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Commutative Algebras of Toeplitz Operators on the Pluriharmonic Bergman Space

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Abstract

We study Toeplitz operators acting on the weighted pluriharmonic Bergman space on the unit ball, denoted here by $b^2_\lambda(\mathbb{B}^n)$. We prove that, besides the C^*-algebra generated by Toeplitz operators with radial symbols, there are commutative Banach algebras generated by Toeplitz operators. These Banach algebras are generated by Toeplitz operators with k-quasi-radial and k-quasi-homogeneous symbols.

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1 Introduction

The pluriharmonic Bergman space on the unit ball is represented in terms of the Bergman and the anti-Bergman spaces. With this in mind we might expect that the study of Toeplitz operators acting on $b^2_\lambda(\mathbb{B}^n)$ is just an extension of the results for the Bergman space setting. However, even for one complex variable, they present unexpected properties. For example, each Fredholm Toeplitz operator with continuous symbol, has Fredholm index zero (see for example [3]). Some other interesting properties of Toeplitz operators acting on $b^2(\mathbb{D})$ (or $b^2(\Pi)$, where Π denotes the upper half-plane) are proved in [4]. There, it is proved

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that Toeplitz operators whose symbols depend only on the vertical variable generate a commutative C^*-algebra, such as in the Bergman space setting. But, if we consider Toeplitz operators with homogeneous symbols they generate a non commutative C^*-algebra.

There are some interesting results concerning to Toeplitz operators acting on the pluriharmonic Bergman space on the unit ball. In particular, as proved in [7], every Toeplitz operator with radial symbol is diagonal. Using some results from [1] we prove here that Toeplitz operators with radial symbols generate a commutative C^*-algebra that is isomorphic to $\mathrm{SO}(\mathbb{Z}_+)$, the algebra of slowly oscillating sequences.

We prove that there exist commutative Banach algebras generated by Toeplitz operators. They are not C^*-algebras and are generated by Toeplitz operators whose symbols are k-quasi-homogeneous.

2 Preliminares and notation

Let $\langle\cdot,\cdot\rangle$ be the hermitian inner product in $\mathbb{C}^n$, i.e, $\langle z,w\rangle = \sum_{j=1}^n z_j\overline{w_j}$, and $|z| = \sqrt{\langle z,z\rangle}$. By $\mathbb{B}^n$ we denote the open unit ball in $\mathbb{C}^n$, $\mathbb{B}^n = \{z = (z_1,\dots,z_n) \in \mathbb{C}^n : |z| < 1\}$.

Let $\tau(\mathbb{B}^n)$ be the base of $\mathbb{B}^n$, considered as a Reinhard domain, i.e.,

$$\tau(\mathbb{B}^n) = \left\{(r_1,\dots,r_n) = (|z_1|,\dots,|z_n|) : r^2 = r_1^2+\cdots+r_n^2 \in [0,1)\right\},$$

which belongs to $\mathbb{R}_+^n = \mathbb{R}_+\times\cdots\times\mathbb{R}_+$.

Denote by $dV = dx_1dy_1\dots dx_ndy_n$, where $z_j = x_j+y_j$, $j = 1,2,\dots,n$, the standard Lebesgue measure in $\mathbb{C}^n$; and by dS the corresponding surface measure on $\mathbb{S}^n$. We introduce the one-parameter family of weighted measures

$$d\nu_\lambda(z) = \frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)}(1-|z|^2)^\lambda dV(z), \quad \lambda > -1$$

which are the probability ones.

For a multi-index $\alpha = (\alpha_1,\dots,\alpha_n)$, $\alpha_k \in \mathbb{Z}_+ = \mathbb{N}\cup\{0\}$ we define the length of α by $|\alpha| = \alpha_1+\cdots+\alpha_n$ and its factorial $\alpha!$ by $\alpha! = \alpha_1!\cdots\alpha_n!$. For two multi-indices α and β the notation $\alpha\geq\beta$ means that $\alpha_i\geq\beta_i$ for $i = 1,\dots,n$ and $\alpha\perp\beta$ means that $\sum_{i=1}^n\alpha_i\beta_i = 0$. If $\alpha\geq\beta$ then $\alpha-\beta = (\alpha_1-\beta_1,\dots,\alpha_n-\beta_n)$ and $|\alpha-\beta| = |\alpha|-|\beta|$.

We recall two equalities from [11]. For every $\alpha,\beta\in\mathbb{N}^n$ we have

$$\int_{\mathbb{S}^n}\zeta^\alpha\overline{\zeta}^\beta dS(\zeta) = \delta_{\alpha,\beta}\frac{2\pi^n\alpha!}{(n-1+|\alpha|)!}, \tag{2.1}$$

$$\int_{\mathbb{B}^n}z^\alpha\overline{z}^\beta d\nu_\lambda(z) = \delta_{\alpha,\beta}\frac{\alpha!\,\Gamma(n+\lambda+1)}{\Gamma(n+|\alpha|+\lambda+1)}. \tag{2.2}$$

The weighted Bergman space $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ is the closed subspace of $L^2(\mathbb{B}^n,d\nu_\lambda)$ consisting of all analytic functions in $\mathbb{B}^n$. The Bergman projection B from $L^2(\mathbb{B}^n,d\nu_\lambda)$ onto $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ has the following integral form

$$(B\varphi)(z) = \int_{\mathbb{B}^n}\frac{\varphi(\zeta)d\nu_\lambda(\zeta)}{(1-\langle z,\zeta\rangle)^{n+\lambda+1}}.$$

An interesting subspace of $L^2(\mathbb{B}^n, d\nu_\lambda)$ that is closed related to the Bergman space is the anti-Bergman space $\widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)$ which consists of all anti-analytic functions in $\mathbb{B}^n$. Recall that a complex-valued function f is anti-analytic if $\overline{f}$ is an analytic function. We denote by $\widetilde{B}$ the orthogonal projection from $L^2(\mathbb{B}^n, d\nu_\lambda)$ onto $\widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)$.

A complex-valued function $u(z)$, defined in the unit ball $\mathbb{B}^n$, is called pluriharmonic if

$$\frac{\partial^2 u}{\partial z_i \partial \overline{z_j}} = 0, \quad i, j = 1, \dots, n.$$

The (weighted) pluriharmonic Bergman space $b^2_\lambda(\mathbb{B}^n)$ is the closed subspace of $L^2(\mathbb{B}^n, d\nu_\lambda)$ consisting of all pluriharmonic functions on $\mathbb{B}^n$. It is well known that the reproducing kernel for $b^2_\lambda(\mathbb{B}^n)$ is given by the following formula

$$K(z,w) = \frac{1}{(1 - \langle z, \zeta\rangle)^{n+\lambda+1}} + \frac{1}{(1 - \langle \zeta, z\rangle)^{n+\lambda+1}} - 1.$$

Thus, if we denote by Q the orthogonal projection from $L^2(\mathbb{B}^n, d\nu_\lambda)$ onto $b^2_\lambda(\mathbb{B}^n)$, Q can be written in terms of B and $\widetilde{B}$ in the following form

$$Qu = Bu + \widetilde{B}u - Bu(0). \tag{2.3}$$

For a function $a \in L_\infty(\mathbb{B}^n, d\nu_\lambda)$, the Toeplitz operator $\widehat{T}_a : b^2_\lambda(\mathbb{B}^n) \to b^2_\lambda(\mathbb{B}^n)$ with symbol a is defined by

$$\widehat{T}_a u = Q(au), \quad u \in b^2_\lambda(\mathbb{B}^n).$$

Let $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$ be the subspace of $L^2(\mathbb{B}^n, d\nu_\lambda)$ defined by

$$\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) = \left\{ f \in \mathcal{A}^2_\lambda(\mathbb{B}^n) \mid f(0) = 0 \right\}.$$

From (2.3), it follows that the space $b^2_\lambda(\mathbb{B}^n)$ is the direct sum of $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$ and $\widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)$, that is,

$$b^2_\lambda(\mathbb{B}^n) = \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n).$$

To be more explicit, every $u \in b^2_\lambda(\mathbb{B}^n)$ admits the following representation

$$u = [Bu - Bu(0)] + [(I - B)u + Bu(0)],$$

where $Bu - Bu(0) \in \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$ and $(I - B)u + Bu(0) \in \widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)$.

Given $f, g \in L^2(\mathbb{B}^n, d\nu_\lambda)$, denote by $f \otimes g$ the rank one operator $(f \otimes g)h = \langle h, g\rangle f$.

Consider the unitary operator $U : L^2(\mathbb{B}^n, d\nu_\lambda) \to L^2(\mathbb{B}^n, d\nu_\lambda)$ given by

$$U(f)(z) = f(\overline{z}) = f(\overline{z_1}, \overline{z_2}, \dots, \overline{z_n}). \tag{2.4}$$

The notation Γ_a means the Hankel operator $\Gamma_a : \mathcal{A}^2_\lambda(\mathbb{B}^n) \to \mathcal{A}^2_\lambda(\mathbb{B}^n)$ given by $\Gamma_a f = B(aUf)$, $f \in \mathcal{A}^2_\lambda(\mathbb{B}^n)$.

The Bergman projection, B, and the anti-Bergman projection, $\widetilde{B}$ hold the following relation

$$UBU^* = \widetilde{B}. \tag{2.5}$$

Besides, $UaU^*(z) = a(\overline{z})I$ for each $a \in L_\infty(\mathbb{B}^n)$. Observe that $U|_{\widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)} : \widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n) \to \mathcal{A}^2_\lambda(\mathbb{B}^n)$ is an isometric isomorphism.

Using the operator U, given above, we construct the operator $\widehat{U} : b^2_\lambda(\mathbb{B}^n) = \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n) \to \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \mathcal{A}^2_\lambda(\mathbb{B}^n)$ by the formula

$$\widehat{U} = \begin{pmatrix} I & 0 \\ 0 & U \end{pmatrix}.$$

Theorem 2.1. *The Toeplitz operator $\widehat{T}_a$, acting on the space $b^2_\lambda(\mathbb{B}^n) = \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \widetilde{\mathcal{A}}^2_\lambda(\mathbb{B}^n)$ is unitarily equivalent to the operator $\widehat{U}\widehat{T}_a\widehat{U}^*$, acting on the space $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \mathcal{A}^2_\lambda(\mathbb{B}^n)$. The operator $\widehat{U}\widehat{T}_a\widehat{U}^*$ is given by the formula*

$$\widehat{U}\widehat{T}_a\widehat{U}^* = \begin{pmatrix} T_a & \Gamma_a \\ \Gamma_{\widehat{a}} & T_{\widehat{a}} \end{pmatrix}, \tag{2.6}$$

where $a^*(z) = \overline{a(\overline{z})}$ and $\widehat{a}(z) = a(\overline{z})$.

Proof. If $f_1 \in \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$, then

$$\begin{aligned}\widehat{T}_a f_1 &= B(af_1) + \widetilde{B}(af_1) - B(af_1)(0) = T_a f_1 + UBUaf_1 - (1 \otimes \overline{a})f_1 \\ &= T_a f_1 + U\Gamma_{\widehat{a}} f_1 - (1 \otimes \overline{a})f_1 = [T_a - (1 \otimes \overline{a})]f_1 + U\Gamma_{\widehat{a}} f_1.\end{aligned}$$

If $f_2 \in \mathcal{A}^2_\lambda(\mathbb{B}^n)$, then

$$\begin{aligned}\widehat{T}_a U f_2 &= B(aUf_2) + \widetilde{B}(aUf_2) - B(aUf_2)(0) = \Gamma_a f_2 + UB\widehat{a} f_2 - (1 \otimes \overline{a})Uf_2 \\ &= \Gamma_a f_2 + UT_{\widehat{a}} f_2 - (1 \otimes a^*)f_2 = [\Gamma_a - (1 \otimes a^*)]f_2 + UT_{\widehat{a}} f_2.\end{aligned}$$

Then, the operator $\widehat{U}\widehat{T}_a\widehat{U}^*$, acting on $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \mathcal{A}^2_\lambda(\mathbb{B}^n)$, is given by the formula

$$\widehat{U}\widehat{T}_a\widehat{U}^* \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} [T_a - (1 \otimes \overline{a})]f_1 + [\Gamma_a - (1 \otimes a^*)]f_2 \\ \Gamma_{\widehat{a}} f_1 + T_{\widehat{a}} f_2 \end{pmatrix}.$$

Observe that $1 \otimes \overline{a}$ and $1 \otimes a^*$ are complex-valued operators and then, their image is orthogonal to $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$. Which implies that $1 \otimes \overline{a} = 0$ and $1 \otimes a^* = 0$. □

3 Toeplitz Operators with quasi-radial symbols on the pluriharmonic Bergman space

This section is devoted to the study of Toeplitz operators with a special kind of symbols: k-quasi-radial symbols. These symbols generalize radial and separately radial symbols. We follow the notation and definitions of [8]. Let $k = (k_1, \ldots, k_m)$ be a tuple of positive integers such that $k_1 + \cdots + k_m = n$, we rearrange the n coordinates of $z \in \mathbb{B}^n$ in m groups, each one of which has k_j, $j = 1, \ldots, m$, entries and introduce the notation

$$z_{(1)} = (z_{1,1}, \ldots, z_{1,k_1}), z_{(2)} = (z_{2,1}, \ldots, z_{2,k_2}), \ldots, z_{(m)} = (z_{m,1}, \ldots, z_{m,k_m}).$$

Each $z_j = (z_{j,1}, \dots, z_{j,k_j}) \in \mathbb{B}^{k_j}$ is represented in the form

$$z_{(j)} = r_j \zeta_{(j)}, \text{ where } r_j = \sqrt{|z_{j,1}|^2 + \cdots + |z_{j,k_j}|^2} \text{ and } \zeta_{(j)} \in \mathbb{S}^{k_j}.$$

In what follows we assume that $k_1 \leq k_2 \leq \cdots \leq k_m$, and that

$$z_{1,1} = z_1,\ z_{1,2} = z_2, \cdots,\ z_{1,k_1} = z_{k_1},\ z_{2,1} = z_{k_1+1}, \dots, z_{2,k_2} = z_{k_1+k_2}, \dots,\ z_{m,k_m} = z_n. \tag{3.1}$$

We will call such multi-index $k = (k_1, \dots, k_m)$ a *partition* of n.

Definition 3.1. Given a partition $k = (k_1, \dots, k_m)$ of n, a bounded measurable function $a = a(z)$, $z \in \mathbb{B}^n$, will be called k-quasi-radial if it depends only on $r_1, \dots, r_m$.

In particular, if $k = (n)$ a k-quasi-radial function is just a radial function and if $k = (1, 1, \dots, 1)$ a k-quasi-radial function is a separately radial function.

Given a partition $k = (k_1, \dots, k_m)$ and any n-tuple $\alpha = (\alpha_1, \dots, \alpha_n)$, we define

$$\alpha_{(1)} = (\alpha_1 \dots, \alpha_{k_1}), \alpha_{(2)} = (\alpha_{k_1+1}, \dots, \alpha_{k_1+k_2}), \dots, \alpha_{(m)} = (\alpha_{n-k_m+1}, \dots, \alpha_n).$$

Toeplitz operators with k-quasi-radial symbols, acting on the Bergman space on the unit ball, have been studied in [8]. The action of this kind of operators on the basis of the Bergman space is given in the following lemma.

Lemma 3.2. *[8] Given a k-quasi-radial function $a = a(r_1, \dots, r_m)$ we have*

$$T_a z^\alpha = \gamma_{a,k,\lambda}(\alpha) z^\alpha, \quad \alpha \in \mathbb{Z}_+^n,$$

where

$$\begin{aligned}\gamma_{a,k,\lambda}(\alpha) &= \gamma_{a,k,\lambda}(|\alpha_{(1)}|, \dots, |\alpha_{(m)}|)\\ &= \frac{2^m \Gamma(n + |\alpha| + \lambda + 1)}{\Gamma(\lambda+1) \prod_{j=1}^m (k_j - 1 + |\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} a(r_1, \dots, r_m)(1 - |r|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1} dr_j.\end{aligned} \tag{3.2}$$

To study how Toeplitz operators with k-quasi-radial symbols act on the pluriharmonic Bergman space note that the set

$$\mathcal{B} = \left\{ \begin{pmatrix} \sqrt{\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\,\Gamma(n+\lambda+1)}} z^\alpha \\ 0 \end{pmatrix} \right\}_{\alpha>0} \cup \left\{ \begin{pmatrix} 0 \\ \sqrt{\frac{\Gamma(n+|\beta|+\lambda+1)}{\beta!\,\Gamma(n+\lambda+1)}} \bar{z}^\beta \end{pmatrix} \right\}_{\beta\geq 0} \tag{3.3}$$

is an orthonormal basis for $\mathcal{A}_\lambda^{2,0}(\mathbb{B}^n) \oplus \mathcal{A}_\lambda^2(\mathbb{B}^n)$.

Lemma 3.3. *Given a k-quasi-radial function $a = a(r_1, \dots, r_m)$ we have*

$$\widehat{T}_a \begin{pmatrix} z^\alpha \\ \bar{z}^\beta \end{pmatrix} = \begin{pmatrix} \gamma_{a,k,\lambda}(\alpha)\, z^\alpha \\ \gamma_{a,k,\lambda}(\beta)\, \bar{z}^\beta \end{pmatrix}, \tag{3.4}$$

where $\gamma_{a,k\lambda,}$ is given by (3.2) *and $\alpha \in \mathbb{N}$, $\beta \in \mathbb{Z}_+^n$.*

Proof. Let $a = a(r_1,\ldots,r_m)$ be a k-quasi-radial function, then $a(z)^* = \overline{a(\overline{z})} = \overline{a(z)}$, and $\widehat{a}(z) = a(\overline{z}) = a(z)$. From Theorem 2.1, the Toeplitz operator $\widehat{T}_a$, acting on $b^2_\lambda(\mathbb{B}^n)$ is unitarily equivalent to the operator $\widehat{U}\widehat{T}_a\widehat{U}^*$ acting on $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \oplus \mathcal{A}^2_\lambda(\mathbb{B}^n)$. The last operator has the form

$$\widehat{U}\widehat{T}_a\widehat{U}^* = \begin{pmatrix} T_a & \Gamma_a \\ \Gamma_a & T_a \end{pmatrix}.$$

Consider first the operator Γ_a. For $\alpha,\beta \in \mathbb{Z}^n_+$

$$\langle \Gamma_a z^\alpha, z^\beta \rangle = \langle BaUz^\alpha, z^\beta \rangle = \langle a\overline{z}^\alpha, z^\beta \rangle = \frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)} \int_{\mathbb{B}^n} a(r_1,\ldots,r_m)\overline{z}^\alpha\overline{z}^\beta(1-|z|^2)^\lambda dV(z).$$

Let $z_{(j)} = r_j\zeta_{(j)}$, where $r_j \in [0,1)$ and $\zeta_{(j)} \in \mathbb{S}^{k_j}$, $j = 1,\ldots,m$. Then

$$\begin{aligned}\langle \Gamma_a z^\alpha, z^\beta \rangle &= \frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)} \int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{|\alpha_{(j)}|+|\beta_{(j)}|+2k_j-1} dr_j \\ &\quad\times \prod_{j=1}^m \int_{\mathbb{S}^{k_j}} \overline{\zeta_{(j)}}^{\alpha_{(j)}+\beta_{(j)}} dS(\zeta_{(j)}), \\ &= \frac{2^m\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^m(k_j-1)!} \int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{2k_j-1} dr_j.\end{aligned}$$

Thus,

$$\langle \Gamma_a z^\alpha, z^\beta \rangle = \begin{cases} \gamma_{a,k,\lambda}(0) & \text{if } \alpha = \beta = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Now, in the representation (2.6) of $\widehat{U}\widehat{T}_a\widehat{U}^*$, the operator Γ_a acts in two different forms. First, $\Gamma_a : \mathcal{A}^2_\lambda(\mathbb{B}^n) \to \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$, in this case $\Gamma_a z^\alpha$ is orthogonal to $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$ and then, it is the zero operator. Analogously, $\Gamma_a : \mathcal{A}^{2,0}_\lambda(\mathbb{B}^n) \to \mathcal{A}^2_\lambda(\mathbb{B}^n)$ is the zero operator. In summary,

$$\widehat{T}_a \begin{pmatrix} z^\alpha \\ z^\beta \end{pmatrix} = \begin{pmatrix} T_a z^\alpha \\ T_a z^\beta \end{pmatrix} = \begin{pmatrix} \gamma_{a,k,\lambda}(\alpha)\, z^\alpha \\ \gamma_{a,k,\lambda}(\beta)\, z^\beta \end{pmatrix}.$$

□

Hongyan Guan, Liu Liu and Yufeng Lu proved (see [9], Lemma 11) that if a is a bounded separately radial function, then the Toeplitz operator $\widehat{T}_a$, acting on the unweighted pluriharmonic Bergman space holds the following properties

$$\widehat{T}_a z^\alpha = \frac{2^n(n+|\alpha|)!}{\alpha!} \int_{\tau(\mathbb{B}^n)} a(r) r^{2\alpha+1} dr\, z^\alpha, \quad \widehat{T}_a \overline{z}^\alpha = \frac{2^n(n+|\alpha|)!}{\alpha!} \int_{\tau(\mathbb{B}^n)} a(r) r^{2\alpha+1} dr\, \overline{z}^\alpha \tag{3.5}$$

for any multi-index $\alpha \in \mathbb{Z}^n_+$.

If a is a separately radial function ($k = (1,\ldots,1)$), then formulas (3.4) in Lemma 3.3 reduce to properties (3.5). Note that the action of $\widehat{T}_a$ at the function $\overline{z}^\alpha$ in (3.5) corresponds to $\widehat{T}_a \begin{pmatrix} 0 \\ z^\alpha \end{pmatrix}$.

It is well known that if a is a radial function, the Toeplitz operator $\widehat{T}_a$, acting on $b^2_\lambda(\mathbb{B}^n)$ is a diagonal operator (see for example [7]). In this case, formulas (3.4) reduce to the following equation

$$\widehat{T}_a \begin{pmatrix} z^\alpha \\ z^\beta \end{pmatrix} = \begin{pmatrix} \frac{2\Gamma(n+|\alpha|+\lambda+1)}{\Gamma(\lambda+1)(n-1+|\alpha|)!} \int_0^1 a(r)(1-|r|^2)^\lambda r^{2|\alpha|+2n-1}\ z^\alpha \\ \frac{2\Gamma(n+|\beta|+\lambda+1)}{\Gamma(\lambda+1)(n-1+|\beta|)!} \int_0^1 a(r)(1-|r|^2)^\lambda r^{2|\beta|+2n-1}\ z^\beta \end{pmatrix}. \tag{3.6}$$

Corollary 3.4. *The C^*-algebra generated by Toeplitz operators with radial symbols, acting on the pluriharmonic Bergman space, is isomorphic and isometric to the C^*-algebra generated by all operators of multiplication by the sequence* $\left\{\frac{2\Gamma(n+m+\lambda+1)}{\Gamma(\lambda+1)(n-1+m)!}\int_0^1 a(r)(1-|r|^2)^\lambda r^{2m+2n-1}\right\}_m$, *acting on $\ell^2(\mathbb{Z}_+)$.*

The sequence $\left\{\frac{2\Gamma(n+m+\lambda+1)}{\Gamma(\lambda+1)(n-1+m)!}\int_0^1 a(r)(1-|r|^2)^\lambda r^{2m+2n-1}\right\}_m$ is called the eigenvalue sequence of the Toeplitz operator $\widehat{T}_a$.

Denote by $\mathrm{SO}(\mathbb{Z}_+)$ the set of all bounded sequences $x = (x_j)_{j\in\mathbb{Z}_+}$ that slowly oscillate in the sense of Schmidt ([6])

$$\mathrm{SO}(\mathbb{Z}_+) = \left\{ x \in \ell_\infty(\mathbb{Z}_+) |\ \lim_{\frac{j+1}{k+1}\to 1} |x_j - x_k| \right\} = 0.$$

If we consider the logarithmic metric on $\mathbb{Z}_+$ which is defined by the formula $\rho(j,k) = |\ln(j+1) - \ln(j)|$ then, $\mathrm{SO}(\mathbb{Z}_+)$ consists of all bounded sequences that are uniformly continuous with respect to this metric.

It was proved in [1] (Theorem 5.4) that the set

$$\Gamma^n_\lambda = \left\{ \left\{ \frac{2\Gamma(n+m+\lambda+1)}{\Gamma(\lambda+1)(n-1+m)!} \int_0^1 a(r)(1-|r|^2)^\lambda r^{2m+2n-1} \right\}_m |\ a \text{ is a radial function} \right\}$$

is dense in $\mathrm{SO}(\mathbb{Z}_+)$. Using these results we can describe the C^*-algebra generated by all Toeplitz operators with radial symbols, acting on $b^2_\lambda(\mathbb{B}^n)$.

Corollary 3.5. *The C^*-algebra generated by all Toeplitz operators with bounded and radial symbols, acting on $b^2_\lambda(\mathbb{B}^n)$ is isomorphic and isometric to* $\mathrm{SO}(\mathbb{Z}_+)$.

4 Toeplitz operators with k-quasi-homogenous symbols on the pluriharmonic Bergman space

Let $k = (k_1, \dots, k_m)$ be a partition of n. We use the notation $z_{(j)} = r_j\zeta_{(j)}$, $j = 1, \dots, m$ to define the vector

$$\zeta = (\zeta_{(1)}, \dots, \zeta_{(m)}) \in \mathbb{S}^{k_1} \times \cdots \mathbb{S}^{k_m}.$$

Definition 4.1. Let $p, q \in \mathbb{Z}^n_+$ be two multi-indices. A function φ is called k-quasi-homogenous of grade (p,q) if it has the form

$$\begin{aligned} \varphi(z) &= a(r_1, \dots, r_m)\zeta^p \overline{\zeta}^q \\ &= a(r_1, \dots, r_m)\, \zeta_{(1)}^{p_{(1)}} \cdots \zeta_{(m)}^{p_{(m)}} \overline{\zeta}_{(1)}^{q_{(1)}} \cdots \overline{\zeta}_{(m)}^{q_{(m)}}, \end{aligned}$$

where $a(r_1, \dots, r_m)$ is a k-quasi-radial function and $p, q \geq 0$.

Toeplitz operators with k-quasi-homogeneous symbols acting on the Bergman space have been studied in [8], for example, and we have the following result.

Lemma 4.2 ([8]). *Let $a\zeta^p\overline{\zeta}^q$ be a k-quasi-homogenous function. Then for any multi-index α,*

$$T_{a\zeta^p\overline{\zeta}^q} z^\alpha = \begin{cases} \widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha) z^{\alpha+p-q}, & \text{if } \alpha \geq q-p, \\ 0, & \text{if } \alpha \ngeq q-p, \end{cases} \quad \alpha \in \mathbb{Z}_+^n,$$

where

$$\begin{aligned} \widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha) = & \frac{2^m\, \Gamma(n+|\alpha+p-q|+\lambda+1)(\alpha+p)!}{\Gamma(\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}+p_{(j)}|)!(\alpha+p-q)!} \\ & \times \int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{|2\alpha_{(j)}+p_{(j)}-q_{(j)}|+2k_j-1} dr_j. \end{aligned} \tag{4.1}$$

Now, we are in position to describe the action of Toeplitz operators with k-quasi-homogenous symbols on the pluriharmonic Bergman space. To simplify the notation, we will use $\widehat{T}_a$ instead of $\widehat{U}\widehat{T}_a\widehat{U}^*$.

Theorem 4.3. *Let p,q be two multi-indices and k-quasi-homogenous function $a\zeta^p\overline{\zeta}^q$. Then the Toeplitz operator $\widehat{T}_{a\zeta^p\overline{\zeta}^q}$ acts on the bases $\mathcal{B}$ of $\mathcal{A}_\lambda^{2,0}(\mathbb{B}^n)\oplus\mathcal{A}_\lambda^2(\mathbb{B}^n)$ (see (3.3)) as follows*

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} \widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha)\, z^{\alpha+p-q} \\ 0 \end{pmatrix}, & \text{if } \alpha+p>q, \\ \begin{pmatrix} 0 \\ \rho^{p,q}_{a,k,\lambda}(\alpha)\, z^{\alpha+p-q} \end{pmatrix}, & \text{if } q>\alpha+p, \\ 0, & \text{otherwise,} \end{cases}$$

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} 0 \\ z^\beta \end{pmatrix} = \begin{cases} \begin{pmatrix} \sigma^{p,q}_{a,k,\lambda}(\beta)\, z^{p-q-\beta} \\ 0 \end{pmatrix}, & \text{if } p>\beta+q, \\ \begin{pmatrix} 0 \\ \vartheta^{p,q}_{a,k,\lambda}(\beta)\, z^{\beta+q-p} \end{pmatrix}, & \text{if } \beta+q\geq p, \\ 0, & \text{otherwise,} \end{cases}$$

where $\widetilde{\gamma}_{a,k,p,q,\lambda}$ *is given by* (4.1) *and*

$$\begin{aligned}
\rho^{p,q}_{a,k,\lambda}(\alpha) &= \frac{2^m\,\Gamma(n+|q-p-\alpha|+\lambda+1)q!}{\Gamma(\lambda+1)\prod_{j=1}^m(k_j-1+|q_{(j)}|)!(q-p-\alpha)!} \\
&\quad\times\int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda\prod_{j=1}^m r_j^{|q_{(j)}-p_{(j)}|+2k_j-1}dr_j, \\
\sigma^{p,q}_{a,k,\lambda}(\beta) &= \frac{2^m\,\Gamma(n+|p-q-\beta|+\lambda+1)p!}{\Gamma(\lambda+1)\prod_{j=1}^m(k_j-1+|p_{(j)}|)!(p-q-\beta)!} \\
&\quad\times\int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda\prod_{j=1}^m r_j^{|p_{(j)}-q_{(j)}|+2k_j-1}dr_j, \\
\vartheta^{p,q}_{a,k,\lambda}(\beta) &= \frac{2^m\,\Gamma(n+|q-p+\beta|+\lambda+1)(\beta+q)!}{\Gamma(\lambda+1)\prod_{j=1}^m(k_j-1+|\beta_{(j)}+q_{(j)}|)!(q-p+\beta)!} \\
&\quad\times\int_{\tau(\mathbb{B}^m)} a(r_1,\ldots,r_m)(1-|r|^2)^\lambda\prod_{j=1}^m r_j^{|2\beta_{(j)}+q_{(j)}-p_{(j)}|+2k_j-1}dr_j.
\end{aligned}\tag{4.2}$$

Proof. Let $\phi(z)=a(r_1,\ldots,r_m)\zeta^p\overline{\zeta}^q$ be a k-quasi-homogeneous function, then

$$\phi(z)^*=\overline{a}(r_1,\ldots,r_m)\zeta^p\overline{\zeta}^q,\ \ \widehat{\phi}(z)=a(r_1,\ldots,r_m)\overline{\zeta}^p\zeta^q.$$

By Theorem 2.1, the Toeplitz operator $\widehat{T}_\phi$ acting on $\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)\oplus\mathcal{A}^2_\lambda(\mathbb{B}^n)$ has the following form

$$\widehat{T}_\phi=\begin{pmatrix} T_\phi & \Gamma_\phi \\ \Gamma_{\widehat{\phi}} & T_{\widehat{\phi}}\end{pmatrix}.$$

Consider first the operator $\Gamma_\phi:\mathcal{A}^2_\lambda(\mathbb{B}^n)\to\mathcal{A}^{2,0}_\lambda(\mathbb{B}^n)$. For $\alpha\in\mathbb{Z}^n_+$ and $\beta\in\mathbb{N}^n$

$$\langle\Gamma_\phi z^\alpha,z^\beta\rangle=\langle B\phi Uz^\alpha,z^\beta\rangle=\langle\phi\overline{z}^\alpha,z^\beta\rangle=\frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)}\int_{\mathbb{B}^n}a(r_1,\ldots,r_m)\zeta^p\overline{\zeta}^q\overline{z}^\alpha\overline{z}^\beta(1-|z|^2)^\lambda dV(z).$$

Let $z_{(j)}=r_j\zeta_{(j)}$, where $r_j\in[0,1)$ and $\zeta_{(j)}\in\mathbb{S}^{k_j}$, $j=1,\ldots,m$. Then,

$$\begin{aligned}
\langle\Gamma_\phi z^\alpha,z^\beta\rangle &= \frac{\Gamma(n+\lambda+1)}{\pi^n\Gamma(\lambda+1)}\int_{\tau(\mathbb{B}^m)}a(r_1,\ldots,r_m)(1-|r|^2)^\lambda\prod_{j=1}^m r_j^{|\alpha_{(j)}+\beta_{(j)}|+2k_j-1}dr_j \\
&\quad\times\prod_{j=1}^m\int_{\mathbb{S}^{k_j}}\zeta_{(j)}^{p_{(j)}}\overline{\zeta_{(j)}}^{\alpha_{(j)}+\beta_{(j)}+q_{(j)}}dS(\zeta_{(j)}), \\
&= \frac{\delta_{p,\alpha+\beta+q}\,2^m\Gamma(n+\lambda+1)p!}{\Gamma(\lambda+1)\prod_{j=1}^m(k_j-1+|p_{(j)}|)!}\int_{\tau(\mathbb{B}^m)}a(r_1,\ldots,r_m)(1-|r|^2)^\lambda\prod_{j=1}^m r_j^{|\alpha_{(j)}+\beta_{(j)}|+2k_j-1}dr_j.
\end{aligned}$$

Thus, $\langle\Gamma_\phi z^\alpha,z^\beta\rangle$ is non zero if and only if $p=\alpha+\beta+q$. So, by (3.3) we have for $p-q-\alpha>0$,

$$\|z^{p-q-\alpha}\|^2=\frac{\Gamma(n+\lambda+1)(p-q-\alpha)!}{\Gamma(n+|p-q-\alpha|+\lambda+1)!}.$$

Then, for $p > \alpha + q$

$$\Gamma_\phi z^\alpha = \langle \Gamma_\phi z^\alpha, z^{p-q-\alpha} \rangle \frac{z^{p-q-\alpha}}{||z^{p-q-\alpha}||^2} = \sigma^{p,q}_{a,k,\lambda}(\alpha) z^{p-q-\alpha},$$

where

$$\sigma^{p,q}_{a,k,\lambda}(\beta) = \frac{2^m\, \Gamma(n + |p-q-\beta| + \lambda + 1) p!}{\Gamma(\lambda+1) \prod_{j=1}^m (k_j - 1 + |p_{(j)}|)!(p-q-\beta)!} \times \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{|p_{(j)}-q_{(j)}|+2k_j-1} dr_j.$$

Then,

$$\Gamma_\phi z^\alpha = \begin{cases} \sigma^{p,q}_{a,k,\lambda}(\alpha) z^{p-q-\alpha}, & \text{if } p > \alpha + q, \\ 0, & \text{otherwise.} \end{cases} \tag{4.3}$$

Analogous calculations show that

$$\Gamma_{\widehat{\phi}} z^\alpha = \begin{cases} \rho^{p,q}_{a,k,\lambda}(\alpha)\, z^{q-p-\alpha}, & \text{if } q > \alpha + p, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\rho^{p,q}_{a,k,\lambda}(\alpha) = \frac{2^m\, \Gamma(n + |q-p-\alpha| + \lambda + 1) q!}{\Gamma(\lambda+1) \prod_{j=1}^m (k_j - 1 + |q_{(j)}|)!(q-p-\alpha)!} \times \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{|q_{(j)}-p_{(j)}|+2k_j-1} dr_j.$$

Besides, from [8] it is easy to show that

$$T_{\widehat{\phi}} z^\alpha = \begin{cases} \vartheta^{p,q}_{a,k,\lambda}(\alpha)\, z^{\alpha+q-p}, & \text{if } \alpha + q \geq p, \\ 0, & \text{otherwise,} \end{cases}$$

where

$$\vartheta^{p,q}_{a,k,\lambda}(\alpha) = \frac{2^m \Gamma(n + |q-p+\alpha| + \lambda + 1)(\alpha+q)!}{\Gamma(\lambda+1) \prod_{j=1}^m (k_j - 1 + |\alpha_{(j)} + q_{(j)}|)!(q-p+\alpha)!} \times \int_{\tau(\mathbb{B}^m)} a(r_1, \ldots, r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{|2\alpha_{(j)}+q_{(j)}-p_{(j)}|+2k_j-1} dr_j. \quad \square$$

Corollary 4.4. *Especially, if p, q are two (non zero) orthogonal multi-indices, we have*

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q} \begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \begin{cases} \begin{pmatrix} \widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha) z^{\alpha+p-q} \\ 0 \end{pmatrix}, & \text{if } \alpha > q, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q} \begin{pmatrix} 0 \\ z^\beta \end{pmatrix} = \begin{cases} \begin{pmatrix} 0 \\ \vartheta^{p,q}_{a,k,\lambda}(\beta)\, z^{\beta+q-p} \end{pmatrix}, & \text{if } \beta \geq p, \\ 0, & \text{otherwise.} \end{cases}$$

If $k=(n)$ or $k=(1,1,\ldots,1)$, Toeplitz operators with k-quasi-homogeneous symbols, acting on the unweighted pluriharmonic Bergman space on the unit ball, were studied in [9] and [10] respectively.

4.1 Commutativity results for Toeplitz operators with k-quasi-homogenous symbols on the pluriharmonic Bergman space

In this section we present some commuting identities for Toeplitz operators with k-quasi-homogenous symbols acting on the pluriharmonic Bergman space on the unit ball $\mathbb{B}^n$. These results were proven in [8] for the weighted Bergman space on $\mathbb{B}^n$.

Corollary 4.5. *Let $k=(k_1,k_2,\ldots,k_m)$ be a partition of n and p,q be two multi-indices. Then for each pair of non identically zero k-quasi-radial functions a_1 and a_2, the Toeplitz operators $\widehat{T}_{a_1}$ and $\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}$ acting on the pluriharmonic Bergman space $b^2_\lambda(\mathbb{B}^n)$ satisfy*

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}=\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}$$

if and only if $|p_{(j)}|=|q_{(j)}|$ for each $j=1,2,\ldots,m$.

Proof. Let $\alpha\in\mathbb{N}^n$ be given. We suppose that $\alpha>q$, then

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}=\begin{pmatrix}\gamma_{a_1,k,\lambda}(\alpha+p-q)\,\widetilde{\gamma}_{a_2,k,p,q,\lambda}(\alpha)z^{\alpha+p-q}\\ 0\end{pmatrix},$$

$$\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}=\begin{pmatrix}\widetilde{\gamma}_{a_2,k,p,q,\lambda}(\alpha)\,\gamma_{a_1,k,\lambda}(\alpha)z^{\alpha+p-q}\\ 0\end{pmatrix}.$$

Thus,

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}=\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}$$

if and only if

$$\gamma_{a_1,k,\lambda}(\alpha+p-q)\,\widetilde{\gamma}_{a_2,k,p,q,\lambda}(\alpha)=\widetilde{\gamma}_{a_2,k,p,q,\lambda}(\alpha)\,\gamma_{a_1,k,\lambda}(\alpha).$$

Using Theorem 4.1 of [8] we conclude that

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}=\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}\begin{pmatrix} z^\alpha \\ 0\end{pmatrix}$$

if and only if $|p_{(j)}|=|q_{(j)}|$ for $j=1,\ldots,m$. Analogously if $\beta\geq q$,

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\begin{pmatrix} 0 \\ z^\beta\end{pmatrix}=\begin{pmatrix} 0 \\ \gamma_{a_1,k,\lambda}(\beta+q-p)\,\vartheta^{p,q}_{a_2,k,\lambda}(\beta)\,z^{\beta+q-p}\end{pmatrix},$$

$$\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}\begin{pmatrix} 0 \\ z^\beta\end{pmatrix}=\begin{pmatrix} 0 \\ \vartheta^{p,q}_{a_2,k,\lambda}(\beta)\gamma_{a_1,k,\lambda}(\beta)\,z^{\beta+q-p}\end{pmatrix}.$$

Then,

$$\widehat{T}_{a_1}\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\begin{pmatrix} 0 \\ z^\beta\end{pmatrix}=\widehat{T}_{a_2\zeta^p\overline{\zeta}^q}\widehat{T}_{a_1}\begin{pmatrix} 0 \\ z^\beta\end{pmatrix}$$

if and only if

$$\gamma_{a_1,k,\lambda}(\beta+q-p)\,\vartheta^{p,q}_{a_2,k,\lambda}(\beta)=\vartheta^{p,q}_{a_2,k,\lambda}(\beta)\gamma_{a_1,k,\lambda}(\beta).$$

From Theorem 4.1 of [8] last equality holds if and only if $|p_{(j)}|=|q_{(j)}|$ for $j=1,\ldots,m$. $\square$

We note that under the condition $|p_{(j)}| = |q_{(j)}|$, formulas (4.1) and (4.2) are reduced to the following expressions

$$\widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha) = \prod_{j=1}^{m} \left[\frac{(k_j - 1 + |\alpha_{(j)}|)!}{(k_j - 1 + |\alpha_{(j)} + p_{(j)}|)!} \frac{(\alpha_{(j)} + p_{(j)})!}{(\alpha_{(j)} + p_{(j)} - q_{(j)})!} \right] \gamma_{a,k,\lambda}(\alpha),$$
$$\vartheta^{p,q}_{a,k,\lambda}(\beta) = \prod_{j=1}^{m} \left[\frac{(k_j - 1 + |\beta_{(j)}|)!}{(k_j - 1 + |\beta_{(j)} + q_{(j)}|)!} \frac{(\beta_{(j)} + q_{(j)})!}{(\beta_{(j)} + q_{(j)} - p_{(j)})!} \right] \gamma_{a,k,\lambda}(\beta).$$

In the following theorem we obtain the conditions under which two Toeplitz operators with k-quasi-homogeneous symbols, acting on the pluriharmonic Bergman space $b^2_\lambda(\mathbb{B}^n)$, commute.

Let $k = (k_1, \ldots, k_m)$ be a partition of n. We consider any two bounded measurable k-quasi-homogeneous functions $a(r_1, \ldots, r_m)\zeta^p\overline{\zeta}^q$ and $b(r_1, \ldots, r_m)\zeta^u\overline{\zeta}^v$, which satisfy the following conditions

1. $p \perp q$ and $u \perp v$,
2. $|p_{(j)}| = |q_{(j)}|$ and $|u_{(j)}| = |v_{(j)}|$,

for $j = 1, 2, \ldots, m$.

Theorem 4.6. *Let $a(r_1, \ldots, r_m)\zeta^p\overline{\zeta}^q$ and $b(r_1, \ldots, r_m)\zeta^u\overline{\zeta}^v$ be as above. Then the Toeplitz operators $\widehat{T}_{a\zeta^p\overline{\zeta}^q}$ and $\widehat{T}_{b\zeta^u\overline{\zeta}^v}$ commute on each pluriharmonic Bergman space $b^2_\lambda(\mathbb{B}^n)$ if and only if one of the following conditions is fulfilled*

$$\begin{array}{ll} 1.\ p_\ell = q_\ell = 0, & 3.\ p_\ell = u_\ell = 0, \\ 2.\ u_\ell = v_\ell = 0, & 4.\ q_\ell = v_\ell = 0. \end{array} \tag{4.4}$$

for each $\ell = 1, 2 \ldots, n$.

Proof. First, using Corollary 4.4 we have

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\widehat{T}_{b\zeta^u\overline{\zeta}^v}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha + u - v)\widetilde{\gamma}_{b,k,u,v,\lambda}(\alpha) z^{\alpha+u-v+p-q} \\ 0 \end{pmatrix}$$

and

$$\widehat{T}_{b\zeta^u\overline{\zeta}^v}\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \begin{pmatrix} \widetilde{\gamma}_{b,k,u,v,\lambda}(\alpha + p - q)\widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha) z^{\alpha+p-q+u-v} \\ 0 \end{pmatrix},$$

for those multi-indices α such that these expressions are non zero. Consequently,

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\widehat{T}_{b\zeta^u\overline{\zeta}^v}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \widehat{T}_{b\zeta^u\overline{\zeta}^v}\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix}$$

if and only if

$$\widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha + u - v)\widetilde{\gamma}_{b,k,u,v,\lambda}(\alpha) = \widetilde{\gamma}_{b,k,u,v,\lambda}(\alpha + p - q)\widetilde{\gamma}_{a,k,p,q,\lambda}(\alpha).$$

Using Theorem 4.5 of [8] we conclude that for all $\alpha \in \mathbb{N}^n$ and $\lambda > -1$,

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\widehat{T}_{b\zeta^u\overline{\zeta}^v}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix} = \widehat{T}_{b\zeta^u\overline{\zeta}^v}\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} z^\alpha \\ 0 \end{pmatrix}$$

if and only if (4.4) holds.

On the other hand, using again Corollary 4.4 we have

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\widehat{T}_{b\zeta^u\overline{\zeta}^v}\begin{pmatrix} 0 \\ z^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ \vartheta^{p,q}_{a,k,\lambda}(\beta+v-u)\vartheta^{u,v}_{b,k,\lambda}(\beta)z^{\beta+v-u+q-p} \end{pmatrix}$$

and

$$\widehat{T}_{b\zeta^u\overline{\zeta}^v}\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} 0 \\ z^\beta \end{pmatrix} = \begin{pmatrix} 0 \\ \vartheta^{u,v}_{b,k,\lambda}(\beta+q-p)\vartheta^{p,q}_{a,k,\lambda}(\beta)z^{\beta+q-p+v-u} \end{pmatrix},$$

for those multi-indices β when these expressions are non zero. That is,

$$\widehat{T}_{a\zeta^p\overline{\zeta}^q}\widehat{T}_{b\zeta^u\overline{\zeta}^v}\begin{pmatrix} 0 \\ z^\beta \end{pmatrix} = \widehat{T}_{b\zeta^u\overline{\zeta}^v}\widehat{T}_{a\zeta^p\overline{\zeta}^q}\begin{pmatrix} 0 \\ z^\beta \end{pmatrix}$$

if and only if

$$\vartheta^{p,q}_{a,k,\lambda}(\beta+v-u)\vartheta^{u,v}_{b,k,\lambda}(\beta) = \vartheta^{u,v}_{b,k,\lambda}(\beta+q-p)\vartheta^{p,q}_{a,k,\lambda}(\beta).$$

This expression can be reduced to

$$\frac{(\beta+v-u+q)!(\beta+v)!}{(\beta+v-u)!} = \frac{(\beta+q-p+v)!(\beta+q)!}{(\beta+q-p)!},$$

which is valid if and only if (4.4) holds. □

In [8], the author shows that there exist many commutative Banach algebras generated by Toeplitz operators acting on the weighted Bergman space on the unit ball and if they are extended to C^*-algebra they become non commutative C^*-algebras. These algebras are generated by k-quasi-homogeneous symbols. From Theorem 4.6, we can construct commutative Banach algebras generated by Toeplitz operators with k-quasi-homogeneous symbols, acting on $b^2_\lambda(\mathbb{B}^n)$. To describe them we use the notation from [8]. In addition to (3.1) we rearrange the variables z_ℓ and correspondingly the components of multi-indices in p and q such that

(i) for each j with $k_j > 1$, we have

$$p_{(j)} = (p_{j,1},\dots,p_{j,h_j},0,\dots,0) \text{ and } q_{(j)} = (0,\dots,0,q_{j,h_{j+1}},\dots,q_{j,k_j}); \tag{4.5}$$

(ii) if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \le h_{j''}$.

Let $k = (k_1,\dots,k_m)$ be a partition of n, we start with m-tuple $h = (h_1,\dots,h_m)$, where $h_j = 0$ if $k_j = 1$ and $1 \le h_j \le k_j - 1$ if $k_j \ge 1$. In the last case, if $k_{j'} = k_{j''}$ with $j' < j''$, then $h_{j'} \le h_{j''}$.

We denote by $\mathcal{R}_k(h)$ the linear space generated by all k-quasi-homogenous functions $a(r_1,\dots,r_m)\zeta^p\overline{\zeta}^q$, where a is a k-quasi-radial function and the components $p_{(j)}$ and $q_{(j)}$, $j = 1,\dots,m$, of multi-indices p and q are the form (4.5) with

$$p_{j,1} + \cdots + p_{j,h_{j+1}} = q_{j,h_{j+1}} + \cdots + q_{j,k_j},$$

$p_{j,1},\dots,p_{j,h_j},q_{j,h_{j+1}},\dots,q_{j,k_j} \in \mathbb{Z}_+$. The set $\mathcal{R}_k(h)$ contains all k-quasi-radial functions.

Corollary 4.7. *The Banach algebra generated by Toeplitz operators with symbols from $\mathcal{R}_k(h)$ is commutative on each pluriharmonic Bergman space $b^2_\lambda(\mathbb{B}^n)$.*

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