma}^{[2]}(0,1;E(A),E)$ for $f \in L_p(0,1;E)$ and $\left|\arg\lambda\right| \le \varphi$ with sufficiently large $|\lambda|$. Moreover, the following uniform coercive estimate holds

$$\sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon^{\frac{i}{2}} \left\|u^{[i]}\right\|_{L_p(0,1;E)} + \|Au\|_{L_p(0,1;E)} \le C\|f\|_{L_p(0,1;E)}.$$

In a similar way as in [21, Theorem 3.1 ] we obtain

**Theorem A**$_4$. Suppose the Condition 1.1 is satisfied. Then, the operator $B_\varepsilon$ is uniformly $R$-positive in $L_p(0,1;E)$.

## 2. Degenerate elliptic equations with parameters

Consider the BVP for the following degenerate partial DOE with parameters

$$\sum_{k=1}^{n} \varepsilon_k a_k(x_k) \frac{\partial^{[2]} u}{\partial x_k^2} + A(x)u + \lambda u + \sum_{k=1}^{n} \varepsilon_k^{\frac{1}{2}} A_k(x) \frac{\partial^{[1]} u}{\partial x_k} = f(x), \tag{2.1}$$

$$L_{k1} u = \sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}} \delta_{ki} u_{x_k}^{[i]}(G_{k0}) = 0, L_{k2} u = \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}} \beta_{ki} u_k^{[i]}(G_{kb}) = 0,$$

for $x^{(k)} \in G_k$, where $A(x)$ and $A_k(x)$ are linear operators, $u = u(x)$, $\varepsilon_k$ are small parameters, $\delta_{ki}, \beta_{ki}$ are complex numbers, $\lambda$ is a complex parameter, $m_{kj} \in \{0,1\}$ and

$$\frac{\partial^{[i]}u}{\partial x_k^i} = \left[x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}} \frac{\partial}{\partial x_k}\right]^i u(x), \ 0 \leq \alpha_{1k}, \alpha_{2k} < 1,$$

$$\sigma_{ik} = \frac{i}{2} + \frac{1}{2p_k (1 - \alpha_{0k})}, \ \alpha_{0k} = \min\{\alpha_{1k}, \alpha_{2k}\},$$

$a_k$ are complex-valued functions and

$$x = (x_1, x_2, ..., x_n) \in G = \prod_{k=1}^{n} (0, b_k),$$

$$G_{k0} = (x_1, x_2, ..., x_{k-1}, 0, x_{k+1}, ..., x_n), \ p_k \in (1, \infty),$$

$$G_{kb} = (x_1, x_2, ..., x_{k-1}, b_k, x_{k+1}, ..., x_n),$$

$$x^{(k)} = (x_1, x_2, ..., x_{k-1}, x_{k+1}, ..., x_n) \in G_k = \prod_{j \neq k} \left(0, b_j\right).$$

Let

$$\alpha = \alpha(x) = \prod_{k=1}^{n} x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}}.$$

**Remark 2.0.** Under the substitutions

$$\tau_k = \int_0^{x_k} x_k^{-\alpha_k} (b_k - x_k)^{-\alpha_k} dx_k, \ k = 1, 2, ..., n$$

the spaces $L_p(G; E)$ and $W_{p,\alpha}^{[2]}(G; E(A), E)$ are mapped isomorphically onto the weighted spaces $L_{p,\tilde{\alpha}}\left(\tilde{G}; E\right)$ and $W_{p,\tilde{\alpha}}^2\left(\tilde{G}; E(A), E\right)$, respectively, where

$$\tilde{G} = \prod_{k=1}^{n} \left(0, \tilde{b}_k\right), \tilde{b}_k = \int_0^{b_k} x_k^{-\alpha_{1k}} (b_k - x_k)^{-\alpha_{2k}} dx_k, \ \tilde{\alpha}(\tau) = \alpha(x_1(\tau_1), x_2(\tau_2), ..., x_n(\tau_n)).$$

Consider the principal part of (2.1), i.e., consider the problem

$$\sum_{k=1}^{n} \varepsilon_k a_k(x_k) \frac{\partial^{[2]}u}{\partial x_k^2} + A(x)u + \lambda u = f(x), \tag{2.2}$$

$$\sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}} \delta_{ki} u_{x_k}^{[i]}(G_{k0}) = 0, \ \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}} \beta_{ki} u_k^{[i]}(G_{kb}) = 0.$$

**Condition 2.1** Assume;

(1) $E$ is the Banach space satisfying the multiplier condition with respect to $p$ and the weight function $\gamma(x) = \prod_{k=1}^{n} x_k^{\alpha_{1k}} (b_k - x_k)^{\alpha_{2k}}$, where $0 \le \alpha_{1k}, \alpha_{2k} < 1 - \frac{1}{p_k}$, $p_k \in (1, \infty)$, $\delta_{km_{k1}} \ne 0$, $\beta_{km_{k2}} \ne 0$;

(2) $A(x)$ is a uniformly $R$-positive operator in $E$, $A(x) A^{-1}(\bar{x}) \in C\left(\bar{G}; L(E)\right)$, $x \in G$;

(3) $a_k(x_k) \in C^{(m)}([0, b_k])$ and $a_k(x_k) < 0$ for $x_k \in [0, b_k]$.

First, we prove the separability properties of the problem (2.2):

**Theorem 2.1.** Let the Conditions 2.1 hold. Then, problem (2.2) has a unique solution $u \in W^{[2]}_{\mathbf{p},\alpha}(G; E(A), E)$ for $f \in L_{\mathbf{p}}(G; E)$, $\left|\arg \lambda\right| \le \varphi$ with sufficiently large $|\lambda|$ and the following coercive uniform estimate holds

$$\sum_{k=1}^{n} \sum_{i=0}^{2} |\lambda|^{1-\frac{i}{2}} \varepsilon_k^{\frac{i}{2}} \left\| \frac{\partial^{[i]} u}{\partial x_k^i} \right\|_{L_{\mathbf{p}}(G;E)} + \|Au\|_{L_{\mathbf{p}}(G;E)} \le C \|f\|_{L_{\mathbf{p}}(G;E)}. \tag{2.3}$$

**Proof.** Consider the BVP

$$(L + \lambda) u = a_1(x_1) \varepsilon_1 D^{[2]}_{x_1} u(x_1) + (A(x_1) + \lambda) u(x_1) = f(x_1), \tag{2.4}$$

$$L_{1j} u = 0, \ j = 1, 2, \ x_1 \in (0, b_1),$$

where $L_{1j}$ are boundary conditions of type (2.2) on $(0, b_1)$. By virtue of Theorem $A_3$, problem (2.4) has a unique solution $u \in W^{[2]}_{p_{1,\alpha_1}}(0, b_1; E(A), E)$ for $f \in L_{p_1}(0, b_1; E)$, $\left|\arg \lambda\right| \le \varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds

$$\sum_{j=0}^{2} |\lambda|^{1-\frac{j}{2}} \varepsilon_1^{\frac{j}{2}} \left\| u^{[j]} \right\|_{L_{p_1}(0,b_1;E)} + \|Au\|_{L_{p_1}(0,b_1;E)} \le C \|f\|_{L_{p_1}(0,b_1;E)}.$$

Now, let us consider the following BVP

$$\sum_{k=1}^{2} \varepsilon_k a_k(x_k) D^{[2]}_k u(x_1, x_2) + A(x_1, x_2) u(x_1, x_2) + \lambda u(x_1, x_2) = f(x_1, x_2), \tag{2.5}$$

$$L_{k1} u = 0, \ L_{k2} u = 0, \ k = 1, 2, \ x_1, x_2 \in G_2 = (0, b_1) \times (0, b_2).$$

Let $\mathbf{p}_2 = (p_1, p_2)$ and $\alpha(2) = (\alpha_1, \alpha_2)$. Since $L_{p_2}\left(0, b_2; L_{p_1}(0, b_1); E\right) = L_{\mathbf{p}_2}(G_2; E)$, the BVP (2.5) can be expressed as

$$a_2 \varepsilon_2 D^{[2]}_2 u(x_2) + (B_{\varepsilon_1}(x_2) + \lambda) u(x_2) = f(x_2), \ L_{2j} u = 0, j = 1, 2,$$

for $x_1 \in (0, b_1)$, where $B_{\varepsilon_1}$ is a differential operator in $L_{p_1}(0, b_1; E)$ for $x_2 \in (0, b_2)$, generated by problem (2.4). By virtue of [1, Theorem 4.5.2 ], $L_{p_1}(0, b_1; E) \in UMD$ for $p_1 \in (1, \infty)$. Hence, by [22, Corollary 4.1], the space $L_{p_1}(0, b_1; E)$ satisfies the multiplier condition. Moreover, the Theorem $A_4$ implies the uniform $R$-positivity of operator $B_{\varepsilon_1}$. Hence, by Theorem $A_3$, problem (2.5) has a unique solution $u \in W^{[2]}_{\mathbf{p}_2,\alpha(2)}(G_2; E(A); E)$ for $f \in L_{\mathbf{p}_2}(G_2; E)$, $\left|\arg \lambda\right| \le \varphi$ with sufficiently large $|\lambda|$ and (2.3) holds for $n = 2$. By continuing this we obtain the assertion.

**Theorem 2.2.** Let the Conditions 2.1 hold and let $A_k(x)A^{-(\frac{1}{2}-\nu)}(x) \in C\left(\bar{G};L(E)\right)$ for $0<\nu<\frac{1}{2}$. Then, problem (2.1) has a unique solution $u \in W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)$ for $f \in L_{\mathbf{p}}(G;E)$, $\left|\arg\lambda\right| \leq \varphi$ with sufficiently large $|\lambda|$ and the coercive uniform estimate holds

$$\sum_{k=1}^{n}\sum_{i=0}^{2}|\lambda|^{1-\frac{i}{2}}\varepsilon_k^{\frac{i}{2}}\left\|\frac{\partial^{[i]}u}{\partial x_k^i}\right\|_{L_{\mathbf{p}}(G;E)}+\|Au\|_{L_{\mathbf{p}}(G;E)} \leq C\|f\|_{L_{\mathbf{p}}(G;E)}. \quad (2.6)$$

**Proof.** By assumption and by Theorem $A_1$, for all $h>0$ we have the following Ehrling-Nirenberg-Gagliardo type estimate

$$\|L_1u\|_{L_{\mathbf{p}}(G;E)} \leq h^{\mu}\ \|u\|_{W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)}+h^{-(1-\mu)}\|u\|_{L_{\mathbf{p}}(G;E)}. \quad (2.7)$$

Let $\mathbf{O}_\varepsilon$ denote the operator generated by the problem (2.2) and

$$L_1u=\sum_{k=1}^{n}\varepsilon_k^{\frac{1}{2}}A_k(x)\frac{\partial^{[1]}u}{\partial x_k}.$$

By using the estimate (2.7) we obtain that there is a $\delta \in (0,1)$ such that

$$\left\|L_1(\mathbf{O}_\varepsilon+\lambda)^{-1}\right\|_{B(X)}<\delta.$$

Hence, from perturbation theory of linear operators we obtain the assertion.

**Theorem 2.2.** Let all conditions of Theorem 2.2 hold. Then, problem (2.1) is Fredholm in $L_{\mathbf{p}}(G;E)$ for $\lambda=0$.

**Proof.** Theorem 2.2 implies that the operator $\mathbf{O}_\varepsilon+\lambda$ has a bounded inverse $(\mathbf{O}_\varepsilon+\lambda)^{-1}$ from $L_{\mathbf{p}}(G;E)$ to $W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)$ for sufficiently large $|\lambda|$, that is the operator $\mathbf{O}_\varepsilon+\lambda$ is Fredholm from $W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)$ into $L_{\mathbf{p}}(G;E)$. Then by Theorem $A_2$, Remark 2.0 and in view of perturbation theory of linear operators we obtain that the operator $\mathbf{O}_\varepsilon$ is Fredholm from $W^{[2]}_{\mathbf{p},\alpha}(G;E(A),E)$ into $L_{\mathbf{p}}(G;E)$.

**Example 2.1.** Now, let us consider a special case of (2.1). Let $E=\mathbb{C}$, $G_2=(0,1)\times(0,1)$ and $A=a(x,y)>0$, i.e., consider the problem

$$\varepsilon_1a_1\frac{\partial^{[2]}u}{\partial x^2}+\varepsilon_2a_2\frac{\partial^{[2]}u}{\partial y^2}+b_1\varepsilon_1^{\frac{1}{2}}\frac{\partial u}{\partial x}+b_2\varepsilon_2^{\frac{1}{2}}\frac{\partial u}{\partial y}+au=f(x,y),$$

$$\sum_{i=0}^{m_{11}}\varepsilon_1^{\sigma_i}\delta_{1i}u_x^{[i]}(0,y)=0,\ \sum_{i=0}^{m_{12}}\varepsilon_1^{\sigma_i}\delta_{2i}u_x^{[i]}(0,y)=0,\ x\in(0,1), \quad (2.8)$$

$$\sum_{i=0}^{m_{21}}\varepsilon_2^{\sigma_i}\eta_{1i}u_y^{[i]}(x,1)=0,\ \sum_{i=0}^{m_{22}}\varepsilon_2^{\sigma_i}\eta_{2i}u_y^{[i]}(x,1)=0,\ y\in(0,1),$$

where $\varepsilon_1$ and $\varepsilon_2$ are positive small parameters, $\delta_{ki}$, $\eta_{ki}$ are complex numbers, $a_1=a_1(x)<0$, $a_2=a_2(y)<0$, $x,y\in[0,1]$, $a_k\in C([0,1])$, $a\in C\left(\bar{G}_2\right)$, $b_k\in L_\infty(G_2)$ and

$$\frac{\partial^{[i]}u}{\partial x^i}=\left[x^{\alpha_1}(1-x)^{\alpha_2}\frac{\partial}{\partial x}\right]^i u(x,y),\ 0\leq\alpha_1,\ \alpha_2<1-\frac{1}{p_1},$$

$$\frac{\partial^{[i]}u}{\partial y^i}=\left[y^{\beta_1}(1-y)^{\beta_2}\frac{\partial}{\partial y}\right]^i u(x,y),\,0\leq\beta_1,\,\beta_2<1-\frac{1}{p_2},$$

$$\sigma_i=\frac{1}{2}\left(i+\frac{1}{p}\right),\;m_{kj}\in\{0,1\},\;\mathbf{p}_2{=}(p_1,p_2).$$

**Result 2.1.** Theorem 2.2 implies that for each $f\in L_{\mathbf{p}_2}(G_2)$ and sufficiently large $a$, problem (2.8) has a unique solution $u\in W^{[2]}_{\mathbf{p}_2}(G_2)$ satisfying the coercive estimate

$$\varepsilon_1\left\|\frac{\partial^{[2]}u}{\partial x^2}\right\|_{L_{\mathbf{p}_2}(G_2)}+\varepsilon_2\left\|\frac{\partial^{[2]}u}{\partial y^2}\right\|_{L_{\mathbf{p}_2}(G_2)}+\|u\|_{L_{\mathbf{p}_2}(G_2)}\leq C\|f\|_{L_{\mathbf{p}_2}(G_2)}.$$

# 3 Abstract Cauchy problem for degenerate parabolic equation with parameter

Consider the initial and BVP for degenerate parabolic equation with parameter:

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}\varepsilon_k a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)u+du=f(x,t),\,t\in(0,T),\,x\in G. \tag{3.1}$$

$$\sum_{i=0}^{m_{k1}}\varepsilon_k^{\sigma_{ik}}\delta_{ki}u^{[i]}_{x_k}(G_{k0},t)=0,\sum_{i=0}^{m_{k2}}\varepsilon_k^{\sigma_{ik}}\beta_{ki}u^{[i]}_{k}(G_{kb},t)=0,$$

$$u(x,0)=0,\;t\in(0,T),\,x^{(k)}\in G_k \tag{3.2}$$

where $u=u(x,t)$ is a solution, $\delta_{ki},\beta_{ki}$ are complex numbers, $\varepsilon_k$ are positive parameters, $a_k$ are complex-valued functions on $G$, $A(x)$ is a linear operator in a Banach space $E$, domains $G$, $G_k$, $G_{k0}$, $G_{kb}$, $\sigma_{ik}$ and $x^{(k)}$ are defined in the section 2 and

$$\frac{\partial^{[i]}u}{\partial x_k^i}=\left[x^{\alpha_{1k}}(b_k-x_k)^{\alpha_{2k}}\frac{\partial}{\partial x_k}\right]^i u(x,t),\,d>0.$$

For $\bar{\mathbf{p}}=(p_0,\mathbf{p}),\,\mathbf{p}=(p_1,p_2,,,,p_n)$, $G_T=(0,T)\times G$, $L_{\tilde{\mathbf{p}},\gamma}(G_T;E)$ will denote the space of all $E$-valued weighted $\tilde{\mathbf{p}}$-summable functions with mixed norm.

**Theorem 3.1.** Suppose the Condition 2.1 hold for $\varphi>\frac{\pi}{2}$. Then, for $f\in L_{\mathbf{p}}(G_T;E)$ and sufficientli large $d>0$ problem (3.1)−(3.2) has a unique solution belonging to $W^{1,[2]}_{\bar{\mathbf{p}},\alpha}(G_T;E(A),E)$ and the following coercive estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\bar{\mathbf{p}}}(G_T;E)}+\sum_{k=1}^{2}\varepsilon_k\left\|\frac{\partial^{[2]}u}{\partial x_k^2}\right\|_{L_{\bar{\mathbf{p}}}(G_T;E)}+\|Au\|_{L_{\bar{\mathbf{p}}}(G_T;E)}\leq C\|f\|_{L_{\bar{\mathbf{p}}}(G_T;E)}.$$

**Proof.** The problem (3.1) can be expressed as the following abstract Cauchy problem

$$\frac{du}{dt}+(\mathbf{O}_\varepsilon+d)u(t)=f(t),\;u(0)=0. \tag{3.3}$$

From Theorems $A_4$, 2.1 we get that $\mathbf{O}_\varepsilon$ is $R$-positive in $X=L_{\mathbf{p}}(G;E)$. By [28, §1.14], $\mathbf{O}_\varepsilon$ is a generator of an analytic semigroup in $X$. Then by virtue of [22, Theorem 4.2], problem

(3.3) has a unique solution $u \in W^{1}_{p_0}(0,T;D(\mathbf{O}_\varepsilon),X)$ for $f \in L_{p_0}(0,T;X)$ and sufficiently large $d>0$. Moreover, the following uniform estimate holds

$$\left\|\frac{du}{dt}\right\|_{L_{p_0}(0,T;X)} + \|\mathbf{O}_\varepsilon u\|_{L_{p_0}(0,T;X)} \leq C\|f\|_{L_{p_0}(0,T;X)}.$$

Since $L_{p_0}(G_T;X) = L_{\tilde{\mathbf{p}}}(G_T;E)$, by Theorem 2.1 we have

$$\|(\mathbf{O}_\varepsilon + d)u\|_{L_{p_0}((0,T);X)} = D(\mathbf{O}_\varepsilon).$$

Hence, the assertion follows from the above estimate.

## 4 Degenerate parabolic DOE on the moving domain

Consider the degenerate problem (3.1) – (3.2) on the moving domain $G(s) = \prod_{k=1}^{n}(0,b_k(s))$:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2} + A(x)u + du = f(x,t), \tag{4.1}$$

$$L_{k1}u = \sum_{i=0}^{m_{k1}} \varepsilon_k^{\sigma_{ik}}\delta_{ki}u_{x_k}^{[i]}(G_{k0}(s),t) = 0, L_{k2}u = \sum_{i=0}^{m_{k2}} \varepsilon_k^{\sigma_{ik}}\beta_{ki}u_k^{[i]}(G_{kb}(s),t) = 0,$$
$$u(x,0) = 0,\ t \in (0,T),\ x \in G(s), \tag{4.2}$$

where the end points $b_k(s)$ depend on a parameter $s$, $x_k \in (0,b_k(s))$ and $b_k(s)$ are positive continues function, $G_{k0}(s)$, $G_{kb}(s)$ are domains defined in the section 2, replacing $(0,b_k)$ by $(0,b_k(s))$ and

$$\sigma_{ik} = \frac{i}{2} + \frac{1}{2p(1-\alpha_{0k})},\ \alpha_{0k} = \min\{\alpha_{1k},\alpha_{2k}\},$$

$$\frac{\partial^{[i]}u}{\partial x_k^i} = \left[x^{\alpha_{1k}}(b_k - x_k)^{\alpha_{2k}}\frac{\partial}{\partial x_k}\right]^i u(x,t).$$

Let

$$G_T = G_T(s) = (0,T)\times G(s).$$

Theorem 3.1 implies the following:

**Proposition 4.1.** Assume the Condition 2.1 hold for $\varphi > \frac{\pi}{2}$. Then, problem (4.1) – (4.2) has a unique solution $u \in W^{1,[2]}_{\tilde{\mathbf{p}},\alpha}((G(s));E(A),E)$ for $f \in L_{\mathbf{p}}(G_T(s);E)$ and sufficiently $d>0$. Moreover, the following coercive uniform estimate holds

$$\left\|\frac{\partial u}{\partial t}\right\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} + \sum_{k=1}^{2}\varepsilon_k\left\|\frac{\partial^{[2]}u}{\partial x_k^2}\right\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} + \|Au\|_{L_{\tilde{\mathbf{p}}}(G_T;E)} \leq C\|f\|_{L_{\tilde{\mathbf{p}}}(G_T;E)}. \tag{4.3}$$

**Proof.** Under the substitution $\tau_k = x_k b_k(s)$ the problem (4.1) – (4.2) reduced to the following BVP in fixed domain $G$:

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{n} b_k^{-2}(s)\tilde{a}_k(\tau_k)\frac{\partial^{[2]}u}{\partial \tau_k^2} + \tilde{A}(\tau)u = \tilde{f}(\tau,t),\ t \in (0,T),\ \tau \in G. \tag{4.4}$$

$$\sum_{i=0}^{m_{k1}} b_k^{\sigma_{ik}}(s)\delta_{ki}u_{x_k}^{[i]}(G_{k0},t)=0, \sum_{i=0}^{m_{k2}} b_k^{\sigma_{ik}}(s)\beta_{ki}u_k^{[i]}(G_{kb},t)=0,$$

$$u(x,0)=0,\ t\in(0,T),\ x\in G=\prod_{k=1}^{n}(0,b_k), \tag{4.5}$$

where

$$\tilde{a}_k(\tau)=a_k(x(\tau)),\ \tilde{A}(\tau)=A((x(\tau))),\ \tilde{f}(\tau)=f((x(\tau))),$$
$$x(\tau)=(x_1(\tau_1),x_2(\tau_2),...,x_n(\tau_n)).$$

The problem (4.4) – (4.5) is a particular case of (3.1) – (3.2). So, by virtue of Theorem 3.1 we obtain the required assertion.

# 5 Nonlinear degenerate abstract parabolic problem

In this section, we consider initial and BVP for the following nonlinear degenerate parabolic equation

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n}a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+B\left(\left(t,x,u,D^{[1]}u\right)\right)u=F\left(t,x,u,D^{[1]}u\right), \tag{5.1}$$

$$\sum_{i=0}^{m_{k1}}\delta_{ki}u_{x_k}^{[i]}(G_{k0},t)=0, \sum_{i=0}^{m_{k2}}\beta_{ki}u_k^{[i]}(G_{kb},t)=0,$$

$$u(x,0)=0,\ t\in(0,T),\ x\in G,\ x^{(k)}\in G_k, \tag{5.2}$$

where $u=u(x,t)$ is a solution, $\delta_{ki}$, $\beta_{ki}$ are complex numbers, $a_k$ are complex-valued functions on $[0,b_k]$; domains $G$, $G_k$, $G_{k0}$, $G_{kb}$ and $\sigma_{ik}$, $x^{(k)}$ are defined in the section 2 and

$$D_k^{[i]}u=\frac{\partial^{[i]}u}{\partial x_k^i}=\left[x_k^{\alpha_k}(b_k-x_k)^{\alpha_k}\frac{\partial}{\partial x_k}\right]^i u(x,t),\ 0\le\alpha_k<1.$$

Let $G_T=(0,T)\times G$. Moreover, we let

$$G_0=\prod_{k=1}^{n}(0,b_{0k}),\ G=\prod_{k=1}^{n}(0,b_k),\ b_k\in(0,b_{0k}),$$

$$T\in(0,T_0),\ B_{ki}=\left(W^{2,p}(G_k,E(A),E),L^p(G_k;E)\right)_{\eta_{ik},p},$$

$$\eta_{ik}=\frac{m_{ki}+\frac{1}{p(1-\alpha_k)}}{2},\ B_0=\prod_{k=1}^{n}\prod_{i=0}^{1}B_{ki}.$$

**Remark 5.1.** By virtue of [28, § 1.8.] and the Remark 2.0, operators $u\to\frac{\partial^{[i]}u}{\partial x_k^i}\mid_{x_{k=0}}$are continuous from $W_{p,\alpha}^{[2]}(G;E(A),E)$ onto $B_{ki}$ and there are the constants $C_1$ and $C_0$ such that for $w\in W_{p,\alpha}^{[2]}(G;E(A),E)$, $W=\{w_{ki}\}$, $w_{ki}=\frac{\partial^{[i]}w}{\partial x_k^i}$, $i=0,1$, $k=1,2,...,n$

$$\left\|\frac{\partial^{[i]}w}{\partial x_k^i}\right\|_{B_{ki},\infty}=\sup_{x\in G}\left\|\frac{\partial^{[i]}w}{\partial x_k^i}\right\|_{B_{ki}}\le C_1\|w\|_{W_{p,\alpha}^{[2]}(G;E(A),E)},$$

$$\|W\|_{0,\infty} = \sup_{x\in G} \sum_{k,i} \|w_{ki}\|_{B_{ki}} \le C_0 \|w\|_{W^{[2]}_{p,\alpha}(G;E(A),E)}.$$

**Condition 5.1.** Suppose the following hold:

(1) $E$ is an UMD space and $0 \le \alpha_1,\ \alpha_2 < 1 - \frac{1}{p},\ p \in (1,\infty)$;

(2) $a_k$ are continuous functions on $[0,b_k]$, $a_k(x_k) < 0$, for all $x \in [0,b_k]$, $\delta_{km_{k1}} \neq 0, \beta_{km_{k2}} \neq 0$, $k = 1,2,...,n$;

(3) there exist $\Phi_{ki} \in B_{ki}$ such that the operator $B(t,x,\Phi)$ for $\Phi = \{\Phi_{ki}\} \in B_0$ is $R$-positive in $E$ uniformly with respect to $x \in G_0$ and $t \in [0,T_0]$; moreover,

$$B(t,x,\Phi)B^{-1}\left(t^0,x^0,\Phi\right) \in C\left(\bar{G};L(E)\right),\ t^0 \in (0,T),\ x^0 \in G;$$

(4) $A = B\left(t^0,x^0,\Phi\right)$: $G_T \times B_0 \to L(E(A),E)$ is continuous. Moreover, for each positive $r$ there is a positive constant $L(r)$ such that

$\left\|\left[B(t,x,U) - B\left(t,x,\bar{U}\right)\right]\upsilon\right\|_E \le L(r)\left\|U-\bar{U}\right\|_{B_0}\|A\upsilon\|_E$

for $t \in (0,T)$, $x \in G$, $U,\bar{U} \in B_0, \bar{U} = \{\bar{u}_{ki}\}, \bar{u}_{ki} \in B_{ki}$, $\|U\|_{B_0}, \left\|\bar{U}\right\|_{B_0} \le r, \upsilon \in D(A)$;

(5) the function $F$: $G_T \times B_0 \to E$ such that $F(.,U)$ is measurable for each $U \in B_0$ and $F(t,x,.)$ is continuous for a.a. $t \in (0,T)$, $x \in G$. Moreover, $\left\|F(t,x,U) - F\left(t,x,\bar{U}\right)\right\|_E \le \Psi_r(x)\left\|U-\bar{U}\right\|_{B_0}$ for a.a. $t \in (0,T)$, $x \in G$, $U,\bar{U} \in B_0$ and $\|U\|_{B_0}, \left\|\bar{U}\right\|_{B_0} \le r$; $f(.) = F(.,0) \in L_p(G_T;E)$.

The main result of this section is the following:

**Theorem 5.1.** Let the Condition 5.1 be satisfied. Then there is $T \in (0,T_0)$ and $b_k \in (0,b_{0k})$ such that problem (5.1) – (5.2) has a unique solution belonging to $W^{1,[2]}_{p,\alpha}(G_T;E(A),E)$.

**Proof.** Consider the following linear problem

$$\frac{\partial w}{\partial t} + \sum_{k=1}^{n} a_k(x_k)\frac{\partial^{[2]}w}{\partial x_k^2} + du = f(x,t),\ x \in G,\ t \in (0,T),$$

$$\sum_{i=0}^{m_{k1}} \delta_{ki} w^{[i]}_{x_k}(G_{k0},t) = 0, \sum_{i=0}^{m_{k2}} \beta_{ki} w^{[i]}_{k}(G_{kb},t) = 0, \tag{5.3}$$

$$w(x,0) = 0, t \in (0,T),\ x \in G,\ x^{(k)} \in G_k,\ d > 0.$$

By Theorem 3.1and in view of Proposition 4.1 there is a unique solution $w \in W^{1,[2]}_{p,\alpha}(G_T;E(A),E)$ of the problem (5.3) for $f \in L_p(G_T;E)$ and sufficiently large $d > 0$ and it satisfies the following coercive estimate

$$\|w\|_{W^{1,[2]}_{p,\alpha}(G_T;E(A),E)} \le C_0\|f\|_{L_p(G_T;E)},$$

uniformly with respect to $b \in (0\ ,\ b_0]$, i.e., the constant $C_0$ does not depends on $f \in L_p(G_T;E)$ and $b \in (0\ b_0]$ where

$$A(x) = B(x,0),\ f(x) = F(x,0),\ x \in (0,b).$$

We want to solve the problem (5.1) – (5.2) locally by means of maximal regularity of the linear problem (5.3) via the contraction mapping theorem. For this purpose, let $w$ be a solution of the linear BVP (5.3). Consider a ball

$$B_r = \{\upsilon \in Y,\ \upsilon - w \in Y_1,\ \|\upsilon - w\|_Y \le r\}.$$

For given $\upsilon \in B_r$, consider the following linearized problem

$$\frac{\partial u}{\partial t}+\sum_{k=1}^{n} a_k(x_k)\frac{\partial^{[2]}u}{\partial x_k^2}+A(x)=F(x,V)+[B(x,0)-B(x,V)]\upsilon,$$

$$\sum_{i=0}^{m_{k1}}\delta_{ki}w_{x_k}^{[i]}(G_{k0},t)=0,\sum_{i=0}^{m_{k2}}\beta_{ki}w_k^{[i]}(G_{kb},t)=0, \tag{5.4}$$

$$w(x,0)=0,\ t\in(0,T),\ x\in G,\ x^{(k)}\in G_k.$$

where $V=\{\upsilon_{ki}\}$, $\upsilon_{ki}\in B_{ki}$. Define a map $Q$ on $B_r$ by $Q\upsilon=u$, where $u$ is solution of (5.4). We want to show that $Q(B_r)\subset B_r$ and that $Q$ is a contraction operator provided $T$ and $b_k$ are sufficiently small, and $r$ is chosen properly. In view of separability properties of the problem (5.3) we have

$$\|Q\upsilon-w\|_Y=\|u-w\|_Y\leq C_0\{\|F(x,V)-F(x,0)\|_X+$$

$$\|[B(0,W)-B(x,V)]\upsilon\|_X\}.$$

By assumption (4) we have

$$\|[B(0,W)\upsilon-B(x,V)]\upsilon\|_X\leq\sup_{x\in[0,b]}\Big\{\|[B(0,W)-B(x,W)]\upsilon\|_{L(E_0,E)}$$

$$+\|B(x,W)-B(x,V)\|_{L(E_0,E)}\|\upsilon\|_Y\Big\}\leq$$

$$\Big[\delta(b)+L(R)\|W-V\|_{\infty,E_0}\Big][\|\upsilon-w\|_Y+\|w\|_Y]\leq$$

$$\{\delta(b)+L(R)[C_1\|\upsilon-w\|_Y+\|\upsilon-w\|_Y]$$

$$[\|\upsilon-w\|_Y+\|w\|_Y]\}\leq\delta(b)+L(R)[C_1r+r][r+\|w\|_Y],$$

where

$$\delta(b)=\sup_{x\in[0,b]}\|[B(0,W)-B(x,W)]\|_{B(E_0,E)}.$$

By assumption (5) we get

$$\|F(x,V)-F(x,0,)\|_E\leq\delta(b)+$$

$$\|F(x,V)-F(x,W)\|_E+\|F(x,W)-F(x,0)\|_E\leq$$

$$\delta(b)+\mu_R[\|\upsilon-w\|_Y+\|w\|_Y]$$

$$\mu_R C_1[\|\upsilon-w\|_Y+\|w\|_Y]\leq\mu_R[C_1r+\|w\|_Y],$$

where $R=C_1r+\|w\|_Y$ is a fixed number. In view of above estimates, by suitable choice of $\mu_R$, $L_R$ and for sufficiently small $T\in(0,T_0)$ and $b_k\in(0\ ,b_{0k}]$ we have

$$\|Q\upsilon-w\|_Y\leq r,$$

i.e.

$$Q(B_r) \subset B_r.$$

Moreover, in a similar way we obtain

$$\|Q\upsilon - Q\bar{\upsilon}\|_Y \leq C_0 \{\mu_R C_1 + M_a + L(R)[\|\upsilon - w\|_Y + C_1 r] +$$

$$L(R) C_1 [r + \|w\|_Y] \|\upsilon - \bar{\upsilon}\|_Y\} + \delta(b).$$

By suitable choice of $\mu_R$, $L_R$ and for sufficiently small $T \in (0, T_0)$ and $b_k \in (0, b_{0k})$ we obtain $\|Q\upsilon - Q\bar{\upsilon}\|_Y < \eta \|\upsilon - \bar{\upsilon}\|_Y$, $\eta < 1$, i.e. $Q$ is a contraction operator. Eventually, the contraction mapping principle implies a unique fixed point of $Q$ in $B_r$ which is the unique strong solution $u \in W_{p,\alpha}^{1,[2]}(G_T; E(A), E)$.

## 6 Cauchy problem for nonlinear system of degenerate parabolic equations

Consider the initial and BVP for the system of nonlinear parabolic equations of infinite order

$$\frac{\partial u_m}{\partial t} = \sum_{k=1}^{n} a_k(x) \frac{\partial^{[2]} u_m}{\partial x_k^2} + \sum_{j=1}^{N} d_{mj}(x) u_j(x,t)$$

$$+ \sum_{k=1}^{n} \sum_{j=1}^{N} b_{kj}(x) \frac{\partial^{[1]} u_j}{\partial x_k} + F_m(x,t,u), \tag{6.1}$$

$$\sum_{i=0}^{m_{k1}} \delta_{ki} D_k^{[i]} u_m(G_{k0}, t) = 0, \sum_{i=0}^{m_{k2}} \beta_{ki} D_k^{[i]} u_m(G_{kb}, t) = 0,$$

$$u_m(x,0) = 0,\ x \in G,\ t \in (0,T),\ m = 1,2,...,N,\ N \in \mathbb{N}, \tag{6.2}$$

where $u = (u_1, u_2, ..., u_N)$, $m_{kj} \in \{0,1\}$, $\delta_{ki}, \beta_{ki}$ are complex numbers, $a_k$ are complex valued functions,

$$D_k^{[i]} u = \frac{\partial^{[i]} u}{\partial x_k^i} = \left[ x_k^{\alpha_k} (b_k - x_k)^{\alpha_k} \frac{\partial}{\partial x_k} \right]^i u(x,t),\ 0 \leq \alpha_k < 1,$$

$$x = (x_1, x_2, ..., x_n) \in G = \prod_{k=1}^{n} (0, b_k),\ m_{kj} \in \{0,1\},$$

$$G_{k0} = (x_1, x_2, ..., x_{k-1}, 0, x_{k+1}, ..., x_n),\ q \in (1,\infty),$$

$$G_{kb} = (x_1, x_2, ..., x_{k-1}, b_k, x_{k+1}, ..., x_n);$$

and

$$\theta_{ki} = \frac{m_{ki} + \frac{1}{p(1-\alpha_k)}}{2},\ s_{ki} = s(1-\theta_{ki}),\ s > 0,\ B_{ki} = l_q^{s_{ki}},\ i = 0,1,$$

$$B_0 = \prod_{k,i} B_{ki},\ \alpha_{km_{k1}} \neq 0, \beta_{km_{k2}} \neq 0,\ k = 1,2,...,n.$$

Let $A$ be the operator in $l_q(N)$ defined by

$$D(A) = l_q(N),\ A = \left[d_{mj}(x)\right],\ d_{mj}(x) = g_m(x) 2^{sj},\ m, j = 1,2,...,N,$$

where

$$l_q(N) = \left\{ u = \left\{u_j\right\},\ j = 1,2,...N, \|u\|_{l_q(N)} = \left( \sum_{j=1}^{N} \left|u_j\right|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$l_q(A) = \left\{ u \in l_q(N), \|u\|_{l_q(A)} = \|Au\|_{l_q(N)} = \left( \sum_{j=1}^{N} \left|2^{sj} u_j\right|^q \right)^{\frac{1}{q}} < \infty \right\},$$

$$x \in G,\ 1 < q < \infty,\ N = 1,2,...,\infty.$$

Let $b_{kj}(x) = M_{kj}(x) 2^{\sigma j}$ and

$$B = B\left(L_p\left(G; l_q(N)\right)\right).$$

From Theorem 5.1 we obtain the following result

**Theorem 6.1.** Let the following condition hold:

(1) $a_k$ are continuous functions on $\bar{G}$ and $a_k(x) < 0$;

(2) $s \geq \frac{2np(2-q)}{q(p-1)}, 0 < \sigma < s_0,\ s_0 = \frac{s(p-1)}{2p}$, and

$$0 \leq \alpha_k < 1 - \frac{1}{p},\ p, q \in (1, \infty);$$

(3) $g_j \in C\left(\bar{G}\right), N_{kj} \in C\left(\bar{G}\right)$; $d_{ii}(x) > 0$ and eigenvalues of the matrix $[d_{mi}(x)]$ are positive for all $x \in \bar{G},\ m, i = 1,2,...,N$ ; there is a positive constant $C$ such that

$$\sum_{k=1}^{n} \sum_{j=1}^{N} M_{kj}^{q_1}(x) \leq C \sum_{j=1}^{N} g_j^{q_1}(x) < \infty,\ x \in G,\ \frac{1}{q} + \frac{1}{q_1} = 1;$$

(4) the function $F(.,\upsilon) = (F_1(.,\upsilon), ..., F_N(.,\upsilon))$ is measurable for each $\upsilon \in B_{0p}$ and the function $F(x,.)$ for a.a. $x \in G$ is continuous and $f(.) = F(.,0) \in L_p\left(G; l_q\right)$; for each $R > 0$ there is a function $\Psi_R \in L_\infty(G)$ such that

$$\left\|F(x,U) - F\left(x,\bar{U}\right)\right\|_{l_q} \leq \Psi_R(x) \left\|U - \bar{U}\right\|_{l_q(A)}$$

a.a. $x \in G$ and

$$U, \bar{U} \in B_{0p}, \|U\|_{B_{0p}} \leq R, \|U\|_{B_{0p}} \leq R, U = \left\{u_{kj}\right\}, \bar{U} = \left\{\bar{u}_{kj}\right\}, u_{kj}, \bar{u}_{kj} \in B_{0p}.$$

Then there is $T \in (0, T_0)$ and $b_k \in (0, b_{0k})$ such that problem (6.1) – (6.2) has a unique solution $u = \{u_m(x)\}_1^N$ that belongs to space $W_p^{1,2}\left(G_T, l_q(A), l_q\right)$.

**Proof.** By virtue of [26], the $l_q(N)$ is a UMD space. It is easy to see that the operator $A$ is $R$-positive in $l_q(N)$. Then by using the conditions (1)-(3) we get that the condition (5) of Theorem 5.1 is hold. So in view of the Theorem 5.1 we obtain the assertion.

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