Communications in **M**athematical **A**nalysis
Volume 17, Number 2, pp. 178–188 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Quasi-radial Operators on the Weighted Bergman Space over the Unit Ball

Alma García *
Department of Mathematics
CINVESTAV
Mexico City, MEXICO

Nikolai Vasilevski†
Department of Mathematics
CINVESTAV
Mexico City, MEXICO

(Communicated by Vladimir Rabinovich)

Abstract

We study the so-called *quasi-radial* operators, i.e., the operators that are invariant under the subgroup of the unitary group $\mathfrak{U}(n)$ formed by the block-diagonal matrices with unitary blocks of fixed dimensions. The quasi-radial Toeplitz operators appear naturally and play a crucial role under the study of the commutative Banach (not C^*) algebras of Toeplitz operators [1, 8]. They form an intermediate class of operators between the Toeplitz operators with radial $a = a(r)$, $r = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$, and separately-radial $a = a(|z_1|, \ldots, |z_n|)$ symbols.

AMS Subject Classification: 47B35

Keywords: Banach algebra, Berezin transform, Toeplitz operator, quasi-radial operator, quasi-radialization.

1 Introduction

In this note we consider the Toeplitz operators acting on the standard weighted Bergman space over the unit ball in $\mathbb{C}^n$. It is a well established fact that the invariance of symbols under a certain subgroup of biholomorphisms of the unit ball determines many of the properties of the corresponding Toeplitz operators. In particular, the invariance under the maximal compact subgroup $\mathfrak{U}(n)$, that consists of all unitary $n \times n$ matrices, leads to the *radial*

*E-mail address: garcialmaitzel@gmail.com
†E-mail address: nvasilev@duke.math.cinvestav.mx

symbols $a = a(r)$, $r = \sqrt{|z_1|^2 + \ldots + |z_n|^2}$. The corresponding Toeplitz operators generate a commutative C^*-algebra, they are diagonal with respect to the standard monomial basis with the eigenvalue sequences that depend only on the length of multi-indices. At the same time the maximal commutative C^*-algebra, that contains Toeplitz operators with radial symbols, is generated by Toeplitz operators with *separately radial* symbols $a = a(|z_1|, \ldots, |z_n|)$, those that are invariant under the action of the torus $\mathbb{T}^n$, the diagonal subgroup of $\mathfrak{U}(n)$. These operator are diagonal with respect to the standard monomial basis with the eigenvalue sequences that depend on the basis multi-indices.

In Section 2 we study an intermediate class of operators, the so-called *quasi-radial* operators, that are invariant under the subgroup of $\mathfrak{U}(n)$ formed by the block-diagonal matrices with unitary blocks of fixed dimensions. The corresponding Toeplitz operators appear naturally and play a crucial role under the study of the commutative Banach (not C^*) algebras of Toeplitz operators [1, 8].

In Section 3 we give explicit formulas for (p, λ)-Berezin and Berezin transforms for quasi-radial operators. These transforms prove to be useful tools in approximation of bounded operators via Toeplitz operators [4, 5, 6, 7].

2 Quasi-radial operators

Let $\mathbb{B}^n$ be the open unit ball in $\mathbb{C}^n$, $n \in \mathbb{N}$, and let dv denote the standard volume form on $\mathbb{B}^n$. For $\lambda > -1$, we introduce the one-parameter family of the weighted measures

$$dv_\lambda(z) = \frac{\Gamma(n+\lambda+1)}{\pi^n \Gamma(\lambda+1)} (1 - |z|^2)^\lambda dv(z).$$

The weighted Bergman space $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is the closed subspace of $L_2(\mathbb{B}^n, dv_\lambda)$ that consists of all functions analytic in $\mathbb{B}^n$. Given a function $a \in L_\infty(\mathbb{B}^n)$, the Toeplitz operator T_a with symbol a and acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is defined by

$$T_a \phi = B_\lambda(a\phi), \qquad \phi \in \mathcal{A}^2_\lambda(\mathbb{B}^n),$$

where

$$(B_\lambda \phi)(z) = \int_{\mathbb{B}^n} \frac{\phi(\zeta)\, dv_\lambda(\zeta)}{(1 - \langle z, \zeta \rangle)^{n+\lambda+1}}$$

is the orthogonal Bergman projection of $L_2(\mathbb{B}^n, dv_\lambda)$ onto $\mathcal{A}^2_\lambda(\mathbb{B}^n)$.

Recall that the reproducing kernel of $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ is defined by

$$K^\lambda_z(w) = \frac{1}{(1 - \langle w, z \rangle)^{n+\lambda+1}} = \sum_{|\alpha|=0}^{\infty} \frac{\Gamma(n + |\alpha| + \lambda + 1)}{\alpha! \Gamma(n+\lambda+1)} \overline{z}^\alpha w^\alpha.$$

Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a tuple such that $k_i \in \mathbb{N}$ for $i = 1, \ldots, m$ and $k_1 + k_2 + \ldots + k_m = n$. Given such tuple $\mathbf{k}$ we rearrange the n coordinates of $z \in \mathbb{B}^n$ in m groups, each of which has k_j entries ($j = 1, \ldots, m$), and introduce the notation:

$$z_{(1)} = (z_{1,1}, \ldots, z_{1,k_1}), \;\; z_{(2)} = (z_{2,1}, \ldots, z_{2,k_2}), \;\; \ldots, \;\; z_{(m)} = (z_{m,1}, \ldots, z_{m,k_m}).$$

We assume that $k_1 \le k_2 \le \ldots \le k_m$ and that

$$z_{1,1} = z_1, \;\; z_{1,2} = z_2, \;\; \ldots, \;\; z_{1,k_1} = z_{k_1}, \;\; z_{2,1} = z_{k_1+1}, \;\; \ldots, \;\; z_{m,k_m} = z_n,$$

That is

$$z = (z_1, \ldots, z_n) = (z_{(1)}, \ldots, z_{(m)}), \qquad \text{with} \quad z_{(j)} \in \mathbb{C}^{k_j}.$$

In the same way for any tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ we let

$$\alpha_{(1)} = (\alpha_1, \ldots, \alpha_{k_1}), \;\; \alpha_{(2)} = (\alpha_{k_1+1}, \;\; \ldots, \;\; \alpha_{k_1+k_2}), \;\; \ldots, \;\; \alpha_{(m)} = (\alpha_{n-k_m+1}, \ldots, \alpha_n),$$

and $z^{\alpha} = z_{(1)}^{\alpha_1} \cdots z_{(m)}^{\alpha_m}$.

Denote by $\mathfrak{U}(l)$ the compact group of all $l \times l$ complex unitary matrices U equipped with the Haar measure. We introduce the compact subgroup $\mathfrak{U}'(\mathbf{k}) \subset \mathfrak{U}(n)$ that consists of all $n \times n$ complex block diagonal unitary matrices

$$U = \begin{pmatrix} A_{k_1} & 0 & \ldots & 0 \\ 0 & A_{k_2} & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_{k_m} \end{pmatrix}, \qquad \text{where} \quad A_{k_j} \in \mathfrak{U}(k_j), \quad j = 1, \ldots, m;$$

equipped with the measure dU being the product of the Haar measures of $\mathfrak{U}(k_j)$, $j = 1, \ldots, m$. Note that

$$\mathfrak{U}'(\mathbf{k}) \simeq \mathfrak{U}(k_1) \times \mathfrak{U}(k_2) \times \ldots \times \mathfrak{U}(k_m).$$

For each $U \in \mathfrak{U}'(\mathbf{k})$, consider the unitary operator $V_U f(w)$ on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$ defined by

$$V_U f(w) = f(Uw). \tag{2.1}$$

An operator $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$ is called **k**-*quasi-radial* if

$$S\, V_U = V_U S$$

for all $U \in \mathfrak{U}'(n)$. The **k**-*quasi-radialization* of S is defined by

$$Q\text{-}Rad(S) := \int_{\mathfrak{U}'(\mathbf{k})} V_U^* S\, V_U dU,$$

where the integral is taken in the weak sense.
Obviously, the operator $Q\text{-}Rad(S)$ is **k**-quasi-radial, and if S is a **k**-quasi-radial operator, then $Q\text{-}Rad(S) = S$.

For $a = a(z_{(1)}, \ldots, z_{(m)}) \in L_\infty(\mathbb{B}^n)$, the **k**-*quasi-radialization* of a is defined by

$$q\text{-}rad(a)(z) := \int_{\mathfrak{U}'(\mathbf{k})} a(Uz) dU.$$

Note that $q\text{-}rad(a)$ is a **k**-quasi-radial function, and for Toeplitz operators we have that

$$Q\text{-}Rad(T_a) = T_{q\text{-}rad(a)}.$$

Recall that the standard basis $\{e_\alpha : \alpha \in \mathbb{Z}_+^n\}$ of $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ is given by

$$e_\alpha(z) := \sqrt{\frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)}} z^\alpha;$$

i.e.,

$$e_{(\alpha_{(1)},\ldots,\alpha_{(m)})}(z_{(1)},\ldots,z_{(m)}) = \sqrt{\frac{\Gamma(n+|\alpha_{(1)}|+\ldots+|\alpha_{(m)}|+\lambda+1)}{\alpha_{(1)}!\cdots\alpha_{(m)}!\Gamma(n+\lambda+1)}} z_{(1)}^{\alpha_{(1)}}\cdots z_{(m)}^{\alpha_{(m)}}.$$

For the next result we need the following lemma (see [2, 3] for the proof).

Lemma 2.1. *The set of Toeplitz operators with bounded measurable symbols is dense in the algebra of all bounded operators on $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ with respect to strong operator topology.*

An important characterization of the **k**-quasi-radial operators gives

Proposition 2.2. *An operator $S \in \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ is* **k***-quasi-radial if and only if it is diagonal with respect to the standard basis $\{e_\alpha\}$ and its eigenvalue sequence $\gamma(k_1,\ldots,k_m) = \{\gamma(\alpha_{(1)},\ldots,\alpha_{(m)}) : \alpha_{(j)} \in \mathbb{Z}_+^{k_j}\}$ has the form $\gamma(\alpha_{(1)},\ldots,\alpha_{(m)}) = \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)$ for some bounded sequence $\tilde{\gamma} \in \ell_\infty(\mathbb{Z}_+^m)$, that is*

$$S e_\alpha = \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)e_{(\alpha_{(1)},\ldots,\alpha_{(m)})}.$$

Proof. Let $S \in \mathcal{L}(\mathcal{A}_\lambda^2(\mathbb{B}^n))$ be a diagonal operator with

$$S e_\alpha = \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)e_{(\alpha_{(1)},\ldots,\alpha_{(m)})}.$$

For each $(t_1,\ldots,t_m) \in \mathbb{Z}_+^m$ consider the finite dimensional subspace $H_{(t_1,\ldots,t_m)}$ of $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ defined by

$$H_{(t_1,\ldots,t_m)} = span\{e_{(\alpha_{(1)},\ldots,\alpha_{(m)})} : |\alpha_{(j)}| = t_j, j \in \{1,\ldots,m\}\},$$

then for all $f \in H_{(t_1,\ldots,t_m)}$ we have

$$S f = \tilde{\gamma}_t f = \tilde{\gamma}(t_1,\ldots,t_m)f.$$

Furthermore each subspace $H_{(t_1,\ldots,t_m)}$ is invariant under the operators V_U with $U \in \mathfrak{U}'(\mathbf{k})$ as $V_U f(z) = f(Uz) \in H_{(t_1,\ldots,t_m)}$ for each $f \in H_{(t_1,\ldots,t_m)}$.

Thus

$$(S V_U f)(z) = S(f(Uz)) = \tilde{\gamma}_t f(Uz) = V_U(\tilde{\gamma}_t f)(z) = (V_U S f)(z),$$

for all $U \in \mathfrak{U}'(\mathbf{k})$, $f \in H_{(t_1,\ldots,t_m)}$, and $t = (t_1,\ldots,t_m) \in \mathbb{Z}_+^m$, and thus S is **k**-quasi-radial.

Conversely, suppose that S is a **k**-quasi-radial operator. Using Lemma 2.1 we select a sequence $\{a_\ell\}_{\ell\in\mathbb{Z}_+} \subset \mathcal{L}_\infty(\mathbb{B}^n)$ such that

$$\lim_{\ell\to\infty} T_{a_\ell} = S$$

in SOT.

By Banach-Steinhaus theorem we know that there is $C < \infty$ such that $\|T_a\| < C$ for all Toeplitz operator T_a with bounded symbol. By this fact and the Lebesgue dominated convergence theorem we have

$$\lim_{\ell\to\infty} \int_{\mathfrak{U}'(\mathbf{k})} V_U^* T_{a_\ell} V_U dU = \int_{\mathfrak{U}'(\mathbf{k})} V_U^* S V_U dU,$$

and therefore

$$\lim_{\ell\to\infty} T_{q\text{-}rad(a_\ell)} = \lim_{\ell\to\infty} Q\text{-}Rad(T_{a_\ell}) = Q\text{-}Rad(S) = S.$$

Assume that each a_ℓ is a $\mathbf{k}$-quasi-radial function and therefore T_{a_ℓ} is a diagonal operator with $T_{a_\ell} e_{(\alpha_{(1)},\ldots,\alpha_{(m)})} = \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)^{(\ell)} e_{(\alpha_{(1)},\ldots,\alpha_{(m)})}$ for all $\alpha_{(j)} \in \mathbb{Z}_+^{k_j}$,
where $\tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)^{(\ell)}$ is the eigenvalue sequence of T_{a_ℓ}. Thus,

$$S\, e_{(\alpha_{(1)},\ldots,\alpha_{(m)})} = \lim_{\ell\to\infty} T_{a_\ell} e_{(\alpha_{(1)},\ldots,\alpha_{(m)})} = \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|) e_{(\alpha_{(1)},\ldots,\alpha_{(m)})},$$

with $\tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|) = \lim_{\ell\to\infty} \tilde{\gamma}(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)^{(\ell)}$.
That is the eigenvalue sequence of the operator S depends only on $|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|$. □

Corollary 2.3. *The set of all bounded* $\mathbf{k}$*-quasi-radial operators acting on* $\mathcal{A}_\lambda^2(\mathbb{B}^n)$ *is a* C^**−algebra which is isometrically isomorphic to* $\ell_\infty(\mathbb{Z}_+^m)$*. The isomorphism is given by the following mapping*

$$S \longmapsto \tilde{\gamma}(S),$$

where $\tilde{\gamma}(S)$ *is the eigenvalue sequence of the operator* S *of the last proposition.*

Apart of the classical Toeplitz operators, an important class of $\mathbf{k}$-quasi-radial operators is provided by Toeplitz operators whose symbols are complex finite regular $\mathbf{k}$−quasi-radial measures.

Via polar coordinates in each $\mathbb{C}^{k_j}$ the punctured ball $\mathbb{B}^n \setminus \{0\}$ can be represented as $\tau(\mathbb{B}^m) \times (S^{2k_1-1} \times S^{2k_2-1} \times \ldots \times S^{2k_m-1})$, where $\tau(\mathbb{B}^m) = \{r = (r_1,\ldots,r_m) \in \mathbb{R}_+^m : 0 \le |r| \le 1\}$ is the base of the unit ball $\mathbb{B}^m$ and $S^{2k_j-1} \subset \mathbb{C}^{k_j}$ denotes the (real) $(2k_j - 1)$−dimensional unit sphere for each $j \in \{1,\ldots,m\}$. The measure ν is said to be $\mathbf{k}$-quasi-radial if it has the form

$$\nu = \mu \otimes (\sigma_1 \otimes \ldots \otimes \sigma_m),$$

where μ is a complex finite regular Borel measure on $\tau(\mathbb{B}^m)$ and each σ_j is the standard $O(2k_j)$−invariant positive probabilistic measure on S^{2k_j-1}.

A description of the eigenvalue sequence of a Toeplitz operator with symbol being a $\mathbf{k}$-quasi-radial measure is given in the following proposition.

Proposition 2.4. *Let* T_ν *be a Toeplitz operator with symbol* $\nu = \mu \otimes (\sigma_1 \otimes \ldots \otimes \sigma_m)$ *being a* $\mathbf{k}$*-quasi-radial measure. Then* T_ν *is diagonal respect to the standard basis* $\{e_\alpha\}$ *and its eigenvalue sequence* $\gamma_{\nu,\lambda} = \{\gamma_{\nu,\lambda}(\alpha)\}_{\alpha\in\mathbb{Z}_+^n}$ *has the form*

$$\gamma_{\nu,\lambda,\mathbf{k}}(\alpha_{(1)},\ldots,\alpha_{(m)}) = \frac{\Gamma(n+|\alpha|+\lambda+1)\prod_{j=1}^m \Gamma(k_j)}{\Gamma(n+\lambda+1)\prod_{j=1}^m (k_j - 1 + |\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r). \tag{2.2}$$

In particular, $\gamma_{\nu,\lambda,\mathbf{k}}(\alpha_{(1)},\ldots,\alpha_{(m)})$ *depends only on* $|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|$*, and thus, according to Proposition 1.1,* T_ν *is a* $\mathbf{k}$*-quasi-radial operator.*

Proof. Let $w = (w_{(1)},\dots,w_{(m)}) \in \mathbb{B}^n$, where each $w_{(j)} = r_j\zeta_j$ with $r_j \in [0,1)$ and $\zeta \in S^{2k_j-1}$. Then,

$$
\begin{aligned}
T_\nu z^\alpha &= \int_{\mathbb{B}^n} \frac{w^\alpha d\nu(w)}{(1-\langle z,w\rangle)^{n+\lambda+1}} = \int_{\mathbb{B}^n} \sum_{|\beta|=0}^{\infty} \frac{\Gamma(n+|\beta|+\lambda+1)}{\beta!\Gamma(n+\lambda+1)} z^\beta \overline{w}^\beta w^\alpha d\nu(w) \\
&= \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} z^\alpha \int_{\mathbb{B}^n} \overline{w}_{(1)}^{\alpha_{(1)}} w_{(1)}^{\alpha_{(1)}} \cdots \overline{w}_{(m)}^{\alpha_{(m)}} w_{(m)}^{\alpha_{(m)}} d\nu(w)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} z^\alpha \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r_1,\dots,r_m) \prod_{j=1}^m \int_{S^{2k_j-1}} \zeta_j^{\alpha_{(j)}} \overline{\zeta}_j^{\alpha_{(j)}} d\sigma_j(\zeta_j) \\
&= \frac{\Gamma(n+|\alpha|+\lambda+1)}{\alpha!\Gamma(n+\lambda+1)} z^\alpha \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r) \prod_{j=1}^m \frac{(k_j-1)!\alpha_{(j)}!}{(k_j-1+|\alpha_{(j)}|)!} \\
&= \frac{\Gamma(n+|\alpha|+\lambda+1)\prod_{j=1}^m \Gamma(k_j)}{\Gamma(n+\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r)\cdot z^\alpha.
\end{aligned}
$$

Therefore,

$$
\gamma_{\nu,\lambda,\mathbf{k}}(\alpha_{(1)},\dots,\alpha_{(m)}) = \frac{\Gamma(n+|\alpha|+\lambda+1)\prod_{j=1}^m \Gamma(k_j)}{\Gamma(n+\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r). \qquad \square
$$

Any Toeplitz operator T_a with symbol being a bounded measurable $\mathbf{k}$-quasi-radial function $a = a(r_1,\dots,r_m)$ can be considered as a Toeplitz operator $T_{\nu_{a,\lambda,\mathbf{k}}}$ whose symbol is the following $\mathbf{k}$-quasi-radial measure

$$
\begin{aligned}
d\nu_{a,\lambda,\mathbf{k}} &= d\mu \otimes d(\sigma_1 \otimes \ldots \otimes \sigma_m) \\
&= \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^m \Gamma(k_j)} 2^m a(r_1,\dots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{2k_j-1} dr \otimes d(\sigma_1 \otimes \ldots \otimes \sigma_m). \qquad (2.3)
\end{aligned}
$$

Note that substitution of this measure into (2.2) returns the known result for the eigenvalues of T_a:

$$
\begin{aligned}
\gamma_{a,\lambda,\mathbf{k}}^{(n)}(|\alpha_{(1)}|,\dots,|\alpha_{(m)}|) &= \frac{\Gamma(n+|\alpha|+\lambda+1)\prod_{j=1}^m \Gamma(k_j)}{\Gamma(n+\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} d\mu(r) \\
&= \frac{\Gamma(n+|\alpha|+\lambda+1)\prod_{j=1}^m \Gamma(k_j)}{\Gamma(n+\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \\
&\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2|\alpha_{(j)}|} \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^m \Gamma(k_j)} 2^m a(r_1,\dots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{2k_j-1} dr \\
&= \frac{2^m\Gamma(n+|\alpha|+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^m (k_j-1+|\alpha_{(j)}|)!} \int_{\tau(\mathbb{B}^m)} a(r_1,\dots,r_m)(1-|r|^2)^\lambda \prod_{j=1}^m r_j^{2|\alpha_{(j)}|+2k_j-1} dr.
\end{aligned}
$$

3 (p,λ)–Berezin transform of k-quasi-radial operators

Recall [4] that the (p,λ)-Berezin transform of an operator S, acting on $\mathcal{A}^2_\lambda(\mathbb{B}^n)$, is defined by

$$(B_{p,\lambda}S)(z) = \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\langle S_z w^\alpha, w^\alpha\rangle_\lambda;$$

where $C_{p,\lambda} := \begin{pmatrix} p \\ |\alpha| \end{pmatrix}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha_1!\ldots\alpha_n!}$, such that $\sum_{|\alpha|=0}^{p} C_{p,\alpha}z^\alpha\overline{w}^\alpha = (1-\langle z,w\rangle)^p$, $S_z := \mathcal{U}_z S\,\mathcal{U}_z$, $\mathcal{U}_z$ is a self-adjoint and unitary operator defined by

$$(\mathcal{U}_z f)(w) = \frac{(1-|z|^2)^{\frac{n+\lambda+1}{2}}}{(1-\langle w,z\rangle)^{n+\lambda+1}}(f\circ\phi_z)(w),$$

and ϕ_z denotes the standard biholomorphism of $\mathbb{B}^n$ that interchanges the points 0 and z.

Let $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$, we have

$$q\text{-}rad\circ B_{p,\lambda}S(z) = \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}'(\mathbf{k})}\langle S_{Uz}w^\alpha, w^\alpha\rangle_\lambda dU, \tag{3.1}$$

where $S_{Uz} = \mathcal{U}_{Uz}S\,\mathcal{U}_{Uz}$ for all $U \in \mathfrak{U}'(\mathbf{k})$ and for $f \in L_2(\mathbb{B}^n, dv_\lambda)$ we have

$$\begin{aligned}(\mathcal{U}_{Uz}f)(w) &= \frac{(1-|Uz|^2)^{\frac{n+\lambda+1}{2}}}{(1-\langle w,Uz\rangle)^{n+\lambda+1}} f\circ\phi_{Uz}(w)\\ &= \frac{(1-|z|^2)^{\frac{n+\lambda+1}{2}}}{(1-\langle U^*w,z\rangle)^{n+\lambda+1}} f\circ U\circ\phi_z\circ U^*(w)\\ &= (V_{U^*}\circ\mathcal{U}_z\circ V_U f)(w),\end{aligned}$$

where the operator V_U is defined by (2.1).

Therefore, $S_{Uz} = V_{U^*}\circ\mathcal{U}_z\circ V_U\circ S\circ V_{U^*}\circ\mathcal{U}_z\circ V_U$, and from (3.1) we have

$$\begin{aligned}q\text{-}rad\circ B_{p,\lambda}(S)(z) &= \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}(k_1)\times\ldots\times\mathfrak{U}(k_m)}\langle S_{Uz}w^\alpha, w^\alpha\rangle_\lambda dU\\ &= \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}(k_1)\times\ldots\times\mathfrak{U}(k_m)}\langle V_{U^*}\circ\mathcal{U}_z\circ V_U\circ S\circ V_{U^*}\circ\mathcal{U}_z\circ V_U(w^\alpha), w^\alpha\rangle_\lambda dU\\ &= \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}(k_1)\times\ldots\times\mathfrak{U}(k_m)}\langle (V_U\circ S\circ V_{U*})_z V_U(w^\alpha), V_U(w^\alpha)\rangle_\lambda dU\\ &= \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}(k_1)\times\ldots\times\mathfrak{U}(k_m)}\langle (V_U\circ S\circ V_{U*})_z (Uw)^\alpha, (Uw)^\alpha\rangle_\lambda dU\\ &= \frac{c_{\lambda+p}}{c_\lambda}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\int_{\mathfrak{U}(k_1)\times\ldots\times\mathfrak{U}(k_m)}\langle (V_U\circ S\circ V_{U*})_z w^\alpha, w^\alpha\rangle_\lambda dU\\ &= B_{p,\lambda}\circ Q\text{-}Rad(S)(z).\end{aligned}$$

Lemma 3.1. *The* **k***-quasi-radialization "commutes" with the* $(p,\lambda)-$*Berezin transform for all* $\lambda > -1$ *and* $p \in \mathbb{Z}_+$*; i.e.,*

$$q\text{-}rad \circ B_{p,\lambda}(S) = B_{p,\lambda} \circ Q\text{-}Rad(S),$$

for $S \in \mathcal{L}(\mathcal{A}^2_\lambda(\mathbb{B}^n))$*. In particular,* S *is a* **k***-quasi-radial operator if and only if* $B_{p,\lambda}(S)$ *is a* **k***-quasi-radial function.*

Proof. If S is a **k**-quasi-radial operator, then

$$q\text{-}rad \circ B_{p,\lambda}(S) = B_{p,\lambda} \circ Q\text{-}Rad(S) = B_{p,\lambda}(S).$$

So, $B_{p,\lambda}(S)$ s a **k**-quasi-radial function.

On the other hand, if $B_{p,\lambda}(S)$ is a **k**-quasi-radial function

$$B_{p,\lambda} \circ Q\text{-}Rad(S) = q\text{-}rad \circ B_{p,\lambda}(S) = B_{p,\lambda}(S);$$

since $B_{p,\lambda}$ is one-to-one on bounded operators we can conclude that $Q\text{-}Rad(S) = S$. □

For the next proposition we need the following formula.

Lemma 3.2. *Let* $\alpha, \beta \in \mathbb{Z}^n_+$*, then*

$$\sum_{|\alpha|=j} \frac{(\alpha+\beta)!}{\alpha!\beta!} = \binom{n+j+|\beta|-1}{j}.$$

Proof. Lemma 3.10 in [3]. □

The next proposition provides the expression of the (p,λ)-Berezin transform of a **k**-quasi-radial operator in terms of its eigenvalue sequence.

Proposition 3.3. *Let* S *be a* **k***-quasi-radial operator with the eigenvalue sequence* $\{\gamma(|\alpha_{(1)}|,\dots,|\alpha_{(m)}|) : (\alpha_{(1)},\dots,\alpha_{(m)}) \in \mathbb{Z}^{k_1}_+ \times \cdots \times \mathbb{Z}^{k_m}_+\}$*. Then its* (p,λ)*-Berezin transform is given by*

$$\begin{aligned}
(B_{p,\lambda}S)(z) &= 2^m \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1} \sum_{\sum_{j=1}^m t_j=0}^{p} (-1)^{\sum_{j=1}^m t_j} \frac{p!}{(p-\sum_{j=1}^m t_j)!} \\
&\times \sum_{\sum_{j=1}^m q_j=0}^{\infty} \left[\frac{\Gamma(n+\sum_{j=1}^m q_j+p+\lambda+1)}{\Gamma(n+p+\lambda+1)}\right]^2 \left[\frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^m (k_j-1+t_j+q_j)!}\right] \qquad (3.2) \\
&\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^m r_j^{2(t_j+q_j)+2k_j-1}(1-r^2)^\lambda dr \binom{n+\sum_{j=1}^m t_j+\sum_{j=1}^m q_j-1}{\sum_{j=1}^m t_j} \gamma(t_1+q_1,\dots,t_m+q_m) \frac{|z|^{2\sum_{j=1}^m q_j}}{\left(\sum_{j=1}^m q_j\right)!}.
\end{aligned}$$

Proof. Note that

$$\langle S(w^\alpha K_z^{p+\lambda}), w^\alpha K_z^{p+\lambda}\rangle_\lambda = \sum_{|\beta|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\beta!\Gamma(n+p+\lambda+1)}\right]^2 |z^\beta|^2 \langle S w^{\alpha+\beta}, w^{\alpha+\beta}\rangle_\lambda$$

$$= \sum_{|\beta_{(1)}|+\ldots+|\beta_{(m)}|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\beta!\Gamma(n+p+\lambda+1)}\right]^2 |z^\beta|^2 \gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|)$$

$$\times \int_{\mathbb{B}^n} w^{\alpha+\beta}\overline{w}^{\alpha+\beta} dv_\lambda(w)$$

$$= \sum_{|\beta_{(1)}|+\ldots+|\beta_{(m)}|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\beta!\Gamma(n+p+\lambda+1)}\right]^2 |z^\beta|^2$$

$$\times \gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|) c_\lambda \prod_{j=1}^{m}\int_{\mathbb{S}^{k_j}} \zeta^{\alpha_{(j)}+\beta_{(j)}}\overline{\zeta}^{\alpha_{(j)}+\beta_{(j)}} d\zeta$$

$$\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr$$

$$= \sum_{|\beta_{(1)}|+\ldots+|\beta_{(m)}|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\beta!\Gamma(n+p+\lambda+1)}\right]^2 \frac{2^m(\alpha+\beta)!}{\prod_{j=1}^{m}(k_j-1+|\alpha_{(j)}|+|\beta_{(j)}|)}$$

$$\times \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)}\gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|)$$

$$\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr |z^\beta|^2.$$

Thus we have

$$(B_{p,\lambda}S)(z) = \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1}\sum_{|\alpha|=0}^{p} C_{p,\alpha}\langle S(w^\alpha K_z^{p+\lambda}), w^\alpha K_z^{p+\lambda}\rangle_\lambda$$

$$= \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1}\sum_{|\alpha_{(1)}|+\ldots+|\alpha_{(m)}|=0}^{p}\binom{p}{|\alpha|}(-1)^{|\alpha|}\frac{|\alpha|!}{\alpha!}$$

$$\times \sum_{|\beta_{(1)}|+\ldots+|\beta_{(m)}|=0}^{\infty}\left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\beta!\Gamma(n+p+\lambda+1)}\right]^2 \frac{2^m(\alpha+\beta)!}{\prod_{j=1}^{m}(k_j-1+|\alpha_{(j)}|+|\beta_{(j)}|)}$$

$$\times \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)}\gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|)$$

$$\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr |z^\beta|^2$$

$$
\begin{aligned}
&= 2^m \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1} \sum_{|\alpha_{(1)}|+\ldots+|\alpha_{(m)}|=0}^{p} \binom{p}{|\alpha|} (-1)^{|\alpha|} \frac{|\alpha|!}{\alpha!} \\
&\times \sum_{|\beta_{(1)}|+\ldots+|\beta_{(m)}|=0}^{\infty} \left[\frac{\Gamma(n+|\beta|+p+\lambda+1)}{\Gamma(n+p+\lambda+1)}\right]^2 \frac{1}{\prod_{j=1}^{m}(k_j-1+|\alpha_{(j)}|+|\beta_{(j)}|)} \\
&\times \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)} \gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|) \\
&\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr \frac{(\alpha+\beta)!}{\beta!^2}|z^\beta|^2 dr \\
&= 2^m \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1} \sum_{t_1+\ldots+t_m=0}^{p} \binom{p}{t_1+\ldots+t_m} (-1)^{t_1+\ldots+t_m} \\
&\times (t_1+\ldots+t_m)! \sum_{q_1+\ldots+q_m=0}^{\infty} \left[\frac{\Gamma(n+\sum q_j+p+\lambda+1)}{\Gamma(n+p+\lambda+1)}\right]^2 \\
&\times \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^{m}(k_j-1+t_j+q_j)} \frac{\gamma(|\alpha_{(1)}|+|\beta_{(1)}|,\ldots,|\alpha_{(m)}|+|\beta_{(m)}|)}{(q_1+\ldots+q_m)!} \\
&\times \sum_{|\beta|=q_1+\ldots+q_m} \frac{(q_1+\ldots+q_m)!}{|\beta|!}|z^\beta|^2 \sum_{|\alpha|=t_1+\ldots+t_m} \frac{(\alpha+\beta)!}{\alpha!\beta!} \\
&\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr.
\end{aligned}
$$

By the multinomial theorem and the last lemma we conclude that

$$
\begin{aligned}
(B_{p,\lambda}S)(z) &= 2^m \frac{c_{\lambda+p}}{c_\lambda}(1-|z|^2)^{p+\lambda+n+1} \sum_{\sum t_j=0}^{p} (-1)^{\sum t_j} \frac{p!}{(p-\sum t_j)!} \\
&\times \sum_{\sum q_j=0}^{\infty} \left[\frac{\Gamma(n+\sum q_j+p+\lambda+1)}{\Gamma(n+p+\lambda+1)}\right]^2 \frac{\Gamma(n+\lambda+1)}{\Gamma(\lambda+1)\prod_{j=1}^{m}(k_j-1+t_j+q_j)} \\
&\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2(|\alpha_{(j)}|+|\beta_{(j)}|)+2k_j-1}(1-r^2)^\lambda dr \binom{n+\sum_{j=1}^m t_j+\sum_{j=1}^m q_j-1}{\sum_{j=1}^m t_j} \\
&\times \gamma(t_1+q_1,\ldots,t_m+q_m) \frac{|z|^{2\sum_{j=1}^m q_j}}{\left(\sum_{j=1}^m q_j\right)!}.
\end{aligned}
$$

□

Corollary 3.4. *Let S be a $\mathbf{k}-$quasi-radial operator with the eigenvalue sequence*

$$\{\gamma(|\alpha_{(1)}|,\ldots,|\alpha_{(m)}|)\}_{(\alpha_{(1)},\ldots,\alpha_{(m)})\in\mathbb{Z}_+^m}.$$

Then its Berezin transform $B_\lambda(S) := B_{0,\lambda}(S)$ *is a quasi-radial function and is given by*

$$(B_\lambda S)(z) = 2^m(1-|z|^2)^{\lambda+n+1} \sum_{\sum q_j=0}^{\infty} \frac{\Gamma(n+\sum q_j+\lambda+1)^2}{\Gamma(\lambda+1)\Gamma(n+\lambda+1)\prod_{j=1}^{m}(k_j-1+q_j)!}$$
$$\times \int_{\tau(\mathbb{B}^m)} \prod_{j=1}^{m} r_j^{2q_j+2k_j-1}(1-r^2)^\lambda \gamma(q_1,\ldots,q_m) \frac{|z|^{2\sum q_j}}{\left(\sum q_j\right)!} dr.$$

Acknowledgments

The work was partially supported by PROMEP (Mexico) via "Proyecto de Redes" and by CONACYT Project 180049.

References

[1] W. Bauer, N. Vasilevski, On the structure of commutative Banach algebras generated by Toeplitz operators on unit ball. Quasi-elliptic case. I: Generating subalgebras. *J. Functional Analysis*, **265** (2013), pp 2956-2990.

[2] M. Engliš, Density of algebras generated by Toeplitz operators on Bergman spaces, *Ark. Mat.*, **30** no. 2 (1992), pp 227-243.

[3] W. Bauer, C. Herrera, and N. Vasilevski, Eigenvalue characterization of radial operators on weighted Bergman spaces over the unit ball. *Integr. Equ. Oper. Theory*, **78**, no. 2 (2014), pp 271-300.

[4] W. Bauer, C. Herrera, and N. Vasilevski, (m, λ)-Berezin transform and approximation of operators on weighted Bergman spaces over the unit ball. In: *Operator Theory in Harmonic and Non-commutative Analysis*, Operator Theory: Advances and Applications **240** (2014), pp 45-68.

[5] M. Mitkovski, D. Suárez, and B. D. Wick, The essential norm of operators on $A^p_\alpha(\mathbb{B}_n)$. *Integr. Equ. Oper. Theory*, **75**, no. 2 (2013), pp 197-233.

[6] K. Nam, D. Zheng, and C. Zhong, m-Berezin transform and compact operators. *Rev. Mat. Iberoamericana*, **22**(3) (2000), pp 867-892.

[7] D. Suárez, Approximation and symbolic calculus for Toeplitz algebras on the Bergman space. *Rev. Mat. Iberoamericana*, **20**(2) (2004), pp 563-610.

[8] N. Vasilevski, Quasi-Radial Quasi-Homogeneous Symbols and Commutative Banach Algebras of Toeplitz Operators. *Integr. Equ. Oper. Theory*, **66**, no. 1 (2010), pp 141-152.

[9] K. Zhu, *Spaces of holomorphic functions in the unit ball*. Springer Verlag 2005.