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Toeplitz Operators with Piecewise Quasicontinuous Symbols

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Abstract

In the paper, QC stands for the C^*-algebra of quasicontinuous functions on $\partial\mathbb{D}$ defined by D. Sarason in [10]. For a fixed subset $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ of the unit circle $\partial\mathbb{D}$, we define the algebra PC of piecewise continuous functions in $\partial\mathbb{D} \setminus \Lambda$ with one-sided limits at each point $\lambda_k \in \Lambda$. We define PQC as the C^*-algebra generated by both PC and QC.

$\mathcal{A}^2(\mathbb{D})$ stands for the Bergman space of the unit disk $\mathbb{D}$, that is, the space of square integrable and analytic functions defined on $\mathbb{D}$. Let $\mathcal{K}$ denote the ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$. Our goal is to describe the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$, where $\mathcal{T}_{PQC}$ is the C^*-algebra generated by Toeplitz operators acting on $\mathcal{A}^2(\mathbb{D})$ whose symbols are certain extensions of functions in PQC. A function defined on $\partial\mathbb{D}$ can be extended to the disk in many ways, the more natural extensions are the harmonic and the radial ones. In the final part of this paper we prove that the description of $\mathcal{T}_{PQC}$ does not depend on the extension chosen.

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1 Introduction

We consider the C^*-algebra of quasicontinuous functions QC, which consists of all functions f on the unit circle $\partial\mathbb{D}$ such that both, f and its complex conjugate $\bar{f}$, belong to $H^\infty + C$. Here H^∞ denotes the set (algebra) of boundary functions for bounded analytic functions on the unit disk $\mathbb{D}$, and C stands for the algebra of continuous functions on $\partial\mathbb{D}$. The space QC has two natural extensions to the disk, namely, the radial and the harmonic extension, we denote these extensions by QC_R and QC_H, respectively.

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We use $\mathcal{A}^2(\mathbb{D})$ to denote the Bergman space of $L^2(\mathbb{D})$ which consists of all analytic functions on $\mathbb{D}$. We denote by $B_{\mathbb{D}}$ the Bergman projection $B_{\mathbb{D}} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ and by $\mathcal{K}$ the ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$.

Recall that, for a bounded function f on $\mathbb{D}$, the Toeplitz operator T_f acting on $\mathcal{A}^2(\mathbb{D})$ is defined by the formula $T_f(g) = B_{\mathbb{D}}(fg)$. For a linear subspace $\mathcal{A} \subset L^\infty(\mathbb{D})$ we denote by $\mathcal{T}_{\mathcal{A}}$ the (closed) operator C^*-algebra generated by Toeplitz operators with defining symbols in $\mathcal{A}$.

In this paper we describe the Calkin algebras $\mathcal{T}_{QC_R}/\mathcal{K}$ and $\mathcal{T}_{QC_H}/\mathcal{K}$. We use the characterization of QC as the set of bounded functions with vanishing mean oscillation to prove that the Calkin algebras $\mathcal{T}_{QC_R}/\mathcal{K}$ and $\mathcal{T}_{QC_H}/\mathcal{K}$ are commutative, moreover they are isomorphic.

For a finite set of points $\Lambda := \{\lambda_1, \dots, \lambda_n\}$ of $\partial\mathbb{D}$, we define the space of piecewise continuous functions $PC := PC_\Lambda$ as the algebra of continuous functions on $\partial\mathbb{D} \setminus \Lambda$ with one-sided limits at each point $\lambda_k \in \Lambda$.

We denote by PQC the C^*-algebra generated by both PC and QC. We use an extension of PQC to the disk and thus define the Toeplitz operator algebra $\mathcal{T}_{PQC} \subset \mathcal{B}(\mathcal{A}^2(\mathbb{D}))$. There are several ways to extend the functions in PQC to $\mathbb{D}$; two of them are: the radial extension, PQC_R, and the harmonic extension, PQC_H. The main goal of this paper is the description of the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$, which is stated in Theorem 3.15. In Section 4, we prove that the result does not depend on the extension chosen for PQC, that is, up to compact operators $\mathcal{T}_{PQC_R}$ and $\mathcal{T}_{PQC_H}$ are the same C^*-algebra.

2 Preliminaries

First of all, we set some notation that will be used throughout the paper. Any mathematical symbol not described here will be used in its more common sense. We denote by $\mathbb{D}$ the unit disk and by $\partial\mathbb{D}$ its boundary, the unit circle. The sets $\mathbb{D}$ and $\partial\mathbb{D}$ are endowed with the standard topology and with the Lebesgue measures $dz = dxdy$ and $d\theta$, where the point $z = x + iy$ belongs to $\mathbb{D}$ and $e^{i\theta}$ belongs to $\partial\mathbb{D}$. All the functions in the paper are considered as complex-valued.

This section includes some basic facts about the space of Vanishing Mean Oscillation functions on $\partial\mathbb{D}$, denoted here by VMO. The importance of this space lies in the fact that $QC = VMO \cap L^\infty$ (see [9]). For the convenience of the reader we recall the relevant material from [10] omitting proofs, thus making the exposition self-contained.

We define the following spaces of functions on $\partial\mathbb{D}$:

- $L^\infty := L^\infty(\partial\mathbb{D})$ = the algebra of bounded measurable functions $f : \partial\mathbb{D} \to \mathbb{C}$,
- $H^\infty := H^\infty(\partial\mathbb{D}) =$ the algebra of radial limits of bounded analytic functions defined on $\mathbb{D}$,
- $C := C(\partial\mathbb{D})$ = the algebra of continuous functions on $\partial\mathbb{D}$.

Definition 2.1 ([10], Page 818)**.** We define the C^*-algebra of quasicontinuous functions QC as the algebra of all bounded functions f on $\partial\mathbb{D}$ such that both, f and its complex conjugate $\bar{f}$, belong to $H^\infty + C$, that is;

$$QC := (H^\infty + C) \cap \left(\overline{H}^\infty + C\right).$$

Some of the statements below are formulated for segments in the real line, but they can also be formulated for arcs in $\partial\mathbb{D}$.

By an interval on $\mathbb{R}$ we always mean a finite interval. The length of the interval I will be denoted by $|I|$.

For $f \in L^1(I)$, the average of f over I is given by

$$I(f) := |I|^{-1} \int_I f(t)dt. \tag{2.1}$$

For $a > 0$, we set

$$M_a(f, I) := \sup_{J \subset I, |J| < a} \frac{1}{|J|} \int_J |f(t) - J(f)| dt.$$

Note that $0 \le M_a(f, I) \le M_b(f, I)$ if $a \le b$, thus let $M_0(f, I) := \lim_{a \to 0} M_a(f, I)$.

Definition 2.2 ([10], Page 818)**.** A function $f \in L^1(I)$ is of vanishing mean oscillation in the interval I (or the arc I), if $M_0(f, I) = 0$. The set of all vanishing mean oscillation functions on I is denoted by $VMO(I)$.

In particular, if we replace I by $\partial\mathbb{D}$ in previous definitions, we get $VMO := VMO(\partial\mathbb{D})$.

A useful characterization of the space VMO is as follows: a function f belongs to VMO if and only if for any $\epsilon > 0$ there exists $\delta > 0$, depending on ϵ, such that

$$|J|^{-2} \int_J \int_J |f(t) - f(s)| ds dt < \epsilon,$$

for every interval $J \subset I$ with $|J| < \delta$.

Definition 2.3 ([10], Page 818)**.** Let f be an integrable function defined in an open interval containing the point λ. The integral gap of f at λ is defined by

$$\gamma_\lambda(f) := \limsup_{\delta \to 0} \left| \delta^{-1} \int_{\lambda}^{\lambda+\delta} f(t)dt - \delta^{-1} \int_{\lambda-\delta}^{\lambda} f(t)dt \right|.$$

If f belongs to $VMO(I)$, then $\gamma_\lambda(f) = 0$ for each interior point λ of I. The most important use of Definition 2.3 is established in the following lemma:

Lemma 2.4 ([10], Lemma 2)**.** *Let $I = (a, b)$ be an open interval, λ a point of I, and f a function on I which belongs to both $VMO((a, \lambda))$ and $VMO((\lambda, b))$. If $\gamma_\lambda(f) = 0$, then f belongs to $VMO(I)$.*

We denote by $M(QC)$ the space of all non-trivial multiplicative linear functionals on QC, endowed with its Gelfand topology. In the same way we define $M(C)$ and identify it with $\partial\mathbb{D}$ via the evaluation functionals. Since C is a subset of QC, every functional in $M(QC)$ induces, by restriction, a functional in C.

Here and subsequently, f_0 denotes the function $f_0(\lambda) = \lambda$. Stone-Weierstrass Theorem implies that f_0 and the function $f(\lambda) = 1$ generate the C^*-algebra of all continuous functions on $\partial\mathbb{D}$.

Definition 2.5 ([10], Page 822)**.** For every $\lambda \in \partial\mathbb{D}$, we denote by $M_\lambda(QC)$ the set of all functionals x in $M(QC)$ such that $x(f_0) = \lambda$, that is:

$$M_\lambda(QC) := \{x \in M(QC) : x(f_0) = f_0(\lambda) = \lambda\}.$$

In other words, x belongs to $M_\lambda(QC)$ if the restriction of x to the continuous functions is the evaluation functional at the point λ.

Definition 2.6 ([10], Page 822)**.** We let $M_\lambda^+(QC)$ denote the set of $x \in M_\lambda(QC)$ with the property that $f(x) = 0$ whenever f in QC is a function such that $\lim\limits_{t\to\lambda^+} f(t) = 0$. $M_\lambda^-(QC)$ is defined in an analogous way.

Let f be a bounded function on $\partial\mathbb{D}$. The harmonic extension of f to the unit disk is denoted by f_H and it is given by the formula

$$f_H(z) := f_H(r,\theta) := \frac{1}{2\pi}\int\limits_{\partial\mathbb{D}} P_r(\theta-\lambda)f(\lambda)d\lambda, \tag{2.2}$$

where

$$P_r(\theta) := \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right) = \frac{1-r^2}{1-2rcos(\theta)+r^2}$$

is the Poisson kernel for the unit disk.

For every point z in $\mathbb{D}$ we define a functional in QC by the following rule: $z(f) = f_H(z)$, thus, we consider $\mathbb{D}$ as a subset of the dual space of QC. Under this identification we have the inclusion of $M(QC)$ into the weak-star closure of $\mathbb{D}$ (see [10, Lemma 7]).

Lemma 2.7. *Let f be a function in QC which is continuous at the point λ_0. Then $x(f) = f(\lambda_0)$ for every functional x in $M_{\lambda_0}(QC)$.*

Proof. Consider the case where the function f is continuous at λ_0 and such that $f(\lambda_0) = 0$. Let x be a point in $M_{\lambda_0}(QC)$. For $\epsilon > 0$ there is $\delta_0 > 0$ such that $|f(\lambda)| < \epsilon$ for all λ in the arc $V_{\lambda_0} = (\lambda_0 - \delta_0, \lambda_0 + \delta_0)$. The values taken by f_H should be small if we evaluate points in $\mathbb{D}$ of an open disk with center at λ_0, *i.e.,* there is a δ_1 such that $|f_H(z)| < \epsilon/2$ if $z \in \mathbb{D}$ and $\operatorname{dist}(z,\lambda_0) < \delta_1$.

We construct a neighbourhood V_x in QC^* with parameters f, f_0, ϵ_1, where $\epsilon_1 = \min\{\delta_0, \delta_1, \epsilon\}$.

By Lemma 7 in [10], there is a point z in $\mathbb{D}$ such that $z \in V_x$, that is

$$|f_H(z) - x(f)| < \epsilon_1 < \epsilon \text{ and } |f_0(z) - f_0(\lambda_0)| = |z-\lambda_0| < \epsilon_1 < \delta_1.$$

This implies that $\operatorname{dist}(z,\lambda_0) \le \delta_1$ and then $|f_H(z)| < \epsilon/2$.

Now we estimate $x(f)$,

$$|x(f)| \le |x(f) - f_H(z)| + |f_H(z)| < \epsilon,$$

consequently $x(f) = 0$.

In the general case, when $f(\lambda_0) \neq 0$, we apply the previous argument to the function $g = f - f(\lambda_0)$. For g we obtain $0 = x(g) = x(f) - f(\lambda_0)$ and thus $x(f) = f(\lambda_0)$ for all $x \in M_{\lambda_0}(QC)$. ◻

For $z \neq 0$ in $\mathbb{D}$, we let I_z denote the closed arc of $\partial\mathbb{D}$ whose center is $z/|z|$ with length $2\pi(1-|z|)$. For completeness, $I_0 = \partial\mathbb{D}$.

Lemma 2.8 ([10], Lemma 5)**.** *For f in QC and any positive number ϵ, there is a positive number δ such that $|f_H(z) - I_z(f)| < \epsilon$ whenever $1-|z| < \delta$.*

The average of a function f over an arc I defines a linear functional on QC. Let us identify each arc I with the "averaging" functional in QC, the set of all these functionals is denoted by $\mathcal{G}$. By Lemma 7 in [10] and Lemma 2.8 we come to the following lemma.

Lemma 2.9 ([10], Page 822)**.** *$M(QC)$ is the set of points in the weak-star closure of $\mathcal{G}$ which is not in $\mathcal{G}$ itself.*

For $\lambda \in \partial\mathbb{D}$ we denote by $\mathcal{G}_\lambda^0$ the set of all arcs I in $\mathcal{G}$ with center at λ. Let $M_\lambda^0(QC)$ be the set of functionals in $M_\lambda(QC)$ that lie in the weak-star closure of $\mathcal{G}_\lambda^0$. By Lemma 2.8, the set $M_\lambda^0(QC)$ coincides with the set of functionals in $M_\lambda(QC)$ that lie in the weak-star closure of the radius of $\mathbb{D}$ terminating at λ.

In [10], D. Sarason splits the space $M_\lambda(QC)$ into three sets: $M_\lambda^0(QC)$, $M_\lambda^+(QC) \setminus M_\lambda^0(QC)$ and $M_\lambda^-(QC) \setminus M_\lambda^0(QC)$. These three sets are mutually disjoint due to the next lemma:

Lemma 2.10 ([10], Lemma 8)**.** *$M_\lambda^+(QC) \cup M_\lambda^-(QC) = M_\lambda(QC)$ and $M_\lambda^+(QC) \cap M_\lambda^-(QC) = M_\lambda^0(QC)$.*

The result in Lemma 2.10 allow us to draw a figure of the the maximal ideal space of QC. We consider the unit circle as the interval $[0, 2\pi]$, where the points 0 and 2π represent the same point. At each point λ in $[0, 2\pi]$ we draw a segment representing the fiber $M_\lambda(QC)$. The segment $M_\lambda(QC)$ is split into two parts, the upper part $M_\lambda^+(QC)$ and the lower part $M_\lambda^-(QC)$. Their intersection is $M_\lambda^0(QC)$, the central part of the fiber.

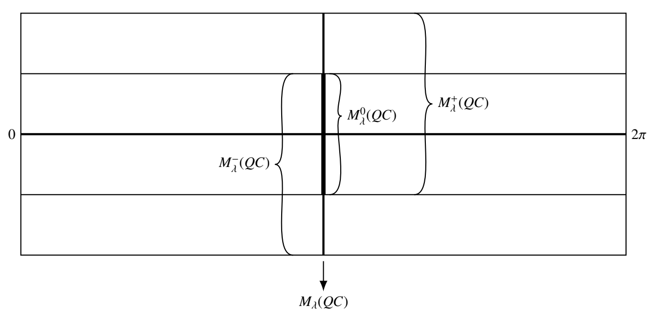

Figure 1: Maximal ideal space of QC.

3 Toeplitz operators with piecewise quasicontinuous symbols on the Bergman space

This section deals with Toeplitz operators with symbols in certain extension of PQC acting on $\mathcal{A}^2(\mathbb{D})$. The C^*-algebra PQC is generated by the space PC of piecewise continuous functions, and QC, the space of quasicontinuous functions, both extended from $\partial\mathbb{D}$ to the unit disk $\mathbb{D}$. The main result of this section (Theorem 3.15) describes the Calkin algebra $\hat{\mathcal{T}}_{PQC} := \mathcal{T}_{PQC}/\mathcal{K}$ as the C^*-algebra of continuous sections over a bundle ξ constructed from the operator algebra $\mathcal{T}_{PQC}$.

Definition 3.1. Let $\Lambda := \{\lambda_1, \lambda_2, \ldots, \lambda_n\}$ be a fixed set of n different points on $\partial\mathbb{D}$. Define $PC := PC_\Lambda$ as the set of continuous functions on $\partial\mathbb{D} \setminus \Lambda$ with one-sided limits at every point λ_k in Λ.
For a function a in PC we set $a_k^+ := \lim\limits_{\lambda \to \lambda_k^+} a(\lambda)$ and $a_k^- := \lim\limits_{\lambda \to \lambda_k^-} a(\lambda)$, following the standard positive orientation of $\partial\mathbb{D}$.

Definition 3.2. PQC is defined as the C^*-algebra generated by both PC and QC.

Our interest is to describe a certain Toeplitz operator algebra acting on the Bergman space $\mathcal{A}^2(\mathbb{D})$. For this end we need to extend the functions in PQC to the whole disk. There are two most natural ways of such extensions

- the harmonic extension g_H, given by the Poisson formula (2.2),
- the radial extension g_R, defined by $g_R(r,\theta) = g(\theta)$.

In this section we use the radial extension, however, we emphasize that the main result of this paper does not depend on the extensions mentioned above (Theorem 4.11).

Recall that the Bergman space $\mathcal{A}^2(\mathbb{D})$ is the closed subspace of $L^2(\mathbb{D})$ which consists of all analytic functions on $\mathbb{D}$. Being closed, the space $\mathcal{A}^2(\mathbb{D})$ has the orthogonal projection $B_\mathbb{D} : L^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$, called the Bergman projection. Let $\mathcal{K}$ stand for the (closed) ideal of compact operators acting on $\mathcal{A}^2(\mathbb{D})$.

Given a function g in $L^\infty(\mathbb{D})$, the Toeplitz operator $T_g : \mathcal{A}^2(\mathbb{D}) \to \mathcal{A}^2(\mathbb{D})$ with generating symbol g is defined by $T_g(f) = B_\mathbb{D}(gf)$.

In [13], K. Zhu describes the largest C^*-algebra $Q \subset L^\infty(\mathbb{D})$ such that the map

$$\begin{array}{cccc} \psi & : & Q & \to & \mathcal{B}(\mathcal{A}^2(\mathbb{D}))/\mathcal{K} \\ & & f & \mapsto & T_f + K, \end{array}$$

is a C^*-algebra homomorphism. This algebra is closely related to QC because both can be described using spaces of functions of vanishing mean oscillation.

Definition 3.3 ([13], Page 633)**.** Consider

$$\Gamma := \{f \in L^\infty(\mathbb{D}) : T_f T_g - T_{fg} \in \mathcal{K} \ \text{ for all } g \in L^\infty(\mathbb{D})\}.$$

Let $Q := \bar{\Gamma} \cap \Gamma$.

For z in $\mathbb{D}$, we define

$$S_z := \{w \in \mathbb{D} : \ |w| \geq |z| \ \text{ and } \ |\arg(z) - \arg(w)| \leq 1 - |z|\}.$$

The area of S_z, denoted by $|S_z|$, is $\pi(1-|z|)^2(1+|z|)$.

Definition 3.4 ([13], Page 621)**.** A function f in $L^1(\mathbb{D})$ belongs to $VMO_\partial(\mathbb{D})$, the space of functions of vanishing mean oscillation near the boundary of $\mathbb{D}$, if

$$\lim_{|z|\to 1^-} \frac{1}{|S_z|}\int_{S_z}\left| f(w) - \frac{1}{|S_z|}\int_{S_z} f(u)dA(u)\right| dA(w) = 0.$$

Theorem 3.5 ([13], Theorem 13)**.** *The algebra Q is the set of bounded functions of vanishing mean oscillation near the boundary, i.e.,*

$$Q = VMO_\partial(\mathbb{D}) \cap L^\infty(\mathbb{D}).$$

For a proof we refer the reader to [13].

Lemma 3.6. *Let f be a function in QC. Then, the function f_R belongs to Q.*

Proof. According to Theorem 3.5 and Definition 3.4, we need to estimate

$$\frac{1}{|S_z|}\int_{S_z}\left| f(w) - \frac{1}{|S_z|}\int_{S_z} f(u)dA(u)\right| dA(w). \tag{3.1}$$

Using polar coordinates we get that this quantity is equal to

$$\frac{2}{|I_z|^2}\int_{I_z}\int_{I_z} |f(\theta) - f(\phi)| dA(\theta)dA(\phi). \tag{3.2}$$

If z is close to the boundary, then the measure of $|I_z|$ is small. Hence, the expression in (3.2) goes to zero because f is in QC. This implies that the expression in (3.1) goes to zero if $|z|$ goes to 1, thus f_R is in Q as required. □

In Lemma 4.5 we prove that the harmonic extension f_H also belongs to Q, but the tools needed for the proof of this fact are not established yet.

From now on, and until further notice, we use only the radial extension of a function in PQC. To simplify the notation, we use PQC to denote functions defined on $\partial\mathbb{D}$ as well as radial extensions of such functions. Moreover g will denote both, the function on $\partial\mathbb{D}$ and its radial extension to $\mathbb{D}$.

By $\mathcal{T}_{PQC}$ we denote the closed C^*-algebra generated by Toeplitz operators with symbols in PQC acting on $\mathcal{A}^2(\mathbb{D})$. We use $\hat{\mathcal{T}}_{PQC}$ to denote the Calkin algebra $\mathcal{T}_{PQC}/\mathcal{K}$. The main goal of this paper is to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$.

We use the Douglas-Varela Local Principle (DVLP for short) to describe the C^*-algebra $\hat{\mathcal{T}}_{PQC}$. A complete description of this principle can be found, for example, in [11, Chapter 1].

Let $\mathcal{A}$ be a C^*-algebra with identity, $\mathcal{Z}$ be some of its central C^*-subalgebras with the same identity, T be the compact of maximal ideals of $\mathcal{Z}$. Furthermore, let J_t be the maximal ideal of $\mathcal{Z}$ corresponding to the point $t \in T$, and $J(t) := J_t \cdot \mathcal{A}$ be the two-sided closed ideal in the algebra $\mathcal{A}$ generated by J_t.

We define $E_t := \mathcal{A}/J(t)$ as the local algebra at the point t. $[a]_t$ stands for the class of the element a in the quotient algebra E_t. Two elements a, b of $\mathcal{A}$ are called locally equivalent at the point $t \in T$ if $[a]_t = [b]_t$ in E_t.

Using the spaces $E := \bigcup_{t\in T} E_t$ and T, there is a standard procedure to construct the C^*-bundle $\xi = (p, E, T)$, where $p : E \to T$ is a projection such that $p(E_t) = \{t\}$. This procedure gives to E a compatible topology such that the function $\hat{a} : T \to E$ with $\hat{a}(t) = [a]_t \in E_t$ is continuous for each a in $\mathcal{A}$.

A function $\gamma : T \to E$ is called a section of the C^*-bundle ξ, if $p(\gamma(t)) = t$. Let $\Gamma(\xi)$ denote the C^*-algebra of all bounded continuous sections defined on ξ.

Theorem 3.7 (Douglas-Varela Local Principle)**.** *The C^*-algebra $\mathcal{A}$ is isomorphic and isometric to the C^*-algebra $\Gamma(\xi)$ of all bounded continuous sections. Where ξ is the C^*-bundle constructed from $\mathcal{A}$ and its central algebra $\mathcal{Z}$.*

Lemma 3.6 and the results in [13] imply that the quotient $\hat{\mathcal{T}}_{QC} = \mathcal{T}_{QC}/\mathcal{K}$ is a commutative C^*-subalgebra of $\hat{\mathcal{T}}_{PQC}$. Thus we use $\hat{\mathcal{T}}_{QC}$ as the central algebra needed to apply the DVLP in the description of $\hat{\mathcal{T}}_{PQC}$. The algebra $\hat{\mathcal{T}}_{QC}$ can be identified with QC

$$\hat{\mathcal{T}}_{QC} = \{T_f + \mathcal{K} | f \in QC\},$$

hence we localize by points in $M(QC)$. We first construct the system of ideals parametrized by points x in $M(QC)$.

Definition 3.8. For every point $x \in M(QC)$, we define the maximal ideal of $\hat{\mathcal{T}}_{QC}$, $J_x := \{f \in QC : f(x) = 0\} = \{T_f + \mathcal{K} | f(x) = 0\}$. The ideal $J(x)$ is defined as the set $J_x \cdot \mathcal{T}_{PQC}/\mathcal{K}$.

We set the notation $\hat{\mathcal{T}}_{PQC}(x) := \hat{\mathcal{T}}_{PQC}/J(x)$ for the local algebra at the point x. We identify T_f with its class $\hat{T}_f$ via the natural projection $\pi : \mathcal{T}_{PQC} \to \hat{\mathcal{T}}_{PQC}$, then the local behaviour of T_f actually means the local behaviour of $\hat{T}_f$.

Lemma 3.9. *Let f be a function in QC and x a point of $M(QC)$. The Toeplitz operator T_f is locally equivalent, at the point x, to the complex number $f(x)$ (realized as the operator $f(x)I$).*

Proof. Let x be a point in $M(QC)$ and f be a function in QC. The function $f - f(x)$ belongs to $J(x)$, thus, the operator $T_f - T_{f(x)} = T_{f-f(x)}$ is zero in $\hat{\mathcal{T}}_{PQC}(x)$. This means that the operator T_f is locally equivalent to the operator $T_{f(x)} = f(x)I$ and then, the operator T_f is locally equivalent to the complex number $f(x)$. □

Lemma 3.10. *Let x be a point of $M_\lambda(QC)$ with $\lambda \notin \Lambda$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number $a(\lambda)$ (realized as the operator $a(\lambda)I$).*

The proof is very similar to the proof of Lemma 3.9 and is omitted.

For the case when $x \in M_{\lambda_k}(QC)$, we use Lemma 2.10 to split the fiber $M_{\lambda_k}(QC)$ into three disjoints sets: $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$, $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and $M^0_{\lambda_k}(QC)$.

Lemma 3.11. *Let x be a point of $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a^+_k (realized as the operator $a^+_k I$).*

Proof. Let a be a function in PC for which $a_k^+ = 0$. If x belongs to $M_{\lambda_k}^+(QC) \setminus M_{\lambda_k}^0(QC)$, then x belongs to $M_{\lambda_k}^+(QC)$ and does not belong to $M_{\lambda_k}^-(QC)$. This implies the existence of a function g in QC such that $\lim\limits_{\lambda \to \lambda_k^-} g(\lambda) = 0$ and $g(x) = 1$.

The product ag is continuous at λ_k and $ag(\lambda_k) = 0$. The difference $T_a - T_{ag}$ can be rewritten as $T_{(1-g)a} = T_{1-g}T_a + K$ where K is a compact operator. Since the function $1-g$ vanishes at x, T_{1-g} belongs to J_x, and then $T_a - T_{ag}$ belongs to $J(x)$. Thus, we conclude that, locally, the Toeplitz operator with symbol a is equivalent to the Toeplitz operator with symbol ag.

At the same time, the Toeplitz operator T_{ag} is locally equivalent to the complex number $0 = ag(\lambda_k)$, hence, the operator T_a is locally equivalent to the complex number $a_k^+ = 0$.

For the general case, if the function a in PC has limit $a_k^+ \neq 0$, we construct the function $b(\lambda) = a(\lambda) - a_k^+$. The function b has lateral limit $b_k^+ = 0$, fulfilling the initial assumption of the proof. The Toeplitz operator $T_b = T_a - a_k^+ I$ is locally equivalent to the complex number 0, thus the Toeplitz operator T_a is locally equivalent to the complex number a_k^+. □

Similarly the following lemma holds.

Lemma 3.12. *Let x be a point of $M_{\lambda_k}^-(QC) \setminus M_{\lambda_k}^0(QC)$ and a be a function in PC. Then, the Toeplitz operator T_a, in the local algebra $\hat{\mathcal{T}}_{PQC}(x)$, is equivalent to the complex number a_k^- (realized as the operator $a_k^- I$).*

Now we analyze the case when x belongs to central part of the fiber $M_{\lambda_k}(QC)$, *i.e.,* $x \in M_{\lambda_k}^0(QC)$. For this case we use some results regarding Toeplitz operators with zero-order homogeneous symbols defined in the upper half-plane Π.

We consider $\mathcal{A}^2(\Pi)$ as the Bergman space of Π, that is, the (closed) space of square integrable and analytic functions on Π. B_Π stands for the Bergman projection $B_\Pi : L^2(\Pi) \to \mathcal{A}^2(\Pi)$.

Denote by $\mathcal{A}_\infty$ the C*-algebra of bounded measureable homogeneous functions on Π of zero-order, or functions depending only in the polar coordinate θ. We introduce the Toeplitz operator algebra $\mathcal{T}_{\mathcal{A}_\infty}$ generated by all Toeplitz operators

$$T_a : \phi \in \mathcal{A}^2(\Pi) \mapsto B_\Pi(a\phi) \in \mathcal{A}^2(\Pi)$$

with defining symbols $a(r,\theta) = a(\theta) \in \mathcal{A}_\infty$.

Theorem 3.13 ([11], Theorem 7.2.1)**.** *For $a = a(\theta) \in \mathcal{A}_\infty$, the Toeplitz operator T_a acting in $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I$ acting on $L^2(\mathbb{R})$. The function $\gamma_a(s)$ is given by*

$$\gamma_a(s) = \frac{2s}{1 - e^{-2s\pi}} \int_0^\pi a(\theta) e^{-2s\theta} d\theta.$$

Let $\partial\mathbb{D}_k^+$ denote the upper half of the circumference and $\mathbb{D}_k^+$ the upper half of the disk $\mathbb{D}$, both determined by the diameter passing through λ_k and $-\lambda_k$. Denote by $\partial\mathbb{D}_k^-$ and $\mathbb{D}_k^-$ the complement of $\partial\mathbb{D}_k^+$ and $\mathbb{D}_k^+$ on $\partial\mathbb{D}$ and $\mathbb{D}$, respectively, and by $\chi_{\partial,k}^+$ and χ_k^+ the characteristic function of $\partial\mathbb{D}_k^+$ and $\mathbb{D}_k^+$, respectively

Recall that all the piecewise continuous functions can be written as a linear combination of the characteristic function χ_k^+ and a continuous function, more specifically, for each a in PC there is a continuous function s such that

$$a(\lambda) = a_k^+\chi_k^+ + a_k^-(1-\chi_k^+) + s(\lambda),$$

and $s(\lambda_k) = 0$.

Due to the property described in the previous paragraph, we conclude that the local algebra at the point $x \in M_{\lambda_k}^0(QC)$ is isomorphic to the C^* algebra generated by $T_{\chi_k^+}$ and the identity I.

Let ϕ be a Möbius transformation which sends the upper half-plane to the unit disk and such that: $\phi(0) = \lambda_k$, $\phi(i) = 0$ and $\phi(\infty) = -\lambda_k$. Using the function ϕ we construct a unitary transformation W which sends $L^2(\mathbb{D})$ onto $L^2(\Pi)$. Under the unitary transformation W, the Toeplitz operator with symbol h, acting on $\mathcal{A}^2(\mathbb{D})$, is unitary equivalent to the Toeplitz operator $T_{h(\phi(w))}$ acting on $\mathcal{A}^2(\Pi)$.

By Theorem 3.13, the Toeplitz operator with symbol $\chi_k^+(\phi(w))$, acting on $\mathcal{A}^2(\Pi)$, is unitary equivalent to the multiplication operator by the function $\gamma_{\chi_k^+(\phi(w))}$, acting on $L^2(\mathbb{R})$. Following the unitary equivalences we conclude the unitary equivalence between $T_{\chi_k^+}$ and $\gamma_{\chi_k^+(\phi(w))}$, thus the algebra generated by $T_{\chi_k^+}$ and I, is isomorphic to the algebra generated by $\gamma_{\chi_k^+(\phi(w))}$ and the function 1.

By Corollary 7.2.2 in [11], the function $\gamma_{\chi_k^+(\phi)}(s) = \frac{1-e^{-s\pi}}{1-e^{-2s\pi}} = \frac{1}{1+e^{-2s\pi}}$ is continuous in $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$, the two point compactification of $\mathbb{R}$. The function $\gamma_{\chi_k^+(\phi)}(s)$ and the identity function 1 generate the algebra of continuous functions on $\bar{\mathbb{R}}$ [11, Corollary 7.2.6].

Using the change of variables $t = \frac{1}{1+e^{-2s\pi}}$, which is a homeomorphism between $[0,1]$ and $\bar{\mathbb{R}}$, we conclude that the local algebra $\hat{\mathcal{T}}_{PQC}(x)$ is isomorphic to $C[0,1]$ for every $x \in M_{\lambda_k}^0(QC)$; further, such isomorphism, denoted here by ψ, acts on the generator $T_{\chi_k^+}$ as follows:

$$T_{\chi_k^+} \mapsto t.$$

This implies that the Toeplitz operator with symbol a in PC is sent to $C([0,1])$, via ψ, to the function $a_k^-(1-t) + a_k^+ t$. Thus we come to the following lemma.

Lemma 3.14. *If x belongs to $M_{\lambda_k}^0(QC)$, then the local algebra generated by the Toeplitz operators with symbols in PQC is isometric and isomorphic to the algebra of all continuous functions in* $[0,1]$.

With the set $M(QC)$, we construct the C*-bundle $\xi_{\mathrm{PQC}} := (p, E, M(QC))$. We use the description of the local algebras given by Lemmas 3.9, 3.10, 3.11, 3.12 and 3.14 to construct the bundle $E := \bigcup\limits_{x \in M(QC)} E_x$ where

- $E_x = \mathbb{C}$, if $x \in M_\lambda(QC)$ with $\lambda \notin \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M_{\lambda_k}^+(QC) \setminus M_{\lambda_k}^0(QC)$, $\lambda_k \in \Lambda$,
- $E_x = \mathbb{C}$, if $x \in M_{\lambda_k}^-(QC) \setminus M_{\lambda_k}^0(QC)$, $\lambda_k \in \Lambda$,
- $E_x = C([0,1])$, if $x \in M_{\lambda_k}^0(QC)$, $\lambda_k \in \Lambda$.

The function p is the natural proyection from E onto $M(QC)$.

Let $\Gamma(\xi_{PQC})$ denote the algebra of all bounded continuous sections of the bundle ξ_{PQC}. Applying the DVLP (Theorem 3.7) we come to the following theorem:

Theorem 3.15. *The C^*algebra $\hat{\mathcal{T}}_{PQC}$ is isometric and isomorphic to the C^*-algebra of continuous sections over the C^*-bundle ξ_{PQC}.*

As a corollary of Theorem 3.15, the algebra $\hat{\mathcal{T}}_{PQC}$ is commutative, thus there exists a compact space $X = M(\hat{\mathcal{T}}_{PQC})$, such that $\hat{\mathcal{T}}_{PQC} \cong C(X) = C(M(\hat{\mathcal{T}}_{PQC}))$. The compact space $M(\hat{\mathcal{T}}_{PQC})$ can be constructed using the irreducible representations of $\hat{\mathcal{T}}_{PQC}$.

Let $\partial\hat{\mathbb{D}}$ be the set $\partial\mathbb{D}$ cut at the points λ_k of Λ. The pair of points of $\partial\hat{\mathbb{D}}$ which correspond to the point λ_k are denoted by λ_k^+ and λ_k^-, following the positive orientation of $\partial\mathbb{D}$. Let $I^n := \sqcup_{k=1}^n [0,1]_k$ be the disjoint union of n copies of the interval $[0,1]$.

Denote by Σ the union of $\partial\hat{\mathbb{D}}$ and I_n with the point identification

$$\lambda_k^- \equiv 0_k \qquad \lambda_k^+ \equiv 1_k,$$

where 0_k and 1_k are the boundary points of $[0,1]_k$, $k = 1, \ldots, n$.

Let $M(\hat{\mathcal{T}}_{PQC}) := \bigcup\limits_{\lambda\in\Sigma} M_\lambda(\mathcal{T}_{PQC})$ where each fiber corresponds to

$$\begin{aligned}
M_\lambda(\hat{\mathcal{T}}_{PQC}) :=& M_\lambda(QC) \text{ if } \lambda \in \partial\hat{\mathbb{D}}, && \lambda \notin \Lambda\\
M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}) :=& \left(M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)\right) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda,\\
M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}) :=& \left(M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)\right) \cup M^0_{\lambda_k}(QC), && \lambda_k \in \Lambda,\\
M_t(\hat{\mathcal{T}}_{PQC}) :=& M^0_{\lambda_k}(QC) \text{ if } t \in (0,1)_k, && k = 1, \ldots, n.
\end{aligned}$$

With the help of Figure 1, we draw the maximal ideal space for $\hat{\mathcal{T}}_{PQC}$. The idea is to duplicate the set $M^0_{\lambda_k}(QC)$ and then connect this two copies by the interval $[0,1]_k$.

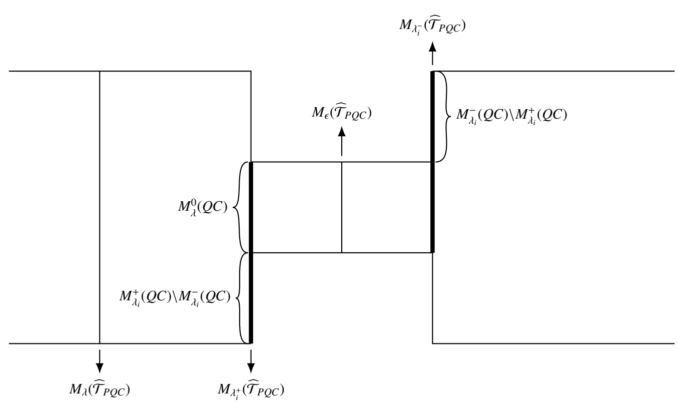

Figure 2: Maximal ideal space of $\widehat{\mathcal{T}}_{PQC}$.

We use the topology of $M(QC)$ to describe the topology of $M(\hat{\mathcal{T}}_{PQC})$. We only describe the topology of the fibers $M_{\lambda_k^\pm}(\hat{\mathcal{T}}_{PQC})$ and $M_t(\hat{\mathcal{T}}_{PQC})$, since the topology on the other fibers corresponds to the topology of $M_\lambda(QC)$.

For x in $M(QC)$, let $\Omega(x)$ denote the family of open neighbourhoods of x. For $x \in M_\lambda(QC)$ and N in $\Omega(x)$, let $N_\lambda = N \cap M_\lambda(QC)$, and let N_{λ^+} and N_{λ^-} denote the sets of points in N that lie above the semicircles $\partial\mathbb{D}_k^+$ and $\partial\mathbb{D}_k^-$, respectively.

Consider the fiber $M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC})$. The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^+}$ form neighbourhoods of $x \in M^+_{\lambda_k}(QC) \setminus M^-_{\lambda_k}(QC)$.

Let $\Omega_+(x)$ be the set of neighbourhoods N in $\Omega(x)$ satisfying $N = N_\lambda \cup N_{\lambda^+}$. The sets

$$(N_{\lambda_k} \times (1-\epsilon, 1]) \cup N_{\lambda_k^+} \qquad N \in \Omega_+(x), \quad \text{and} \quad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M^0_{\lambda_k}(QC)$.

The open neighbourhoods for points in the fiber $M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC})$ are constructed analogously.

The sets N in $\Omega(x)$ satisfying $N = N_{\lambda_k} \cup N_{\lambda_k^-}$ form neighbourhoods of $x \in M^-_{\lambda_k}(QC) \setminus M^+_{\lambda_k}(QC)$.

The sets

$$(N_{\lambda_k} \times [0, \epsilon)) \cup N_{\lambda_k^-} \qquad N \in \Omega_-(x), \quad \text{and} \quad 0 < \epsilon < 1,$$

form open neighbourhoods of points x in $M^0_{\lambda_k}(QC)$.

Each set $M^0_{\lambda_k}(QC) \times (0,1)$ is open in $M(\hat{\mathcal{T}}_{PQC})$ and carries the product topology.

Theorem 3.16. *Let $X := M(\hat{\mathcal{T}}_{PQC})$ with the topology described above. The algebra $\hat{\mathcal{T}}_{PQC}$ is isomorphic to the algebra of continuous functions over X, the isomorphism acts on the generators in the following way:*

- *For generators which symbols are functionsusy0403 a in PC*

$$\Phi(\hat{T}_a)(x) = \begin{cases} a(\lambda), & \text{if } x \in M_\lambda(\hat{\mathcal{T}}_{PQC}) \text{ with } \lambda \notin \Lambda; \\ a_k^+, & \text{if } x \in M_{\lambda_k^+}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-, & \text{if } x \in M_{\lambda_k^-}(\hat{\mathcal{T}}_{PQC}); \\ a_k^-(1-t) + a_k^+ t, & \text{if } x \in M_t(\hat{\mathcal{T}}_{PQC}). \end{cases}$$

- *For generators which symbols are functions f in QC, $\Phi(\hat{T}_f)(x) = f(x)$.*

4 Independence of the result on the extension chosen

In this section we prove that the description of the algebra $\hat{\mathcal{T}}_{PQC}$ does not depend of the extension chosen for functions in PQC. Recall that PQC is the algebra generated by PC and QC. This algebra is defined on $\partial\mathbb{D}$ and then extended to the whole disk by two different ways:

- the harmonic extension g_H given by the Poisson formula (2.2),

- the radial extension g_R, defined by $g_R(r, \theta) = g(\theta)$.

Let a be a function in PC. At the point $x \in M_\lambda(QC)$, for $\lambda \notin \Lambda$; the Toeplitz operator T_{a_R} is locally equivalent to the complex number $a(\lambda)$. The same still true if we use the harmonic extension a_H, thus T_{a_H} and T_{a_R} are locally equivalent at the point $x \in M_\lambda(QC)$, $\lambda \notin \Lambda$.

For points x in $M^+_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$ (respectively $M^-_{\lambda_k}(QC) \setminus M^0_{\lambda_k}(QC)$), the Toeplitz operators T_{a_R} and T_{a_H} are equivalent to the number a^+_k (Respectively a^-_k), and then, the local algebras are the same.

Now, we analyze the case when x belongs to $M^0_{\lambda_k}(QC)$. Let $\hat{a}$ be a function in PC, we construct a function a such that $a = \hat{a}^+_k$ in $\partial\mathbb{D}^+_k$ and $a = \hat{a}^-_k$ in $\partial\mathbb{D}^-_k$. The Toeplitz operator with symbol $\hat{a}_H$ is locally equivalent to T_{a_H}.

As in Section 3, we use a Möbius transformation ϕ to generate a unitary operator between $L^2(\mathbb{D})$ and $L^2(\Pi)$. For the function a in PC described earlier, the function $a_H(\phi(z))$ is harmonic in Π and corresponds to the harmonic extension of $a(\phi(t))$.

The harmonic extension of $a(\phi(t))$ is $a^\Pi_H := \frac{\theta}{\pi}(a^-_k - a^+_k) - a^+_k$, which is a zero-order homogeneous function on Π. By Theorem 3.13, the Toeplitz operator $T_{a^\Pi_H}$ is unitary equivalent to the multiplication operator $\gamma_{a^\Pi_H}$. The function $\gamma_{a^\Pi_H}$ is given by

$$\gamma_{a^\Pi_H} = A\left(\frac{1}{2s\pi} - \frac{1}{e^{-2s\pi} - 1}\right) + B,$$

for suitable complex constants A and B. Corollary 7.2.7 of [11] shows that the algebra generated by $\gamma_{a^\Pi_H}$ and the identity is the algebra of continuous functions on $\bar{\mathbb{R}}$.

Following the unitary equivalences from T_{a_H} to $\gamma_{a^\Pi_H}$ and making a change of variables, we have that the algebra generated by T_{a_H} is isomorphic to the algebra of continuous functions over the segment $[0,1]$.

We already know, from Theorem 3.14, that the Toeplitz operator with symbol a_R generates the algebra of continuous functions over $[0,1]$ as well, so the local algebras, at the point $x \in M^0_{\lambda_k}(QC)$, generated by T_{a_H} and T_{a_R} are the same. We have thus proved

Theorem 4.1. *Consider the algebra PC defined on $\partial\mathbb{D}$ and its extensions PC_R and PC_H. The local algebras $\hat{\mathcal{T}}_{PC_R}(x)$ and $\hat{\mathcal{T}}_{PC_H}(x)$ are the same for every point $x \in M(QC)$.*

To show the same theorem for functions f in QC we need to establish some definitions related to the space Q in Definition 3.3. Further information on the theorems and definitions below can be found in [13].

Definition 4.2. For a function $g \in L^\infty(\mathbb{D})$ we define its Berezin transform $\tilde{g}$ by the formula

$$\tilde{g}(z) := \int_{\mathbb{D}} g(w)\frac{1-|w|^2}{(1-z\bar{w})^2} dA(w).$$

Note that $\tilde{g}$ belongs to $L^\infty(\mathbb{D})$ and $||\tilde{g}||_\infty \leq ||g||_\infty$.

Definition 4.3. Define B as the set of bounded functions on $\mathbb{D}$ such that its Berezin transform goes to zero as z approaches the boundary of $\mathbb{D}$, that is,

$$B := \{f \in L^\infty(\mathbb{D}) \ : \lim_{|z|\to 1^-} \tilde{f}(z) = 0\}.$$

In [1], S. Axler and D. Zheng proved that a Toeplitz operator T_g, with bounded symbol g, is compact if and only if g is in B.
The next lemma is due to D. Sarason and is a combination of some results in [10].

Theorem 4.4. *The set Q in Definition 3.3 is described as*

$$Q = \{f \in L^\infty(\mathbb{D}) : \lim_{|z|\to 1} \widetilde{|f|}^2(z) - |\tilde{f}(z)|^2 = 0\}.$$

The set $B \cap Q$ is an ideal of Q and, for $f \in Q$, the Toeplitz operator T_f is compact if and only if f belongs to $B \cap Q$.

Lemma 4.5. *For a function f in QC, the function f_H belongs to Q.*

Proof. For this proof we use two facts:

1. The Berezin transform of a harmonic function is the function itself, in our case, $\tilde{f}_H = f_H$.
2. By [7], the harmonic extension is asymptotically multiplicative in QC, that is

$$\lim_{|z|\to 1^-} |f|^2_H(z) - |f_H(z)|^2 = 0.$$

Now we proceed with the proof:

$$\begin{aligned} \widetilde{|f_H|^2}(z) - |\widetilde{f_H}(z)|^2 &\le \left|\widetilde{|f_H|^2}(z) - |f|^2_H(z)\right| + \left||f|^2_H(z) - |f_H(z)|^2\right| \\ &\le \left|\widetilde{|f_H|^2(z) - |f|^2_H}(z)\right| + \left||f|^2_H(z) - |f_H(z)|^2\right|, \end{aligned}$$

the last two summands goes to zero as z approaches the boundary $\partial\mathbb{D}$; the later because of item 2, and the former is due to items 2 and 1. Finally, using Theorem 4.4, we have that f_H is in Q. □

Definition 4.6 ([13], Page 626)**.** For each point z in $\mathbb{D}$ we define

$$S'_z := \left\{ w \in \mathbb{D} \ : \ |w| \ge |z| \quad \text{and} \quad |\arg(z) - \arg(w)| \le \frac{1 - |z|}{2} \right\}.$$

Definition 4.7 ([13], Page 627)**.** For a function f in $L^\infty(\mathbb{D})$ define

$$\hat{f}(z) := \frac{1}{|S'_z|} \int_{S'_z} f(w)\,dA(w).$$

Definition 4.8 ([13], Page 626)**.** Let f be in $L_\infty(\mathbb{D}, dA)$. We say f is in $ESV(\mathbb{D})$ if and only if for any $\epsilon > 0$, and $\sigma \in (0,1)$, there exists $\delta_0 > 0$ such that $|f(z) - f(w)| < \epsilon$ whenever $w \in S'_z$ and $|z|, |w| \in [1-\delta, 1-\delta\sigma]$, with $\delta < \delta_0$.

The notation $ESV(\mathbb{D})$ means eventually slowly varying and was introduced by C. Berger and L. Coburn in [2].

Theorem 4.9 ([13], Theorem 5)**.** *$Q = ESV + Q \cap B$. A decomposition is given by $f = \hat{f} + (f - \hat{f})$. Moreover*

$$ESV(D) \cap B = \{f \in L_\infty(D) \mid \ f(z) \to 0 \text{ as } |z| \to 1^-\}.$$

For a function f in QC_R, we calculate $\hat{f}_R$ and get $\hat{f}_R(z) = I_z(f)$. Then, Theorem 4.9 gives us the decomposition $f_R(z) = I_z(f) + (f_R(z) - I_z(f))$, where $I_z(f)$ belongs to $ESV(\mathbb{D})$ and $f_R(z) - I_z(f)$ belongs to $Q \cap B$.

Lemma 4.10. *Consider the function f in QC. The Toeplitz operator with symbol $f_R - f_H$ is compact.*

Proof. We write $f_R(z) - f_H(z) = (I_z(f) - f_H(z)) + (f_R(z) - I_z(f))$. The first summand goes to zero as $|z|$ goes to 1 by Theorem 2.8. Then by Theorem 4.9, the function $I_z(f) - f_H(z)$ belongs to $ESV(\mathbb{D}) \cap B$. By the decomposition of Q as $ESV(\mathbb{D}) + Q \cap B$ we have that $(f_R(z) - I_z(f)) = f_R(z) - \hat{f}_R(z)$ belongs to $Q \cap B$.

In summary, the function $f_R(z) - f_H(z)$ belongs to $Q \cap B$ and then the Toeplitz operator with symbol $f_R - f_H$ is compact. □

Now we establish the main result of this section: the algebra described in Theorem 3.15 does not depend on the extension chosen for the symbols in PQC.

Theorem 4.11. *Let PQC_R and PQC_H denote, respectively, the radial and the harmonic extension to the disk of functions in PQC. Then, the Calkin algebras $\mathcal{T}_{PQC_R}/\mathcal{K}$ and $\mathcal{T}_{PQC_H}/\mathcal{K}$ are the same.*

Proof. The proof follows from Lemma 4.1 and 4.10. □

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