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C^*-algebra of Angular Toeplitz Operators on Bergman Spaces over the Upper Half-plane

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Abstract

We consider the C*-algebra generated by Toeplitz operators acting on the Bergman space over the upper half-plane whose symbols depend only on the argument of the variable. This algebra is known to be commutative, and it is isometrically isomorphic to a certain algebra of bounded complex-valued functions on the real numbers. In the paper we prove that the latter algebra consists of all bounded functions f that are very slowly oscillating on the real line in the sense that the composition of f with sinh is uniformly continuous with respect to the usual metric.

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1 Introduction

Let $\Pi = \left\{z = re^{i\theta}:\ r > 0,\ \theta \in (0,\pi)\right\}$ be the upper half-plane and let $d\mu = r\,dr\,d\theta$ be the Lebesgue measure on Π. Recall that the Bergman space $\mathcal{A}^2(\Pi)$ is a reproducing kernel Hilbert subspace of $L_2(\Pi, d\mu)$ which consists of all square integrable analytic functions on Π. The reproducing kernel of this space has the form

$$K_w(z) = -\frac{1}{\pi(\overline{w}-z)^2}, \tag{1.1}$$

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and the orthogonal projection of $L_2(\Pi, d\mu)$ onto $\mathcal{A}^2(\Pi)$ is given by $(Pf)(w) = \langle f, K_w \rangle$.

The Toeplitz operator $T_g\colon \mathcal{A}^2(\Pi) \to \mathcal{A}^2(\Pi)$ with defining symbol $g \in L_\infty(\Pi)$ is given by $T_g f = P(gf)$. Korenblum, Zhu, Grudsky, Karapetyants, Quiroga-Barranco and Vasilevski [15, 7, 8, 9, 17, 18, 19] have found various families of symbols which generate commutative C^*-algebras of Toeplitz operators (see the summarizing book by Vasilevski [27]). These families of defining symbols may be reduced to three model cases: *radial* symbols, functions on the unit disk depending only on $|z|$, *vertical* symbols, functions on the upper half-plane depending on $\operatorname{Im} z$, and *angular* symbols defined on the upper half-plane and depending only on $\theta = \arg z$. In addition, Huang [14] showed that these commutative algebras are maximal.

Suárez [22, 23] made the principal step in the characterization of the commutative algebra generated by Toeplitz operators with bounded radial symbols. His results are complemented and generalized in [1, 2, 10, 16]. Recently, Herrera Yañez, Hutník, Maximenko, and Vasilevski [11, 12] have given a description of the commutative C^*-algebra generated by Toeplitz operators with vertical symbols and show that it is isommetrically isomorphic to the C^*-algebra of functions uniformly continuous on the positive half-line with respect to the *logarithmic metric* $|\ln x - \ln y|$.

This paper is devoted to the studying of the third (angular) case. A function $a \in L_\infty(\Pi)$ is said to be *angular* if there exists $b \in L_\infty(0,\pi)$ such that $a(z) = b(\arg z)$ for almost every $z \in \Pi$. Denote by $\mathcal{A}_\infty$ the C^*-subalgebra of $L_\infty(\Pi)$ that consists of the bounded angular functions. Introduce the operator algebra $\mathcal{T}(\mathcal{A}_\infty)$ generated by all Toeplitz operators T_a acting on the Bergman space $\mathcal{A}^2(\Pi)$ with defining symbols $a \in \mathcal{A}_\infty$.

The main tool to study $\mathcal{T}(\mathcal{A}_\infty)$ is an isometric isomorphism of $\mathcal{A}^2(\Pi)$ onto $L_2(\mathbb{R})$ constructed by Vasilevski [27, section 7.1]:

$$(R\varphi)(x) = \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2x}{1-e^{-2x\pi}}} \int_\Pi (\overline{z})^{-ix-1} \varphi(z)\, d\mu(z), \qquad x \in \mathbb{R}. \tag{1.2}$$

This isomorphism reduces every Toeplitz operator T_a with angular symbol a to the multiplication operator by γ_a acting on $L_2(\mathbb{R})$, where the function $\gamma_a\colon \mathbb{R} \to \mathbb{C}$ is given by

$$\gamma_a(x) = \frac{2x}{1-e^{-2x\pi}} \int_0^\pi a(\theta)\, e^{-2x\theta}\, d\theta, \qquad \lambda \in \mathbb{R}. \tag{1.3}$$

More precisely, if $a \in \mathcal{A}_\infty$, then $RT_aR^* = \gamma_a I$. In particular, this implies that the C^*-algebra $\mathcal{T}(\mathcal{A}_\infty)$ is commutative and isometrically isomorphic to the C^*-algebra generated by the set

$$\mathfrak{G} = \{\gamma_a : a \in L_\infty(0,\pi)\}. \tag{1.4}$$

Hence, a natural task to be done here is to obtain an explicit and intrinsic characterization of the commutative C^*-subalgebra of $L_\infty(\mathbb{R})$ generated by the set $\mathfrak{G}$ and thus describe the operator algebra $\mathcal{T}(\mathcal{A}_\infty)$.

The main result of this paper states that the C^*-algebra generated by $\mathfrak{G}$ coincides with the C^*-algebra $\mathrm{VSO}(\mathbb{R})$ of bounded *very slowly oscillating functions* on $\mathbb{R}$, i.e. the functions that are uniformly continuous with respect to the "arcsinh-metric" $\rho(x,y) = |\operatorname{arcsinh} x - \operatorname{arcsinh} y|$. As a consequence, the operator algebra $\mathcal{T}(\mathcal{A}_\infty)$ is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R})$. We also prove that $\mathcal{T}(\mathcal{A}_\infty)$ is dense in the C^*-algebra of angular operators with respect to the strong operator topology in $\mathcal{B}(\mathcal{A}^2(\Pi))$.

The paper is organized as follows. In Section 2 we define angular operators and give various equivalent characterizations of these operators. In Section 3 we give a criterion of angular Toeplitz operators and show that the closure in the strong operator topology of the algebra angular Toeplitz operators coincides with the C^*-algebra of angular operators. In Sections 4 we introduce the C^*-algebra VSO($\mathbb{R}$) and show that for every angular symbol a the function γ_a belongs to VSO($\mathbb{R}$). In Section 5 prove the main result, i.e. the density in VSO($\mathbb{R}$) of the algebra generated by the functions γ_a.

2 Angular operators

Let $\mathcal{B}(\mathcal{A}^2(\Pi))$ be the algebra of all linear bounded operators acting on the Bergman space $\mathcal{A}^2(\Pi)$. Given $h \in \mathbb{R}_+$, let $D_h \in \mathcal{B}(\mathcal{A}^2(\Pi))$ be the *dilation* operator defined by

$$D_h f(z) = h f(hz), \quad z \in \Pi. \tag{2.1}$$

An operator $V \in \mathcal{B}(\mathcal{A}^2(\Pi))$ is said to be *angular* or *invariant under dilations* if it commutes with all dilation operators. The set of all angular operators will be denoted by $\mathfrak{A}$:

$$\mathfrak{A} := \left\{ V \in \mathcal{B}(\mathcal{A}^2(\Pi)) : \ \forall h \in \mathbb{R}_+, \quad D_h V = V D_h \right\}. \tag{2.2}$$

Example 2.1. Let $h > 0$. We shall prove that the dilation operator D_h is diagonalized by R. Given $\varphi \in \mathcal{A}^2(\Pi)$ and $x \in \mathbb{R}$, by the equation (1.2) we have

$$(RD_h\varphi)(x) = \frac{h^{ix}}{\sqrt{2\pi}} \sqrt{\frac{2x}{1-e^{-2x\pi}}} \int_\Pi (\overline{z})^{-ix-1} \varphi(z)\, d\mu(z) = E_h(x)(R\varphi)(x).$$

Thus

$$RD_hR^* = M_{E_h}. \tag{2.3}$$

Here and in what follows, we denote by E_h the function $\mathbb{R} \to \mathbb{C}$ defined by $E_h(x) = h^{ix}$, and M_{E_h} stays for the multiplication operator by E_h.

The *Berezin transform* [3, 4] of an operator $V \in \mathcal{B}(\mathcal{A}^2(\Pi))$ is the function $\widetilde{V} \colon \Pi \to \mathbb{C}$ defined by

$$\widetilde{V}(z) := \frac{\langle VK_z, K_z\rangle}{\langle K_z, K_z\rangle}. \tag{2.4}$$

The next theorem gives a criterion for a bounded linear operator acting in $\mathcal{A}^2(\Pi)$ to be angular. It is analogous to the corresponding criteria of radial and vertical operators [12, 28].

Theorem 2.2 (**Criterion of angular operators).**
Assume that $V \in \mathcal{B}(\mathcal{A}^2(\Pi))$. The following conditions are equivalent:

i). $V \in \mathfrak{A}$.

ii). $RVR^*M_{E_h} = M_{E_h}RVR^*$ *for all* $h \in \mathbb{R}_+$.

iii). *There exists* ϕ *in* $L_\infty(\mathbb{R})$ *such that* $V = R^*M_\phi R$.

iv). *The Berezin transform* $\widetilde{V}$ *depends on* $\beta = \arg z$ *only.*

Proof. *i*). $\longrightarrow$ *ii*). The proof follows from (2.3).

ii). $\longrightarrow$ *iii*). It follows from the well known characterization of the translation invariant operators (see, for example, [13, Theorem 2.5.10]); see also [12, Lemma 2.1]).

iii). $\longrightarrow$ *iv*). Given $w \in \Pi$, with $w = \rho e^{i\beta}$, consider the Berezin transform of V at the point w:

$$\widetilde{V}(w) = \frac{\langle VK_w, K_w\rangle_{\mathcal{A}^2(\Pi)}}{\langle K_w, K_w\rangle} = \frac{\langle M_\phi RK_w, RK_w\rangle_{L_2(\mathbb{R})}}{\langle K_w, K_w\rangle} = (2\operatorname{Im} w)^2 \int_{\mathbb{R}} \phi(x)\,|RK_w(x)|^2\,dx.$$

A direct computation shows that

$$(RK_w)(x) = \frac{\pi e^{-2x\pi} e^{(i-x)\beta}}{2\rho^{1+ix}} \sqrt{\frac{2x}{1-e^{-2x\pi}}}, \quad x \in \mathbb{R}.$$

Combining the last two formulas we see that $\widetilde{V}(w)$ depends on $\beta = \arg w$ only:

$$\widetilde{V}(w) = 4\pi^2 \sin^2\beta \int_{\mathbb{R}} \phi(x)\, e^{-2x(\beta+2\pi)} \frac{2x}{1-e^{-2x\pi}}\,dx,$$

iv). $\longrightarrow$ *i*). Given $z, w \in \Pi$, and $h \in \mathbb{R}_+$, by (1.1) and (2.1)

$$(D_h K_w)(z) = hK_w(hz) = -\frac{h}{\pi}(\overline{w} - hz)^{-2} = -\frac{1}{h\pi}\left(\frac{\overline{w}}{h} - z\right)^{-2} = \frac{1}{h} K_{\frac{w}{h}}(z).$$

Using this formula we calculate the Berezin transform of the operator $D_{h^{-1}} V D_h$:

$$\widetilde{D_{h^{-1}} V D_h}(w) = \frac{\langle VD_hK_w, D_hK_w\rangle}{\langle D_hK_w, D_hK_w\rangle} = \frac{\langle VK_{\frac{w}{h}}, K_{\frac{w}{h}}\rangle}{\langle K_{\frac{w}{h}}, K_{\frac{w}{h}}\rangle} = \widetilde{V}\left(\frac{w}{h}\right) = \widetilde{V}(w).$$

Since the Berezin transform is injective [21], $D_{h^{-1}} V D_h = V$. □

Corollary 2.3. *The set $\mathfrak{A}$ of all angular operators on $\mathcal{B}(\mathcal{A}^2(\Pi))$ is a commutative C^*-algebra which is isometrically isomorphic to $L_\infty(\mathbb{R})$.*

3 Criterion for a Toeplitz operator to be angular

The next proposition gives a criterion for a Toeplitz operator on the Bergman space $\mathcal{A}^2(\Pi)$ to be angular. In what follows we denote by $\mu_{\mathbb{R}^2}$ the Lebesgue measure in $\mathbb{R}^2$, and $d\mu_{\mathbb{R}^2_+}(x,y) = xydxdy$.

Proposition 3.1. *Let $a \in L_\infty(\Pi)$. The Toeplitz operator T_a is angular if and only if a is angular.*

Proof. Sufficiency. If a is angular, there exists $b \in L_\infty(0,\pi)$ such that $a(z) = b(\arg z)$ for almost every $z \in \Pi$, then for all $h \in \mathbb{R}_+$ we get $a(hz) = b(\arg(hz)) = b(\arg z) = a(z)$ almost all $z \in \Pi$. Hence, the Toeplitz operator T_a acting on $\mathcal{A}^2(\Pi)$ is unitary equivalent to the multiplication operator $\gamma_a I = RT_aR^*$ acting on $L_2(\mathbb{R})$ (see [27, Section 7.2]), in this way T_a is angular by Theorem 2.2.

Necessity. Suppose that $T_a \in \mathfrak{A}$. Hence, for each $h > 0$ one gets that $T_a = D_h T_a D_{h^{-1}} = T_{a_h}$, where $a_h(w) = a(hw)$. Equivalently, $T_{a-a_h} = 0$ for all $h > 0$. This implies that for all $h \in \mathbb{R}_+$ $a(z) = a_h(z) = a(hz)$ a. e. $z \in \Pi$ (see [12, Section 3, Lemma 3.1]), that is,

$$\mu_\Pi(\Delta_h) = 0, \quad \text{where} \quad \Delta_h = \left\{(x,\theta) \in \mathbb{R}_+ \times (0,\pi) : a(xe^{i\theta}) \neq a(hxe^{i\theta})\right\}. \tag{3.1}$$

Define $\Phi : \mathbb{R}_+^2 \times (0,\pi) \to \mathbb{C}$ by

$$\Phi(x,y,\theta) = \begin{cases} 0, & \text{if } a(xe^{i\theta}) = a(ye^{i\theta}); \\ 1, & \text{if } a(xe^{i\theta}) \neq a(ye^{i\theta}), \end{cases}$$

and note that for all $h \in \mathbb{R}_+$

$$\{(x,\theta) \in \Pi : \Phi(x,hx,\theta) \neq 0\} = \left\{(x,\theta) \in \mathbb{R}_+ \times (0,\pi) : a(xe^{i\theta}) \neq a(hxe^{i\theta})\right\} = \Delta_h. \tag{3.2}$$

Accordingly, by (3.1) for all $h \in \mathbb{R}_+$ we get $\Phi(x,hx,\theta) = 0$ a. e. $(x,\theta) \in \Pi$, and by Tonelli's theorem

$$\begin{aligned} \int_{\mathbb{R}_+^2 \times (0,\pi)} \Phi(x,y,\theta) xy \, d\theta \, dx \, dy &\overset{y=hx}{=} \int_{\mathbb{R}_+^2 \times (0,\pi)} \Phi(x,hx,\theta) x^3 h \, d\theta \, dx \, dh \\ &= \int_{\mathbb{R}_+} h \left(\int_\Pi \Phi(x,hx,\theta) x^2 d\mu_\Pi(x,\theta) \right) dh = 0. \end{aligned}$$

Therefore, for almost $\theta \in (0,\pi)$

$$0 = \mu_{\mathbb{R}_+^2}\left(\left\{(x,y) \in \mathbb{R}_+^2 : a(xe^{i\theta}) \neq a(ye^{i\theta})\right\}\right) = \int_{\mathbb{R}_+^2} \Phi(x,y,\theta) xy \, dx \, dy = \int_{\mathbb{R}^2} \Phi(e^t, e^u, \theta) e^{2t} e^{2u} \, dt \, du.$$

It follows that

$$0 = \mu_{\mathbb{R}^2}\left(\left\{(t,u) \in \mathbb{R}^2 : \Phi(e^t, e^u, \theta) \neq 0\right\}\right) = \mu_{\mathbb{R}^2}\left(\left\{(t,u) \in \mathbb{R}^2 : a \circ \exp(t+i\theta) \neq a \circ \exp(u+i\theta)\right\}\right)$$

a. e. $\theta \in (0,\pi)$. Now, there exists a constant $c(\theta)$ such that $a \circ \exp(t+i\theta) = c(\theta)$ for almost $t \in \mathbb{R}$ (see [12, Section 3, Lemma 3.2]), for this reason the bounded function $b : (0,\pi) \to \mathbb{C}$ given by

$$b(\theta) = \begin{cases} c(\theta), & \text{if } \mu_{\mathbb{R}^2}\left(\{(t,u) \in \mathbb{R}^2 : a \circ \exp(t+i\theta) \neq a \circ \exp(u+i\theta)\}\right) = 0; \\ 0, & \text{otherwise}, \end{cases}$$

satisfies the equality $a(z) = b(\arg z)$ for almost all $z \in \Pi$, which means that a is angular. □

Proposition 3.2. *The C^*-algebra $\mathcal{T}(\mathcal{A}_\infty)$ generated by angular Toeplitz operators is dense in the algebra of bounded angular operators $\mathfrak{A}$ with respect to the strong-operator topology in $\mathcal{B}(\mathcal{A}^2(\Pi))$, i.e.*

$$\text{SOT-closure}(\mathcal{T}(\mathcal{A}_\infty)) = \mathfrak{A}. \tag{3.3}$$

Proof. By Theorem 2.2 $\mathfrak{A}$ is isometrically isomorphic to $\{M_f : f \in L_\infty(\mathbb{R})\}$, which is a maximal abelian von Neumann algebra in $\mathcal{B}(L_2(\mathbb{R}))$ (see, for example, [5, Theorem 1.45 and Proposition 12.4]). Thus $\mathfrak{A}$ is maximal abelian von Neumann algebra in $\mathcal{B}(\mathcal{A}^2(\Pi))$. However, the von Neumann algebra $W^*(\mathcal{T}(\mathcal{A}_\infty))$ generated by the Toeplitz algebra $\mathcal{T}(\mathcal{A}_\infty)$ is abelian maximal, and $\mathcal{T}(\mathcal{A}_\infty)$ is dense in $W^*(\mathcal{T}(\mathcal{A}_\infty))$ with respect to the strong operator topology in $\mathcal{B}(\mathcal{A}^2(\Pi))$ (see [14]). Therefore $\mathfrak{A} = \text{SOT-closure}(\mathcal{T}(\mathcal{A}_\infty))$. □

4 VSO-property of the functions γ_a

In this section we formally introduce the C*-algebra VSO($\mathbb{R}$) and prove that $\mathfrak{G}$ is a proper subset of VSO($\mathbb{R}$). We denote by ρ the "arcsinh-metric" on the real line:

$$\rho(x,y) = |\operatorname{arcsinh}(x) - \operatorname{arcsinh}(y)|. \tag{4.1}$$

Definition 4.1. Let $f\colon \mathbb{R} \to \mathbb{C}$. The function $\Omega_{\rho,f}\colon [0,+\infty] \longrightarrow [0,+\infty]$ defined by

$$\Omega_{\rho,f}(\delta) := \sup\{|f(x) - f(y)|\colon\ x,y \in \mathbb{R},\ \rho(x,y) \le \delta\}. \tag{4.2}$$

is called the *modulus of continuity* of f with respect to the metric ρ.

Definition 4.2 (very slowly oscillating functions on the real line). Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a bounded function. We say that f is *very slowly oscillating* if it is uniformly continuous with respect to the metric ρ given by (4.1), i.e., if $\lim\limits_{\delta\to 0} \Omega_{\rho,f}(\delta) = 0$. In other words, f is very slowly oscillating if and only if the composition $f \circ \sinh$ is uniformly continuous with respect the usual metric on $\mathbb{R}$. Denote by VSO($\mathbb{R}$) the set of all such functions.

The following result is well known in more general settings, namely, for bounded uniformly continuous functions on a general metric space. The idea of the proof may be seen in [12].

Proposition 4.3. VSO($\mathbb{R}$) *is a closed C^*-subalgebra of the C^*-algebra $C_b(\mathbb{R})$ of bounded continuous functions $\mathbb{R} \to \mathbb{C}$ with pointwise operations and supremum-norm.*

We are going to prove that $\gamma_a \in \mathrm{VSO}(\mathbb{R})$ for every $a \in L_\infty(0,\pi)$. Recall that

$$\gamma_a(x) = \frac{2x}{1-e^{-2x\pi}} \int_0^\pi a(\theta) e^{-2x\theta}\, d\theta, \qquad x \in \mathbb{R}.$$

In the following proposition we introduce a metric ζ on $\mathbb{R}$ which is in a certain sense, the most "natural" for the functions γ_a. After that we will show that ζ may be estimated from above by the arcsinh-metric ρ.

Proposition 4.4. *Define $\zeta\colon \mathbb{R}^2 \to [0,+\infty)$ by*

$$\zeta(x,y) = \sup_{\substack{a\in L_\infty(0,\pi)\\ \|a\|_\infty = 1}} |\gamma_a(x) - \gamma_a(y)|. \tag{4.3}$$

Then

$$\zeta(x,y) = \int_0^\pi \left| \frac{2x}{1-e^{-2x\pi}} e^{-2x\theta} - \frac{2y}{1-e^{-2y\pi}} e^{-2y\theta} \right| d\theta. \tag{4.4}$$

Proof. For every $a \in L_\infty(0,\pi)$ and $x,y \in \mathbb{R}$ we have

$$|\gamma_a(x) - \gamma_a(y)| \le \|a\|_\infty \int_0^\pi \left| \frac{2x}{1-e^{-2x\pi}} e^{-2x\theta} - \frac{2y}{1-e^{-2y\pi}} e^{-2y\theta} \right| d\theta.$$

On the other hand, if x and y are fixed and $x \ne y$, we define $a_0\colon (0,\pi) \to \mathbb{R}$ by

$$a_0(\theta) = \operatorname{sign}\left(\frac{2x}{1-e^{-2x\pi}} e^{-2x\theta} - \frac{2y}{1-e^{-2y\pi}} e^{-2y\theta} \right)$$

and obtain $a \in L_\infty(0,\pi)$, $\|a\|_\infty = 1$, and

$$\zeta(x,y) = \left|\gamma_{a_0}(x) - \gamma_{a_0}(y)\right| = \int_0^\pi \left|\frac{2x}{1-e^{-2x\pi}} e^{-2x\theta} - \frac{2y}{1-e^{-2y\pi}} e^{-2y\theta}\right| d\theta,$$

which completes the proof. □

Lemma 4.5. *For every $x, y \in \mathbb{R}$, $\zeta(-x,-y) = \zeta(x,y)$.*

Proof. It follows from the identity $\gamma_a(-x) = \gamma_b(x)$ where b is defined by $b(\theta) = a(\pi - \theta)$. □

Lemma 4.6. *Define the functions $\omega\colon \mathbb{R} \to (0,+\infty)$ and $\tau\colon (0,+\infty) \to (0,\infty)$ by*

$$\omega(x) = \frac{2x}{1-e^{-2x\pi}}, \qquad \tau(y) = \frac{\omega'(y)}{\omega(y)} \sqrt{y^2+1}.$$

Then ω is strictly increasing on $\mathbb{R}$ and τ is strictly decreasing on $(0,+\infty)$. Moreover, $\lim\limits_{y\to 0^+} \tau(y) = \pi$, $\lim\limits_{y\to\infty} \tau(y) = 1$.

Proof. All statements of the lemma are easily verified with elementary calculus. □

Lemma 4.7. *For every $x, y \in \mathbb{R}$,*

$$\zeta(x,y) \le 2\pi\rho(x,y). \tag{4.5}$$

Proof. Since both sides of (4.5) are symmetric, we assume that $x < y$. Then

$$\begin{aligned}
\zeta(x,y) &= \int_0^\pi \left|\frac{2x}{1-e^{-2x\pi}} e^{-2x\theta} - \frac{2x}{1-e^{-2x\pi}} e^{-2y\theta} + \frac{2x}{1-e^{-2x\pi}} e^{-2y\theta} - \frac{2y}{1-e^{-2y\pi}} e^{-2y\theta}\right| d\theta \\
&\le \omega(x) \int_0^\pi \left(e^{-2x\theta} - e^{-2y\theta}\right) d\theta + (\omega(y) - \omega(x)) \int_0^\pi e^{-2y\theta}\, d\theta \\
&= \omega(x)\left(\frac{1}{\omega(x)} - \frac{1}{\omega(y)}\right) + (\omega(y) - \omega(x)) \frac{1}{\omega(y)} = 2\left(1 - \frac{\omega(x)}{\omega(y)}\right).
\end{aligned}$$

The elemenatry inequality $1 - \frac{1}{t} \le \ln(t)$ holds for all $t \ge 1$. Applying it to $t = \frac{\omega(y)}{\omega(x)}$ we get

$$1 - \frac{\omega(x)}{\omega(y)} \le \ln(\omega(y)) - \ln(\omega(x)).$$

Consider the case $y > x \ge 0$. By the Cauchy's mean-value theorem, there exists $c \in (x,y)$ such that

$$\frac{\ln\omega(y) - \ln\omega(x)}{\operatorname{arcsinh}(y) - \operatorname{arcsinh}(x)} = \frac{[\ln\omega(x)]'\big|_{x=c}}{\operatorname{arcsinh}'(x)\big|_{x=c}} = \frac{\omega'(c)}{\omega(c)} \sqrt{c^2+1} = \tau(c),$$

where τ is given in Lemma 4.6. Since $\tau(c) \le \pi$ for all $c > 0$, we obtain (4.5) for all $x < y$.

In the case $x < 0 \le y$ we apply the triangular inequality and Lemma 4.5:

$$\zeta(x,y) \le \zeta(x,0) + \zeta(0,y) = \zeta(0,-x) + \zeta(0,y) \le 2\pi(\rho(x,0) + \rho(y,0)) = 2\pi\rho(x,y).$$ □

Theorem 4.8. *For each $a \in L_\infty(0,\pi)$, the function γ_a satisfies $\|\gamma_a\|_\infty \le \|a\|_\infty$ and is Lipschitz continuous with respect to the metric ρ. As a consequence, $\mathfrak{G}$ is a proper subset of* $\mathrm{VSO}(\mathbb{R})$.

Proof. The inequality $\|\gamma_a\|_\infty \le \|a\|_\infty$ follows from the definition of γ_a applying the trivial estimate $|a(\theta)| \le \|a\|_\infty$. The Lipschitz continuity of the functions γ_a is a consequence of Lemma 4.7. In order to justify the last statement of the theorem we note the function σ defined on $\mathbb{R}$ by $\sigma(x) := x^{1/3}/(1+x^2)$ is uniformly continuous, but not Lipschitz continuous, with respect to the usual metric. Therefore the function $\sigma \circ \operatorname{arcsinh}$ belongs to $\mathrm{VSO}(\mathbb{R})$, but it is not Lipschitz continuous with respect to ρ and therefore does not belong to $\mathfrak{G}$. □

5 Main result

Lemma 5.1. *Given $\sigma \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$, there exists $b \in L_\infty(\mathbb{R}_+)$ such that*

$$\sup_{x \in \mathbb{R}_+} \left|\sigma(x) - \gamma_b^v(x)\right| < \varepsilon, \tag{5.1}$$

where

$$\gamma_b^v(y) = 2y \int_0^\infty b(t) e^{-2yt}\, dt, \quad y \in \mathbb{R}_+. \tag{5.2}$$

Proof. Let $x, y \in (0,\infty)$, by the Cauchy's Mean Value Theorem the arcsinh-metric ρ satisfies $\rho(x,y) \le |\ln(x) - \ln(y)|$. So one gets that $\sigma|_{\mathbb{R}_+} \in \mathrm{VSO}(\mathbb{R}_+)$, where $\mathrm{VSO}(\mathbb{R}_+)$ consists of uniformly continuous and bounded functions on $\mathbb{R}_+$ with respect to the metric $\rho_{\ln}(a,b) = |\ln(a) - \ln(b)|$. Therefore, there exists $b \in L_\infty(\mathbb{R}_+)$ such that $\sup_{y \in \mathbb{R}_+} \left|\sigma(y) - \gamma_b^V(y)\right| < \varepsilon$ (see [12]). □

In the same manner as above we can see that given $\sigma \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$ there exists $h \in L_\infty(\mathbb{R}_+)$ such that

$$\sup_{y \in \mathbb{R}_-} \left|f(y) - \gamma_h^v(-y)\right| \le \varepsilon. \tag{5.3}$$

The inequalities (5.1) and (5.3) are very important, because it gives a way to approximate functions in $\mathrm{VSO}(\mathbb{R})$ by functions in the algebra generated by the set of all " spectral functions " γ_a. The following lemmas say us that a function in $\mathrm{VSO}(\mathbb{R})$ can be approximated by functions belonging to $\mathfrak{G}$ in some subset of real numbers.

Lemma 5.2 (**Properties of $\gamma_a(x)$ for large values of x**). *Given $a \in L_\infty(0,\pi)$, we denote by ν_a the function*

$$\nu_a(x) = 2x \int_0^\pi a(\theta) e^{-2x\theta}\, d\theta, \quad x \in \mathbb{R}. \tag{5.4}$$

i). Let $a \in L_\infty(0,\pi)$. Given $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_{x \ge M} |\gamma_a(x) - \nu_a(x)| \le \varepsilon. \tag{5.5}$$

ii). If $a \in L_\infty(0,\pi)$ be such that $a(\theta) = 0$ for all $\theta \in (\pi/2, \pi)$, then $|\gamma_a(x)| \le \dfrac{\|a\|_\infty}{1 + e^{-x\pi}}$ for each $x \in \mathbb{R}$.

iii). If $b \in L_\infty(0,\pi)$ be such that $b(\theta) = 0$ for all $\theta \in (0, \pi/2)$, then $|\gamma_b(x)| \le \dfrac{\|b\|_\infty}{1 + e^{x\pi}}$ for each $x \in \mathbb{R}$.

Proof. *i*). The proof is based on the following observation $\gamma_a(x) = \nu_a(x) + e^{-2x\pi}\gamma_a(x)$. The proof of *ii* and *iii* are straightforward. □

As a consequence of (5.5) and the equality $\gamma_a(-x) = \gamma_b(x)$, where b is given by the formula $b(\theta) = a(\pi-\theta)$, we get that for every $\varepsilon > 0$ there exists $m > 0$ such that

$$\sup_{x\le -m}\left|\gamma_a(x) - \nu_{a(\pi-)}(-x)\right| = \sup_{x\le -m}\left|\gamma_{a(\pi-)}(-x) - \nu_{a(\pi-)}(-x)\right| \le \varepsilon. \tag{5.6}$$

Lemma 5.3. *Let* $\sigma \in \mathrm{VSO}(\mathbb{R})$*. Given* $\varepsilon > 0$*. there exist a generating symbol* $a \in L_\infty(0,\pi)$ *and number* $L > 0$ *such that*

$$\sup_{|x|>L}|\sigma(x) - \gamma_a(x)| \le \varepsilon. \tag{5.7}$$

Proof. Given $\varepsilon > 0$ there is $b \in L_\infty(\mathbb{R}_+)$ such that (5.1) holds. Write $c = \chi_{(0,\pi/2)}b$, hence

$$|f(x) - \gamma_c(x)| \le \left|f(x) - \gamma_b^{\nu}(x)\right| + \left|\gamma_b^{\nu}(x) - \nu_c(x)\right| + |\gamma_c(x) - \nu_c(x)| \le \frac{\varepsilon}{2} + 2\,\|b\|_\infty\, e^{-x\pi}.$$

From this, there exists $k > 0$ such that $\sup_{x\ge k}|f(x) - \gamma_c(x)| \le \varepsilon$. The same argument applied above proves that there are $h \in L_\infty(0,\pi)$ with $h(\theta) = 0$ for each $\theta \in [\pi/2,\pi)$ and $m > 0$ such that $\sup_{y\le -m}|f(y) - \gamma_h(-y)| \le \varepsilon$. However,

$$\gamma_h(-x) = \frac{2xe^{-2x\pi}}{1-e^{-2x\pi}}\int_0^{\pi/2} h(\theta)e^{2x\theta}\,d\theta = \frac{2x}{1-e^{-2x\pi}}\int_{\pi/2}^{\pi} h(\pi-\beta)e^{-2x\beta}\,d\beta.$$

Let $g(\theta) = h(\pi-\theta)$, thus $g \in L_\infty(0,\pi)$ with $g(\theta) = 0$ for each $\theta \in (0,\pi/2]$.

On the other hand, by Proposition 5.2 there are constants $K, M > 0$ such that

$$\sup_{x\le -K}|\gamma_c(x)| \le \frac{\varepsilon}{2}, \quad \sup_{y\ge M}|\gamma_g(y)| \le \frac{\varepsilon}{2}.$$

Taking $a = c + g \in L_\infty(0,\pi)$, and $L = \max\{K, M, k, m\}$, it follows easily that

$$\sup_{|x|>L}|\sigma(x) - \gamma_a(x)| \le \varepsilon,$$

which is the desired conclusion. □

Now, we are ready to prove the main result of the paper which states that the algebra generated by the set of all "spectral functions" is dense in the C^*-algebra $\mathrm{VSO}(\mathbb{R})$. We denote by Γ and $\Gamma_{[0,\pi]}$ the algebras of functions $\mathbb{R} \to \mathbb{C}$ given by

$$\Gamma := \mathcal{A}lg\,\{\gamma_a :\ a \in L_\infty(0,\pi)\} = \mathcal{A}lg\,\mathfrak{G} \tag{5.8}$$

$$\Gamma_{[0,\pi]} := \mathcal{A}lg\left\{\gamma_a :\ a \in L_\infty(0,\pi), \quad \text{such that } \lim_{\theta\to\pi} a(\theta) \text{ and } \lim_{\theta\to 0} a(\theta) \text{ exist}\right\} \tag{5.9}$$

Note that $\Gamma_{[0,\pi]} \subset \Gamma$. It is proved in [27, Lemma 7.2.3] that if $a \in L_\infty(0,\pi)$ and the limits $\lim_{\theta\to 0} a(\theta)$ and $\lim_{\theta\to\pi} a(\theta)$ exist, then γ_a is continuous on $\mathbb{R}$ and

$$\lim_{x\to+\infty}\gamma_a(x) = \lim_{\theta\to 0} a(\theta), \qquad \lim_{x\to-\infty}\gamma_a(x) = \lim_{\theta\to\pi} a(\theta).$$

Thus $\Gamma_{[0,\pi]}$ may be considered as a subset of $C(\overline{\mathbb{R}})$, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$. Furthermore, it is proved in [27, Theorem 7.2.4] that algebra $\Gamma_{[0,\pi]}$ is dense in $C(\overline{\mathbb{R}})$ with respect to the topology generated by the sup-norm $\|\cdot\|_\infty$, i.e.

$$C\left(\overline{\mathbb{R}}\right) = \mathrm{uc}(\Gamma_{[0,\pi]}). \tag{5.10}$$

Theorem 5.4. *The algebra* Γ *is dense in* $\mathrm{VSO}(\mathbb{R})$.

Proof. Let $f \in \mathrm{VSO}(\mathbb{R})$ and $\varepsilon > 0$. By Lemma 5.3 there exists a function $a \in L_\infty(0,\pi)$ and a constant L such that $\sup_{|x|>L} |f(x) - \gamma_a(x)| \leq \frac{\varepsilon}{2}$. Let $g\colon \mathbb{R} \to [0,1]$ be a continuous function with $g(x) = 1$ for each $x \in [-L, L]$ and $g(y) = 0$ for each $y \in [-2L, 2L]$. Define

$$h(x) = (f - \gamma_a)(x)g(x) = \begin{cases} (f - \gamma_a)(x), & \text{if } x \in [-L, L]; \\ 0, & \text{if } x \notin [-2L, 2L]. \end{cases}$$

Note that $h \in C(\overline{\mathbb{R}})$, it follows from (5.10) that there exists $\sigma \in \Gamma_{[0,\pi]}$ such that $\|h - \sigma\|_\infty \leq \varepsilon$. Writing $\gamma = \gamma_a + \sigma \in \Gamma$, one gets $\|f - \gamma\|_\infty = \|(f - \gamma_a) - \sigma\|_\infty \leq \|h - \sigma\|_\infty \leq \varepsilon$. □

The last theorem shows that C^*-algebra of angular Toeplitz operators is isometrically isomorphic to $\mathrm{VSO}(\mathbb{R})$, while by Proposition 3.2 its closure in the strong operator topology coincides with the C^*-algebra of angular operators, which is isometrically isomorphic to $L_\infty(\mathbb{R})$, see Corollary 2.3.

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