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Conditions of Linearizability for Multi-control Systems of the Class C^1

K. V. Sklyar[*]
University of Szczecin, Wielkopolska 15, Szczecin 70-451, Poland

S. Yu. Ignatovich[†]
Department of Differential Equations and Control,
Kharkov National University, Svobody sqr. 4, Kharkov 61077, Ukraine

V. O. Skoryk[‡]
Department of Differential Equations and Control,
Kharkov National University, Svobody sqr. 4, Kharkov 61077, Ukraine

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Abstract

We give the complete description of nonlinear control systems of the class C^1 with multi-dimensional control that are linearizable by means of changes of variables of the class C^2.

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1 Introduction and statement of the problem

In this paper we consider the linearizability problem for systems of the form

$$\dot{x} = f(x,u),\ x \in Q \subset \mathbb{R}^n,\ u \in \mathbb{R}^r, \tag{1.1}$$

where the vector function $f(x,u)$ is continuously differentiable, i.e., $f(x,u) \in C^1(Q \times \mathbb{R}^r)$. System (1.1) is linearizable, if there exists a nonsingular change of variables $z = F(x)$ such that in the new variables the system has a linear (more precisely, an affine) form

$$\dot{z} = Az + Bu + c, \quad z \in \mathbb{R}^n,\ u \in \mathbb{R}^r. \tag{1.2}$$

[*]E-mail address: sklar@univ.szczecin.pl
[†]E-mail address: ignatovich@ukr.net
[‡]E-mail address: skoryk@univer.kharkov.ua

The close statement of the problem concerns feedback linearizability: the system (1.1) is feedback linearizable, if there exists a nonsingular change of variables $z = F(x)$ and a nonsingular change of the control $v = g(x,u)$, which reduce the system to the linear form

$$\dot{z} = Az + Bv.$$

In the class C^∞, the conditions of linearizability and feedback linearizability are well known [6, 2, 7]. However, such smoothness requirements are not necessary: for a special class of *triangular systems* the feedback linearizability problem was considered for the class C^1 [5].

For nonlinear systems (1.1) with one-dimensional control (i.e., with $r = 1$), the conditions of linearizability and feedback linearizability in the class C^1 were obtained in [8]. It turned out that the Lie brackets technique, which is commonly used for C^∞-smoothness systems, *can be* successfully applied in the problem of linearizability. Let us explain this point more specifically. We use the standard notation for the Lie brackets, $[a(x),b(x)] = (b(x))_x a(x) - (a(x))_x b(x)$, and $\mathrm{ad}_a^0 b(x) = b(x)$, $\mathrm{ad}_a^k b(x) = [a(x), \mathrm{ad}_a^{k-1} b(x)]$, $k \geq 1$. Then, if a nonlinear system with one-dimensional control is linearizable, it has the affine form, i.e., $f(x,u) = a(x) + b(x)u$, where vector fields $a(x)$, $b(x)$ are of class $C^1(Q)$ and *all their Lie brackets* $\mathrm{ad}_a^k b(x)$ *necessarily exist and are of class* $C^1(Q)$. It is worth noting that for feedback linearizable systems one should introduce some new vector fields instead of $\mathrm{ad}_a^k b(x)$, since they, generally, do not exist.

The present paper deals with the linearizability problem for systems with multi-dimensional control and complements the approach and the results of [8]. Namely, we study the linearizability problem for systems of the form (1.1), which means the mappability to affine systems of the form (1.2); we suppose that affine systems are controllable and the number of controls cannot be reduced, that is,

$$\mathrm{rank}(B, AB, \ldots, A^{n-1}B) = n \ \text{ and } \ \mathrm{rank} B = r. \tag{1.3}$$

Definition 1.1. We say that a control system of the form (1.1), where $f(x,u) \in C^1(Q \times \mathbb{R}^r)$, is *locally linearizable in the domain* Q, if there exists a change of variables

$$z = F(x) \in C^2(Q) \ \text{ such that } \ \det(F(x))_x \neq 0,\ x \in Q, \tag{1.4}$$

which reduces the system (1.1) to a linear form (1.2), (1.3).

Analogously to [8], we seek a change of variables, which is defined in the domain (not in a neighborhood); however, we require only local invertibility (in this sense our approach is close to [3]). In the next section, we give a criterion of local linearizability, which turns to be close to the criterion in the case C^∞ [7].

2 Conditions of linearizability

Theorem 2.1. *A nonlinear system of the form* (1.1)*, where* $f(x,u) \in C^1(Q \times \mathbb{R}^r)$*, is locally linearizable in the domain* Q *if and only if there exist integers* $\ell_1, \ldots, \ell_r \geq 1$, $\ell_1 + \cdots + \ell_r = n$*, such that the following conditions hold:*

(A) $f(x,u) = a(x) + \sum_{i=1}^{r} b_i(x) u_i$*, where* $a(x), b_1(x), \ldots, b_r(x) \in C^1(Q)$*;*

(B1) *vector functions* $\mathrm{ad}_a^k b_s(x)$*,* $s = 1, \ldots, r$*,* $k = 0, \ldots, \ell_s$*, exist and belong to the class* $C^1(Q)$*;*

(B2) $\mathrm{rank} M(x) = n$ *for* $x \in Q$*, where*

$$M(x) = (b_1(x), \ldots, \mathrm{ad}_a^{\ell_1 - 1} b_1(x), \ldots, b_r(x), \ldots, \mathrm{ad}_a^{\ell_r - 1} b_r(x)),$$

(B3) $[\mathrm{ad}_a^k b_s(x), \mathrm{ad}_a^j b_q(x)] = 0$, $x \in Q$*, for all* $s, q = 1, \ldots, r$*,* $k = 0, \ldots, \ell_s$*,* $j = 0, \ldots, \ell_q$*.*

Proof is almost obvious for C^∞-smooth systems. Our goal is to give arguments, which are correct in the class C^1.

Necessity can be proved completely analogously to [8, Propositions 2 and 4].

Sufficiency. First, we note that (B1) and (B2) imply

$$\mathrm{ad}_a^{\ell_s} b_s(x) = \sum_{k=1}^{r} \sum_{i=0}^{\ell_k - 1} \nu_{k,i}^{s,\ell_s}(x) \mathrm{ad}_a^i b_k(x), \quad x \in Q,$$

where $\nu_{k,i}^{s,\ell_s}(x)$ are some functions defined on Q. Moreover, due to (B1) and (B2), $\nu_{k,i}^{s,\ell_s}(x) \in C^1(Q)$. Now we show that $\nu_{k,i}^{s,\ell_s}(x)$ are constant. For any $1 \le m \le r$ and $0 \le p \le \ell_m - 1$ we have

$$[\mathrm{ad}_a^p b_m(x), \mathrm{ad}_a^{\ell_s} b_s(x)] = \sum_{k=1}^{r} \sum_{i=0}^{\ell_k - 1} \Big((\nu_{k,i}^{s,\ell_s}(x))_x \mathrm{ad}_a^p b_m(x) \Big) \mathrm{ad}_a^i b_k(x) + \sum_{k=1}^{r} \sum_{i=0}^{\ell_k - 1} \nu_{k,i}^{s,\ell_s}(x) [\mathrm{ad}_a^p b_m(x), \mathrm{ad}_a^i b_k(x)] = 0,$$

hence, conditions (B2) and (B3) imply

$$(\nu_{k,i}^{s,\ell_s}(x))_x \mathrm{ad}_a^p b_m(x) = 0 \ \text{ for } \ 1 \le m \le r, \ 0 \le p \le \ell_m - 1.$$

Using (B2) once more, we get $\nu_{k,i}^{s,\ell_s}(x) = const \equiv \nu_{k,i}^{s,\ell_s}$. Thus,

$$\mathrm{ad}_a^{\ell_s} b_s(x) = \sum_{k=1}^{r} \sum_{i=0}^{\ell_k - 1} \nu_{k,i}^{s,\ell_s} \mathrm{ad}_a^i b_k(x), \ 1 \le s \le r.$$

Therefore, $\mathrm{ad}_a^m b_s(x)$ exist and belong to the class $C^1(Q)$ for *all* $m \ge \ell_s$ and $1 \le s \le r$, and moreover,

$$\mathrm{ad}_a^m b_s(x) = \sum_{k=1}^{r} \sum_{i=0}^{\ell_k - 1} \nu_{k,i}^{s,m} \mathrm{ad}_a^i b_k(x), \quad 1 \le s \le r, \ m \ge \ell_s, \tag{2.1}$$

where $\nu_{k,i}^{s,m}$ are certain constants.

Now, let us fix any q such that $1 \le q \le r$. Consider the following system of n partial differential equations

$$\begin{aligned} &(\varphi(x))_x \mathrm{ad}_a^j b_s(x) = 0, \quad 1 \le s \le r, \ 0 \le j \le \ell_s - 1, \ (s,j) \ne (q, \ell_q - 1), \\ &(\varphi(x))_x \mathrm{ad}_a^{\ell_q - 1} b_q(x) = 1, \end{aligned} \tag{2.2}$$

or, in the matrix form,

$$(\varphi(x))_x M(x) = e_p,$$

where e_p is a unit row vector with 1 on the p-th place, $p = \ell_1 + \cdots + \ell_q$. Due to condition (B2), this system can be rewritten as

$$(\varphi(x))_x = h(x), \ \text{ where } \ h(x) = e_p (M(x))^{-1} \in C^1(Q). \tag{2.3}$$

It is well known that the necessary and sufficient condition of solvability of this system is

$$\frac{\partial h_i(x)}{\partial x_j} = \frac{\partial h_j(x)}{\partial x_i}, \quad i, j = 1, \ldots, n. \tag{2.4}$$

Moreover, Q is a domain, and therefore, *is simply connected*, hence, the condition (2.4) implies the solvability of (2.2) in Q [4, Chapter VI]. Let us prove (2.4). Denote by $h^T(x)$ the column vector, which is the transpose of $h(x)$, and denote by $M_k(x)$, $k = 1, \ldots, n$, the columns of the matrix $M(x)$. Let $\langle \cdot, \cdot \rangle$ be the inner product. Then, due to the definition,

$$\langle h^T(x), M_k(x) \rangle = const.$$

Differentiating the both sides of this equality w.r.t. x and then multiplying by $M_s(x)$, we get

$$\langle (h^T(x))_x M_s(x), M_k(x) \rangle + \langle h^T(x), (M_k(x))_x M_s(x) \rangle = 0.$$

Substituting s instead of k and vice versa, we get

$$\langle (h^T(x))_x M_k(x), M_s(x)\rangle + \langle h^T(x), (M_s(x))_x M_k(x)\rangle = 0.$$

Due to condition (B3),

$$[M_s(x), M_k(x)] = (M_k(x))_x M_s(x) - (M_s(x))_x M_k(x) = 0.$$

Hence,

$$\langle (h^T(x))_x M_s(x), M_k(x)\rangle = \langle (h^T(x))_x M_k(x), M_s(x)\rangle \text{ for any } k,s = 1,\dots,n.$$

This means that the matrix $(h^T(x))_x$ is symmetric, i.e., (2.4) holds. Therefore, the system (2.3), or, what is the same, the system (2.2) has a solution; since $h(x)\in C^1(Q)$, this solution is necessarily of class $C^2(Q)$. (It is defined uniquely up to a constant.)

For any $q = 1,\dots,r$, let us choose a solution of the system (2.2) and denote it by $\varphi_q(x)\in C^2(Q)$. We note that equalities (2.1) give

$$(\varphi_q(x))_x \mathrm{ad}_a^m b_s(x) = const \equiv y_{q,s}^m \text{ for } 1\le s\le r,\ m\ge 0, \tag{2.5}$$

where, in particular,

$$\begin{aligned} &y_{q,q}^m = 0 \text{ if } 0\le m\le \ell_q-2,\\ &y_{q,q}^{\ell_q-1} = 1,\\ &y_{q,s}^m = 0 \text{ if } 1\le s\le r,\ s\ne q \text{ and } 0\le m\le \ell_s-1. \end{aligned} \tag{2.6}$$

Below we use the standard notation $L_a^0\varphi(x) = \varphi(x)$ and $L_a^k\varphi(x) = (L_a^{k-1}\varphi(x))_x a(x)$ for $k\ge 1$. Let us prove that $L_a^k\varphi_q(x)$ exist for all $k\ge 0$, and, moreover,

$$L_a^k\varphi_q(x)\in C^2(Q) \text{ for } k\ge 0, \tag{2.7}$$

$$(L_a^k\varphi_q(x))_x \mathrm{ad}_a^j b_s(x) = (-1)^k y_{q,s}^{j+k} \text{ for } 1\le s\le r,\ k\ge 0,\ j\ge 0. \tag{2.8}$$

We use the induction on k. For $k=0$, there is nothing to prove. Suppose (2.7), (2.8) hold for $k=d\ge 0$. Then, using the symmetry of $(L_a^d\varphi_q(x))_{xx}$, we get

$$(L_a^{d+1}\varphi_q(x))_x \mathrm{ad}_a^j b_s(x) = \big((L_a^d\varphi_q(x))_x a(x)\big)_x \mathrm{ad}_a^j b_s(x) =$$

$$= \big((L_a^d\varphi_q(x))_x \mathrm{ad}_a^j b_s(x)\big)_x a(x) - (L_a^d\varphi_q(x))_x \mathrm{ad}_a^{j+1} b_s(x) = (-1)^{d+1} y_{q,s}^{j+d+1} \text{ for } 1\le s\le r,\ j\ge 0,$$

what implies (2.8) for $k=d+1$. Hence,

$$(L_a^{d+1}\varphi_q(x))_x M(x) = const,$$

therefore, (2.7) holds for $k=d+1$. By induction, (2.7), (2.8) are proved.

Let us denote $\sigma_1 = 0$ and $\sigma_q = \ell_1+\dots+\ell_{q-1}$ for $q = 2,\dots,r$, and consider the change of variables $z = F(x)\in C^2(Q)$ of the form

$$z_{\sigma_q+k} = F_{\sigma_q+k}(x) = L_a^{k-1}\varphi_q(x), \quad 1\le q\le r,\ 1\le k\le \ell_q. \tag{2.9}$$

First, we prove that the functions $F_{\sigma_q+k}(x)$ are independent. Assume the converse; then $\det(F(x))_x = 0$ for some $x\in Q$. Hence, there exists a vector $v\ne 0$ such that $(F(x))_x v = 0$. Let us express v as a linear combination of columns of the matrix $M(x)$, i.e., $v = \sum_{s=1}^r\sum_{j=0}^{\ell_s-1}\mu_{s,j}\mathrm{ad}_a^j b_s(x)$. Using (2.8), we get

$$(L_a^{k-1}\varphi_q(x))_x \sum_{s=1}^r\sum_{j=0}^{\ell_s-1}\mu_{s,j}\mathrm{ad}_a^j b_s(x) = \sum_{s=1}^r\sum_{j=0}^{\ell_s-1}\mu_{s,j}(-1)^{k-1} y_{q,s}^{k+j-1} = 0 \text{ for any } 1\le q\le r,\ 1\le k\le \ell_q. \tag{2.10}$$

It is convenient to put $\mu_{s,j} = 0$ if $j<0$. Then, (2.10) and (2.6) imply

$$\sum_{s=1}^r\sum_{j=\ell_s-k+1}^{\ell_s-1}\mu_{s,j} y_{q,s}^{k+j-1} + \mu_{q,\ell_q-k} = 0 \text{ for any } 1\le q\le r,\ 1\le k\le \ell_q.$$

Choosing successively $k = 1,\dots,\max\{\ell_1,\dots,\ell_r\}$ for $q = 1,\dots,r$, we get that the set of numbers $\mu_{s,j}$ is trivial, hence, $v = 0$; this contradicts our supposition. Thus, the functions (2.9) are independent, i.e., $\det(F(x))_x \neq 0$, $x \in Q$.

Let us find the form of the system in the new variables. We fix any $q = 1,\dots,r$. Then for $1 \le k \le \ell_q$ we get

$$\dot{z}_{\sigma_q+k} = (F_{\sigma_q+k}(x))_x\Big(a(x)+\sum_{i=1}^{r} b_i(x)u_i\Big) = (L_a^{k-1}\varphi_q(x))_x a(x) + \sum_{i=1}^{r}(L_a^{k-1}\varphi_q(x))_x b_i(x)u_i =$$

$$= L_a^k\varphi_q(x) + \sum_{i=1}^{r}(L_a^{k-1}\varphi_q(x))_x \mathrm{ad}_a^0 b_i(x)u_i = L_a^k\varphi_q(x) + \sum_{i=1}^{r}(-1)^{k-1}y_{q,i}^{k-1}u_i.$$

For $1 \le k \le \ell_q - 1$ we have $L_a^k\varphi_q(x) = F_{\sigma_q+k+1}(x) = z_{\sigma_q+k+1}$. Let us express $L_a^{\ell_q}\varphi_q(x)$ via z_j. Due to (2.8), we get

$$(L_a^{\ell_q}\varphi_q(x))_x M(x) = w_q, \quad (F(x))_x M(x) = Y,$$

where w_q is a constant row and Y is a constant nonsingular matrix. Then

$$(L_a^{\ell_q}\varphi_q(x))_x M(x) = w_q Y^{-1}(F(x))_x M(x),$$

what gives $L_a^{\ell_q}\varphi_q(x) - w_q Y^{-1}F(x) = const$. Hence, $L_a^{\ell_q}\varphi_q(x) = \sum_{j=1}^{n} p_{qj}z_j + p_{q0}$ for some numbers $p_{q0},\dots,p_{qn}$. Thus,

$$\dot{z}_{\sigma_q+k} = z_{\sigma_q+k+1} + \sum_{i=1}^{r}(-1)^{k-1}y_{q,i}^{k-1}u_i, \quad k = 1,\dots,\ell_q-1,$$

$$\dot{z}_{\sigma_q+\ell_q} = \sum_{j=1}^{n} p_{qj}z_j + p_{q0} + \sum_{i=1}^{r}(-1)^{\ell_q-1}y_{q,i}^{\ell_q-1}u_i, \quad q = 1,\dots,r.$$

This means that the system in the new variables has the form (1.2). Let us note that $(F(x))_x a(x) = AF(x) + c$, $(F(x))_x b_s(x) = B_s$, where B_s is the s-th column of the matrix B. By our supposition, $\ell_1,\dots,\ell_r \ge 1$, hence, condition (B2) implies that $b_1(x),\dots,b_r(x)$ are linearly independent. One can show analogously to [8, Lemma 1] that the condition $F(x) \in C^2(Q)$ gives $(F(x))_x \mathrm{ad}_a^j b_s(x) = (-1)^j A^j B_s$, $j \ge 0$. Since $(F(x))_x$ is nonsingular, we get (1.3) from (B2). □

Remark 2.2. For the case $r = 1$, Theorem 2.1 implies that condition (B4) of [8, Theorem 3] follows from the other conditions of the theorem.

Remark 2.3. In Theorem 2.1, one can try to consider integers $\ell_1,\dots,\ell_n$ depending on the point x. More specifically, suppose Q is covered by several domains, each of which has its own set of numbers $\ell_1,\dots,\ell_n$ satisfying (B1)–(B3) (in the intersection of two such domains the both sets can be used). However, since Q is connected, representation (2.1) shows that all such sets of numbers are suitable for all points of Q.

Remark 2.4. Recall that the *controllability indices* [1, 9] are defined as follows: put $w_0 = 0$, $w_j = \mathrm{rank}(B,\dots,A^{j-1}B)$, $j \ge 1$, then controllability indices are $n_q = \max\{j : w_j - w_{j-1} \ge q\}$, $q = 1,\dots,r$. It is well known that each system (1.2) satisfying (1.3) can be reduced to the canonical form

$$\begin{aligned} &\dot{z}_{\sigma_q+k} = z_{\sigma_q+k+1}, \quad k = 1,\dots,n_q-1,\\ &\dot{z}_{\sigma_q+n_q} = \sum_{j=1}^{n} p_{qj}z_j + p_{q0} + u_{s_q} + \sum_{i:\, n_i<n_q} d_{qi}u_{s_i}, \quad q = 1,\dots,r, \end{aligned} \tag{2.11}$$

where $\sigma_1 = 0$, $\sigma_q = n_1 + \cdots + n_{q-1}$ for $q = 2,\dots,r$, and $\{s_1,\dots,s_r\}$ is a permutation of the set $\{1,\dots,r\}$. We note that the numbers $\ell_1,\dots,\ell_r$ from Theorem 2.1 not necessarily coincide with the controllability indices. For example, for the system

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_4 + u_2,\ \dot{z}_3 = z_4,\ \dot{z}_4 = u_1,$$

one can choose $\ell_1 = 3$, $\ell_2 = 1$ or $\ell_1 = \ell_2 = 2$; however, only the second pair really gives the controllability indices.

Let us re-number $b_1, \dots, b_r$ so that $\ell_1 \geq \cdots \geq \ell_r$. One can show that $\ell_1, \dots, \ell_r$ coincide with the controllability indices if, in addition to conditions of Theorem 2.1,

$$\mathrm{ad}_a^{\ell_q} b_q(x) \in \mathrm{Lin}\{\mathrm{ad}_a^k b_s(x) : 1 \leq s \leq r,\ 0 \leq k \leq \min\{\ell_q, \ell_s - 1\}\},\ x \in Q,\ q = 1, \dots, r. \tag{2.12}$$

Example 2.5. Consider the system of the class C^1

$$\dot{x}_1 = x_2 + x_2^2|x_2|,\ \dot{x}_2 = \tfrac{x_4}{1+3x_2|x_2|} + \tfrac{1}{1+3x_2|x_2|}u_2,\ \dot{x}_3 = \tfrac{x_4}{1-3|x_3|x_3},\ \dot{x}_4 = u_1, \tag{2.13}$$

in the domain $Q = \{x \in \mathbb{R}^4 : x_2 > -\frac{1}{\sqrt{3}},\ x_3 < \frac{1}{\sqrt{3}}\}$. For brevity, denote $f(x) = x + x^2|x|$, $g(x) = x - x^2|x|$, then the system can be rewritten as

$$\dot{x}_1 = f(x_2),\ \dot{x}_2 = \tfrac{x_4}{f'(x_2)} + \tfrac{1}{f'(x_2)}u_2,\ \dot{x}_3 = \tfrac{x_4}{g'(x_3)},\ \dot{x}_4 = u_1.$$

We have

$$a(x) = \begin{pmatrix} f(x_2) \\ \frac{x_4}{f'(x_2)} \\ \frac{x_4}{g'(x_3)} \\ 0 \end{pmatrix},\ b_1(x) = e_4,\ b_2(x) = \begin{pmatrix} 0 \\ \frac{1}{f'(x_2)} \\ 0 \\ 0 \end{pmatrix},\ \mathrm{ad}_a b_1(x) = \begin{pmatrix} 0 \\ -\frac{1}{f'(x_2)} \\ -\frac{1}{g'(x_3)} \\ 0 \end{pmatrix},\ \mathrm{ad}_a b_2(x) = -e_1,\ \mathrm{ad}_a^2 b_1(x) = e_1,$$

and $\mathrm{ad}_a^2 b_2(x) = \mathrm{ad}_a^3 b_1(x) = 0$. Hence, conditions of Theorem 2.1 hold with $\ell_1 = 3$, $\ell_2 = 1$ and $\ell_1 = 2$, $\ell_2 = 2$.

First, let us choose $\ell_1 = 3$, $\ell_2 = 1$. Then $\mathrm{ad}_a^1 b_2(x) \notin \mathrm{Lin}\{\mathrm{ad}_a^0 b_1(x), \mathrm{ad}_a^0 b_2(x), \mathrm{ad}_a^1 b_1(x)\}$, i.e., in this case the condition (2.12) does not hold. A linearizing change of variables is defined by the system

$$(\varphi_1(x))_x b_1(x) = 0, \quad (\varphi_1(x))_x \mathrm{ad}_a b_1(x) = 0, \quad (\varphi_1(x))_x \mathrm{ad}_a^2 b_1(x) = 1, \quad (\varphi_1(x))_x b_2(x) = 0,$$

$$(\varphi_2(x))_x b_1(x) = 0, \quad (\varphi_2(x))_x \mathrm{ad}_a b_1(x) = 0, \quad (\varphi_2(x))_x \mathrm{ad}_a^2 b_1(x) = 0, \quad (\varphi_2(x))_x b_2(x) = 1,$$

what gives

$$\frac{\partial \varphi_1(x)}{\partial x_1} = 1,\ \frac{\partial \varphi_1(x)}{\partial x_2} = 0,\ \frac{\partial \varphi_1(x)}{\partial x_3} = 0,\ \frac{\partial \varphi_1(x)}{\partial x_4} = 0,$$

$$\frac{\partial \varphi_2(x)}{\partial x_1} = 0,\ \frac{\partial \varphi_2(x)}{\partial x_2}\frac{1}{f'(x_2)} = 1,\ \frac{\partial \varphi_2(x)}{\partial x_2}\frac{1}{f'(x_2)} + \frac{\partial \varphi_2(x)}{\partial x_3}\frac{1}{g'(x_3)} = 0,\ \frac{\partial \varphi_2(x)}{\partial x_4} = 0.$$

As a solution, let us choose $\varphi_1(x) = x_1$, $\varphi_2(x) = f(x_2) - g(x_3)$; then a linearizing change of variables can be chosen as

$$z_1 = \varphi_1(x) = x_1, \quad z_2 = L_a\varphi_1(x) = f(x_2), \quad z_3 = \varphi_2(x) = L_a^2\varphi_1(x) = x_4, \quad z_4 = \varphi_2(x) = f(x_2) - g(x_3),$$

and system (2.13) is reduced to

$$\dot{z}_1 = z_2,\ \dot{z}_2 = z_3 + u_2,\ \dot{z}_3 = u_1,\ \dot{z}_4 = u_2.$$

We note that this system is not of the form (2.11).

Now we choose $\ell_1 = 2$, $\ell_2 = 2$; these numbers obviously satisfy the condition (2.12). In this case we get the system

$$(\varphi_1(x))_x b_1(x) = 0, \quad (\varphi_1(x))_x \mathrm{ad}_a b_1(x) = 1, \quad (\varphi_1(x))_x b_2(x) = 0, \quad (\varphi_1(x))_x \mathrm{ad}_a b_2(x) = 0,$$

$$(\varphi_2(x))_x b_1(x) = 0, \quad (\varphi_2(x))_x \mathrm{ad}_a b_1(x) = 0, \quad (\varphi_2(x))_x b_2(x) = 0, \quad (\varphi_2(x))_x \mathrm{ad}_a b_2(x) = 1,$$

what gives

$$\frac{\partial \varphi_1(x)}{\partial x_1} = 0,\ \frac{\partial \varphi_1(x)}{\partial x_2} = 0,\ \frac{\partial \varphi_1(x)}{\partial x_3} = -g'(x_3),\ \frac{\partial \varphi_1(x)}{\partial x_4} = 0,$$

$$\frac{\partial \varphi_2(x)}{\partial x_1} = -1,\ \frac{\partial \varphi_2(x)}{\partial x_2} = 0,\ \frac{\partial \varphi_2(x)}{\partial x_3} = 0,\ \frac{\partial \varphi_2(x)}{\partial x_4} = 0.$$

We can choose $\varphi_1(x) = -g(x_3)$, $\varphi_2(x) = -x_1$; then a linearizing change of variables can be chosen in the form

$$z_1 = \varphi_1(x) = -g(x_3), \quad z_2 = L_a\varphi_1(x) = -x_4, \quad z_3 = \varphi_2(x) = -x_1, \quad z_4 = L_a\varphi_2(x) = -f(x_2),$$

and system (2.13) is reduced to the form

$$\dot{z}_1 = z_2,\ \dot{z}_2 = -u_1,\ \dot{z}_3 = z_4,\ \dot{z}_4 = z_2 - u_2.$$

Multiplying all z_i by -1, we get the system of the form (2.11).

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