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An Algorithm for the Truncated Matrix Hausdorff Moment Problem

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Abstract

In this paper we obtain an algorithm for the truncated matrix Hausdorff moment problem with an odd number of given moments. The coefficients of the corresponding linear fractional matrix transformation can be calculated using the prescribed moments. No conditions besides solvability are assumed for the moment problem. The question of the determinateness of the moment problem is answered by (a part of) the algorithm as well. Several examples are provided.

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1 Introduction

In this paper we consider the truncated matrix Hausdorff moment problem: find a non-decreasing matrix-valued function $M(x) = (m_{k,j}(x))_{k,j=0}^{N-1}$ on $[a,b]$, which is left-continuous in (a,b), $M(a) = 0$, and such that

$$\int_a^b x^n dM(x) = S_n, \qquad n = 0,1,...,\ell, \tag{1.1}$$

where $\{S_n\}_{n=0}^{\ell}$ is a prescribed sequence of Hermitian $(N\times N)$ complex matrices $N \in \mathbb{N}$, $\ell \in \mathbb{Z}_+$. Here $a,b \in \mathbb{R}$: $a < b$. If this problem has a unique solution, it is said to be *determinate*.

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In the opposite case, it is said to be *indeterminate*. A historical review on the results about this problem can be found in [5].

In this paper *we shall analyze the case of an odd number of prescribed moments:* $\ell = 2d$, *with* $d \in \mathbb{N}$. Our main aim is to derive an algorithm which allows us to explicitly solve the moment problem. The coefficients of the corresponding linear fractional matrix transformation can be calculated using the prescribed moments for an arbitrary solvable moment problem (1.1).

Lately, a Nevanlinna-type formula for solutions of the moment problem (1.1) was obtained in [5]. In order to state this result we need some definitions and constructions from [5].

Set

$$\Gamma_k = (S_{i+j})_{i,j=0}^{k} = \begin{pmatrix} S_0 & S_1 & \dots & S_k \\ S_1 & S_2 & \dots & S_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ S_k & S_{k+1} & \dots & S_{2k} \end{pmatrix}, \qquad k \in \mathbb{Z}_+ : \ 2k \le \ell; \tag{1.2}$$

$$\widetilde{\Gamma}_k = (-abS_{i+j} + (a+b)S_{i+j+1} - S_{i+j+2})_{i,j=0}^{k-1}, \qquad k \in \mathbb{N} : \ 2k \le \ell. \tag{1.3}$$

In the case of an odd number of prescribed moments $\ell = 2d$, the result of Choque Rivero, Dyukarev, Fritzsche and Kirstein states that conditions

$$\Gamma_d \ge 0, \ \widetilde{\Gamma}_d \ge 0, \tag{1.4}$$

are necessary and sufficient for the solvability of the matrix moment problem (1.1), see [1, Theorem 1.3, p. 106].

Consider the matrix moment problem (1.1) with $\ell = 2d$, $d \in \mathbb{N}$. Suppose that conditions (1.3) are satisfied. Let $\Gamma_d = (\gamma_{d;n,m})_{n,m=0}^{(d+1)N-1}$, $\gamma_{d;n,m} \in \mathbb{C}$. Notice that

$$\gamma_{d;rN+j,tN+n} = s_{r+t}^{j,n}, \qquad 0 \le j,n \le N-1; \quad 0 \le r,t \le d, \tag{1.5}$$

where

$$S_n = (s_n^{k,\ell})_{k,\ell=0}^{N-1}, \qquad n \in \mathbb{Z}_+$$

are the given moments. By Theorem 2.1 in [5] a finite-dimensional Hilbert space H and a sequence $\{x_n\}_{n=0}^{(d+1)N-1}$ in H exist such that

$$(x_n, x_m) = \gamma_{d;n,m}, \qquad n,m = 0,1,...,(d+1)N-1, \tag{1.6}$$

and $\operatorname{span}\{x_n\}_{n=0}^{(d+1)N-1} = \operatorname{Lin}\{x_n\}_{n=0}^{(d+1)N-1} = H$. Set $H_a = \operatorname{Lin}\{x_n\}_{n=0}^{dN-1}$. Consider the following operators:

$$Ax = \sum_{k=0}^{dN-1} \alpha_k x_{k+N}, \qquad x \in H_a, \ x = \sum_{k=0}^{dN-1} \alpha_k x_k, \ \alpha_k \in \mathbb{C}. \tag{1.7}$$

$$B = \frac{2}{b-a}A - \frac{a+b}{b-a}E_H, \tag{1.8}$$

with $D(A) = D(B) = H_a$. The operators A and B are Hermitian, and B is a contraction in H (i.e. $\|B\| \le 1$), see Proposition 2.1 in [5].

Let $\widehat{B}$ be an arbitrary self-adjoint extension of B in a Hilbert space $\widehat{H} \supseteq H$. Let $R_z(\widehat{B})$ be the resolvent of $\widehat{B}$ and $\{\widehat{E}_\lambda\}_{\lambda\in\mathbb{R}}$ be an orthogonal resolution of unity of $\widehat{B}$. Recall that the operator-valued function $\mathbf{R}_z$: $\mathbf{R}_z h = P^{\widehat{H}}_H R_z(\widehat{B})h$, $h \in H$ is said to be a *generalized resolvent* of B, $z \in \mathbb{C}\backslash\mathbb{R}$. The function $\mathbf{E}_\lambda$: $\mathbf{E}_\lambda h = P^{\widehat{H}}_H \widehat{E}_\lambda h$, $\lambda \in \mathbb{R}$, $h \in H$ is said to be a *spectral function* of the symmetric operator B.
In the case when $\widehat{B}$ is a self-adjoint contraction, the corresponding generalized resolvent $\mathbf{R}_z$: $\mathbf{R}_z h = P^{\widehat{H}}_H R_z(\widehat{B})h$, $h \in H$, is said to be a *generalized sc-resolvent* of B. The sets of generalized sc-resolvents and sc-spectral functions are nonempty.

Denote $\mathcal{D} = D(B)$, $\mathcal{R} = H \ominus \mathcal{D}$. A set of all self-adjoint contractive extensions of B inside H, we denote by $\mathcal{B}_H(B)$. There are the "minimal" element B^μ and the "maximal" element B^M in this set, such that $\mathcal{B}_H(B)$ coincides with the operator segment

$$B^\mu \le \widetilde{B} \le B^M. \tag{1.9}$$

In the case $B^\mu = B^M$ the set $\mathcal{B}_H(B)$ consists of a unique element. This case is said to be *determinate*, whereas the case $B^\mu \neq B^M$ is called *indeterminate*. The case $B^\mu x \neq B^M x$, $x \in \mathcal{R}\backslash\{0\}$ is said to be *completely indeterminate*. The indeterminate case can always be reduced to the completely indeterminate. If $\mathcal{R}_0 = \{x \in \mathcal{R} : \ B^\mu x = B^M x\}$, we may extend B to a linear operator B_e such that

$$B_e x = Bx,\ x \in \mathcal{D}; \quad B_e x = B^\mu x,\ x \in \mathcal{R}_0. \tag{1.10}$$

The sets of generalized sc-resolvents for B and for B_e coincide. Set

$$C = B^M - B^\mu, \tag{1.11}$$

$$Q_\mu(z) = \left(C^{\frac{1}{2}} R^\mu_z C^{\frac{1}{2}} + E_H \right)\Big|_{\mathcal{R}}, \qquad z \in \mathbb{C}\backslash[-1,1], \tag{1.12}$$

where $R^\mu_z = (B^\mu - zE_H)^{-1}$. An operator-valued function $k(z)$ with values in $[\mathcal{R}]$ belongs to the class $R_{\mathcal{R}}[-1,1]$ if

1) $k(z)$ is analytic in $z \in \mathbb{C}\backslash[-1,1]$ and

$$\frac{\operatorname{Im} k(z)}{\operatorname{Im} z} \le 0, \qquad z \in \mathbb{C} : \ \operatorname{Im} z \neq 0;$$

2) For $z \in \mathbb{R}\backslash[-1,1]$, $k(z)$ is a self-adjoint non-negative contraction.

Set $L_N = \operatorname{Lin}\{x_k\}_{k=0}^{N-1}$. Define a linear transformation G from $\mathbb{C}^N$ onto L_N by the following relation:

$$G\vec{e}_k = x_k, \qquad k = 0,1,...,N-1, \tag{1.13}$$

where $\vec{e}_k = (\delta_{0,k}, \delta_{1,k}, ..., \delta_{N-1,k})$. The following result from [5, Theorem 2.4] will be our starting point to construct an algorithm.

Theorem 1.1. *Let the matrix moment problem (1.1) with $\ell = 2d$, $d \in \mathbb{N}$ be given and conditions (1.4) hold. Let the operator B be defined by (1.8). The following statements are true:*

1) If $B^\mu = B^M$, then the moment problem (1.1) has a unique solution. This solution is given by

$$M(x) = (m_{j,n}(x))_{j,n=0}^{N-1}, \quad m_{j,n}(x) = \left(E^\mu_{\frac{2}{b-a}x - \frac{a+b}{b-a}} x_j, x_n\right)_H, \ 0 \le j,n \le N-1, \tag{1.14}$$

where $\{E^\mu_z\}$ is the left-continuous in $[-1,1)$, right-continuous at the point 1, constant outside $[-1,1]$, orthogonal resolution of unity of the operator B^μ.

2) If $B^\mu \neq B^M$, define the extended operator B_e by (1.10); $\mathcal{R}_e = H \ominus D(B_e)$ and $Q'_\mu(z) = \left(C^{\frac{1}{2}} R^\mu_z C^{\frac{1}{2}} + E_H\right)\Big|_{\mathcal{R}_e}$, $z \in \mathbb{C}\backslash[-1,1]$. An arbitrary solution $M(\cdot)$ of the moment problem can be found by the Stieltjes-Perron inversion formula from the following relation

$$\int_{-1}^{1} \frac{1}{t-z} dM^T\left(\frac{(b-a)t + (a+b)}{2}\right) = \mathcal{A}(z) - C(z)k(z)(E_{\mathcal{R}_e} + \mathcal{D}(z)k(z))^{-1}\mathcal{B}(z), \tag{1.15}$$

where $k(z) \in R_{\mathcal{R}_e}[-1,1]$, and on the left-hand side one means the matrix of the corresponding operator in $\mathbb{C}^N$. Here $\mathcal{A}(z), \mathcal{B}(z), C(z), \mathcal{D}(z)$ are analytic operator-valued functions given by

$$\mathcal{A}(z) = G^* P^H_{L_N} R^\mu_z P^H_{L_N} G : \ \mathbb{C}^N \to \mathbb{C}^N, \tag{1.16}$$

$$\mathcal{B}(z) = C^{\frac{1}{2}} R^\mu_z P^H_{L_N} G : \ \mathbb{C}^N \to \mathcal{R}_e, \tag{1.17}$$

$$C(z) = G^* P^H_{L_N} R^\mu_z C^{\frac{1}{2}} : \ \mathcal{R}_e \to \mathbb{C}^N, \tag{1.18}$$

$$\mathcal{D}(z) = Q'_\mu(z) - E_{\mathcal{R}_e} : \ \mathcal{R}_e \to \mathcal{R}_e. \tag{1.19}$$

Moreover, the correspondence between all solutions of the moment problem and $k(z) \in R_{\mathcal{R}_e}[-1,1]$ is one-to-one.

Our algorithm allows the explicit construction of the matrix functions $\mathcal{A}(z)$, $\mathcal{B}(z)$, $C(z)$, $\mathcal{D}(z)$. The algorithm will be illustrated by several examples.

Formulas for the extremal extensions B^μ and B^M were proposed by Krein in [2]. Some other formulas were obtained by Shtraus, see [3], [4] and references therein. We shall present other formulas, under some additional assumptions, which are convenient for a calculation in our algorithm.

Notations. As usual, we denote by $\mathbb{R}, \mathbb{C}, \mathbb{N}, \mathbb{Z}, \mathbb{Z}_+$ the sets of real numbers, complex numbers, positive integers, integers and non-negative integers, respectively; $\mathbb{R}_e = \mathbb{C}\backslash\mathbb{R}$. By $\mathbb{C}_{n\times m}$ we denote the set of all complex $(n\times m)$ matrices, $n,m \in \mathbb{N}$. For a complex (finite or semi-infinite) matrix M, by M^T we mean its transposed matrix, and by M^* we denote the complex conjugate matrix. By I_δ we denote the identity matrix of order $(\delta\times\delta)$, $\delta \in \mathbb{N}$.

For a separable Hilbert space H we denote by $(\cdot,\cdot)_H$ and $\|\cdot\|_H$ the scalar product and the norm in H, respectively. The indices may be omitted in obvious cases.

For a linear operator A in H we denote by $D(A)$ its domain, by $R(A)$ its range, and by A^* we denote its adjoint if it exists. If A is bounded, then $\|A\|$ stands for its operator norm. By $\mathrm{Ker}A$ we mean the set $\{x \in H : \ Ax = 0\}$ (the kernel of A). For a set of elements $\{x_n\}_{n\in B}$ in H, we denote by $\mathrm{Lin}\{x_n\}_{n\in B}$ and $\mathrm{span}\{x_n\}_{n\in B}$ the linear span and the closed linear span (in the norm of H), respectively. Here B is an arbitrary set of indices. For a set $M \subseteq H$ we denote by $\overline{M}$ the closure of M in the norm of H. By E_H we denote the identity operator in H, i.e. $E_H x = x$, $x \in H$. If H_1 is a subspace of H, by $P_{H_1} = P^H_{H_1}$ we denote the operator of the orthogonal projection on H_1 in H. A set of all linear bounded operators which map H into H is denoted by $[H]$. All operators in this paper are assumed to be linear.

2 Formulas for extremal extensions of a symmetric contraction

We shall use the following result, which in the case of matrices is known as a *lemma on the block matrix*.

Theorem 2.1. *Let T be a bounded self-adjoint operator in a Hilbert space H, $D(T) = H$. Suppose that*

$$H = H_1 \oplus H_2, \tag{2.1}$$

where H_1, H_2 are some subspaces of H, and T has the following block representation:

$$T = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}, \tag{2.2}$$

where $A = P^H_{H_1} T P^H_{H_1} : H_1 \mapsto H_1$, $B = P^H_{H_1} T P^H_{H_2} : H_2 \mapsto H_1$, $C = P^H_{H_2} T P^H_{H_2} : H_2 \mapsto H_2$. Assume that $R(A)$ is closed. The operator T is non-negative ($T \geq 0$) if and only if the following conditions hold:

1) *$A \geq 0$;*

2) *There exists a bounded operator X, which maps H_2 into H_1, such that $B = AX$;*

3) *For an operator X, which satisfies 2), we have the inequality*

$$C - X^* A X \geq 0. \tag{2.3}$$

If conditions 1)-3) are satisfied, then the operator X^*AX does not depend on the choice of an operator X, which satisfies 2).

Proof. The proof follows the lines of the proof for the matrix case, see, e.g., [6], with slight necessary modifications.

Necessity. Let $h = \begin{pmatrix} \lambda h_1 \\ \mu h_2 \end{pmatrix}$, $\lambda, \mu \in \mathbb{C}$, $h_1 \in H_1$, $h_2 \in H_2$. Then

$$(Th, h)_H = |\lambda|^2 (Ah_1, h_1)_{H_1} + \mu\overline{\lambda}(Bh_2, h_1)_{H_1} + \lambda\overline{\mu}(B^* h_1, h_2)_{H_2} + |\mu|^2 (Ch_2, h_2)_{H_2}. \tag{2.4}$$

Using conditions for the quadratic form with respect to λ and μ to be non-negative, we get

$$(Ah_1, h_1)_{H_1} \geq 0, \quad (Ch_2, h_2)_{H_2} \geq 0, \qquad h_1 \in H_1,\ h_2 \in H_2; \tag{2.5}$$

$$|(Bh_2, h_1)_{H_1}|^2 \leq (Ah_1, h_1)_{H_1} (Ch_2, h_2)_{H_2}, \qquad h_1 \in H_1,\ h_2 \in H_2. \tag{2.6}$$

In particular, condition 1) of the theorem holds. By (2.6), if $h_1 \in \operatorname{Ker} A$ then $(Bh_2, h_1)_{H_1} = 0$, $\forall h_2 \in H_2$; therefore $BH_2 \perp \operatorname{Ker} A$. Then

$$BH_2 \subseteq \overline{R(A)} = R(A). \tag{2.7}$$

Set $\widetilde{A} = A|_{R(A)}$. The operator $\widetilde{A}$ is a bounded self-adjoint invertible operator on the Hilbert space $R(A)$. Moreover, $R(\widetilde{A}) = R(A)$. Therefore the inverse operator $\widetilde{A}^{-1}$ is bounded. According to (2.7), we may introduce the following operator:

$$X := \widetilde{A}^{-1}B, \tag{2.8}$$

which is bounded and maps H_2 into H_1. Since $AX = A\widetilde{A}^{-1}B = B$, then condition 2) holds for X.

Consider inequality (2.6) with $h_1 = Xh_2 = \widetilde{A}^{-1}Bh_2$. Then $Bh_2 = \widetilde{A}h_1 = Ah_1$, and

$$|(Ah_1,h_1)_{H_1}|^2 \le (Ah_1,h_1)_{H_1}(Ch_2,h_2)_{H_2}, \qquad h_2 \in H_2,\ h_1 = Xh_2.$$

If $(Ah_1,h_1)_{H_1} \neq 0$, we get

$$(Ah_1,h_1)_{H_1} = (X^*AXh_2,h_2)_{H_1} \le (Ch_2,h_2)_{H_2}, \qquad h_2 \in H_2,\ h_1 = Xh_2. \tag{2.9}$$

In the case $(Ah_1,h_1)_{H_1} = 0$, by (2.5) we conclude that relation (2.9) holds as well. Thus, condition 3) is proved.

Sufficiency. By conditions 1),3) it follows that relation (2.5) holds. Inequality (2.6) must be checked to verify that the quadratic form in (2.4) is non-negative and complete the proof of the sufficiency. For arbitrary $h_1 \in H_1$, $h_2 \in H_2$, we may write

$$\begin{aligned}|(h_1,Bh_2)_{H_1}|^2 =& |(h_1,AXh_2)_{H_1}|^2 = |(Ah_1,Xh_2)_{H_1}|^2\\ &\le (Ah_1,h_1)_{H_1}(AXh_2,Xh_2)_{H_1} \le (Ah_1,h_1)_{H_1}(Ch_2,h_2)_{H_1}.\end{aligned}$$

Here we used the property $|(Ah,g)|^2 \le (Ah,h)(Ag,g)$, $h,g \in H_1$.

Let us check the last statement of the theorem. Let X_1,X_2 be bounded operators, which map H_2 into H_1, and satisfy the following relation:

$$B = AX_1, \quad B = AX_2.$$

Then $B^* = X_1^*A = X_2^*A$, and we get

$$X_1^*AX_1 = X_1^*B = X_1^*AX_2 = X_2^*AX_2.$$

□

Theorem 2.2. *Let B be a symmetric contraction in a Hilbert space H, defined on a subspace $\mathcal{D}$, and $\mathcal{R} := H \ominus \mathcal{D}$. Suppose that sets $R\left(P^H_{\mathcal{D}}B \pm E_{\mathcal{D}}\right)$ are closed. The extremal self-adjoint contractive extensions B^μ, B^M have the following block representations with respect to the decomposition $H = \mathcal{D} \oplus \mathcal{R}$:*

$$B^\mu = \begin{pmatrix} B_1 & B_2^* \\ B_2 & D_\mu \end{pmatrix}, \quad B^M = \begin{pmatrix} B_1 & B_2^* \\ B_2 & D_M \end{pmatrix}, \tag{2.10}$$

where $B_1 = P^H_{\mathcal{D}}B :\ \mathcal{D} \mapsto \mathcal{D}$, $B_2 = P^H_{\mathcal{R}}B :\ \mathcal{D} \mapsto \mathcal{R}$,

$$D_\mu = X^*B_1X + X^*X - E_{\mathcal{R}}, \quad D_M = \widetilde{X}^*B_1\widetilde{X} - \widetilde{X}^*\widetilde{X} + E_{\mathcal{R}}, \tag{2.11}$$

Here, the operators X, $\widetilde{X}$ are arbitrary bounded operators which map $\mathcal{R}$ into $\mathcal{D}$, such that

$$B_2^* = B_1X + X, \quad B_2^* = B_1\widetilde{X} - \widetilde{X}. \tag{2.12}$$

The operators D_μ, D_M do not depend on the choice of bounded operators X, $\widetilde{X}$ which map $\mathcal{R}$ into $\mathcal{D}$ and satisfy (2.12).

Proof. Consider an arbitrary self-adjoint contraction $\widetilde{B} \supseteq B$ in the Hilbert space H. The block representation of $\widetilde{B}$ with respect to the decomposition $H = \mathcal{D} \oplus \mathcal{R}$ has the following form:

$$\widetilde{B} = \begin{pmatrix} B_1 & B_2^* \\ B_2 & D \end{pmatrix}, \tag{2.13}$$

where D is a bounded self-adjoint operator in $\mathcal{R}$, $D(D) = \mathcal{R}$. We shall use the following elementary lemma.

Lemma 2.1. *Let A be a bounded self-adjoint operator in a Hilbert space H, $D(A) = H$. The operator A is a contraction ($\|A\| \le 1$) if and only if the following relation holds:*

$$-E_H \le A \le E_H. \tag{2.14}$$

Proof of the lemma. *Necessity.* Let $A = \int_{-1}^{1} \lambda dE_\lambda$, be the spectral representation of A, where $\{E_\lambda\}$ is an orthogonal resolution of the identity of A. Then

$$((E_H - A)h, h)_H = \int_{-1}^{1} (1-\lambda) d(E_\lambda h, h)_H \ge 0, \qquad h \in H,$$

and the second inequality in (2.14) is proved. The first inequality in (2.14) can be similarly derived.
Sufficiency. We have

$$(Ax, x)_H \le (x, x)_H, \quad (Ax, x)_H \ge -(x, x)_H, \qquad \forall x \in H,$$

and therefore

$$|(Ax, x)_H| \le (x, x)_H, \forall x \in H.$$

Then $\|A\| = \sup_{x \in H:\ \|x\|_H = 1} |(Ax, x)| \le 1$. □ (End of the proof of the lemma).

By applying Lemma 2.1 to the contraction $\widetilde{B}$ we obtain that the following relation holds:

$$\begin{pmatrix} B_1 + E_{\mathcal{D}} & B_2^* \\ B_2 & D + E_{\mathcal{R}} \end{pmatrix} \ge 0, \quad \begin{pmatrix} E_{\mathcal{D}} - B_1 & -B_2^* \\ -B_2 & E_{\mathcal{R}} - D \end{pmatrix} \ge 0. \tag{2.15}$$

By employing Theorem 2.1 we obtain that the operator D satisfies the following relation:

$$X^* B_1 X + X^* X - E_{\mathcal{R}} \le D, \quad D \le \widetilde{X}^* B_1 \widetilde{X} - \widetilde{X}^* \widetilde{X} + E_{\mathcal{R}}, \tag{2.16}$$

where operators X, $\widetilde{X}$ are arbitrary bounded operators which map $\mathcal{R}$ into $\mathcal{D}$, and satisfy relation (2.12).

On the other hand, for an arbitrary bounded self-adjoint operator D in $\mathcal{R}$, $D(D) = \mathcal{R}$, which satisfies (2.16), there corresponds a self-adjoint operator of the form (2.13). By Theorem 2.1 relation (2.15) holds, and therefore this operator is a contraction. Consequently, there exists a one-to-one correspondence between contractive self-adjoint extensions of B in H and the operator segment

$$D_\mu \le D \le D_M, \tag{2.17}$$

where D_μ and D_M are defined by (2.11).

Let us check the first equality in (2.10). Denote the operator in the right-hand side of this equality by $B_{\mu,0}$. Since the extremal extension B^{μ} is a contractive self-adjoint extension of B, it has a representation of the form (2.13) with some $D = D_1$ from the segment (2.17). Therefore

$$D_1 \geq D_{\mu}.$$

On the other hand, since B^{μ} is the extremal extension, we have

$$0 \leq B_{\mu,0} - B^{\mu} = \begin{pmatrix} 0 & 0 \\ 0 & D_{\mu} - D_1 \end{pmatrix},$$

and therefore $D_1 \leq D_{\mu}$. Thus, we get $D_1 = D_{\mu}$, and the first equality in (2.10) is proved. The second equality in (2.10) can be derived in a similar manner. □

3 An algorithm

Consider a matrix moment problem (1.1) with $\ell = 2d$, $d \in \mathbb{N}$, and some set of prescribed moments $\{S_n\}_{n=0}^{2d}$. Suppose that conditions of solvability (1.4) are satisfied for the moment problem. Repeating our constructions in the Introduction, below (1.4), we come to the formula (1.15) which describes all solutions of the moment problem. Our aim here is to calculate the matrices of operator-valued functions $\mathcal{A}(z)$, $\mathcal{B}(z)$, $\mathcal{C}(z)$, $\mathcal{D}(z)$, using the prescribed moments.

Observe that

$$\|x_n\|_H^2 = (x_n, x_n)_H = \gamma_{d;n,n}, \qquad 0 \leq n \leq N-1.$$

Taking into account relation (1.5) we conclude that the norms of elements x_n can be explicitly calculated. If

$$\|x_n\|_H = 0, \qquad \forall n: \ 0 \leq n \leq N-1, \tag{3.1}$$

then $S_0 = 0$, and the moment problem (1.1) has a unique solution $M(x) = 0$, $a \leq x \leq b$. Thus, in the case of (3.1) we stop the algorithm. Notice that by Theorem 2.5 in [5], $B^{\mu} = B^{M}$.

Suppose that there exists a non-zero element in a sequence $x_0, x_1, \ldots, x_{N-1}$. Let us apply the Gram-Schmidt orthogonalization procedure to a sequence

$$x_0, x_1, \ldots, x_{dN+N-1}, \tag{3.2}$$

with the removal of linearly dependent elements. We shall obtain an orthonormal basis $\mathfrak{A} = \{f_j\}_{j=0}^{R-1}$ in H, where $1 \leq R \leq dN + N$. Elements $x_0, x_1, \ldots, x_{N-1}$ during the above procedure will form a subset $\mathfrak{A}_0 = \{f_j\}_{j=0}^{\rho-1}$, where $1 \leq \rho \leq R$. Notice that $\mathfrak{A}_0$ is an orthonormal basis in L_N. Moreover, elements $x_0, x_1, \ldots, x_{dN-1}$ during the above procedure will form a subset $\mathfrak{A}_1 = \{f_j\}_{j=0}^{\omega-1}$, where $1 \leq \omega \leq R$. Observe that $\mathfrak{A}_1$ is an orthonormal basis in $D(B) = H_a$. We emphasize that *elements f_j are linear combinations of elements x_k with coefficients expressed in terms of moments using relations (1.6) and (1.5).*

Denote by $\mathcal{G}$ the matrix of the operator G with respect to the bases $\mathfrak{A}_2 := \{\vec{e}_k\}_{k=0}^{N-1}$ and $\mathfrak{A}_0$:

$$\mathcal{G} = ((Ge_k, f_j)_H)_{0 \leq j \leq \rho-1,\ 0 \leq k \leq N-1} = ((x_k, f_j)_H)_{0 \leq j \leq \rho-1,\ 0 \leq k \leq N-1}.$$

Notice that the matrix $\mathcal{G}$ can be explicitly calculated by the given moments.

Case (A): $\omega = R$. In this case $D(B) = H$, and therefore $B^\mu = B^M = B$. By Theorem 1.1 we conclude that the moment problem is determinate. By Remark 2.2 in [5] formula (1.15) holds in this case with $k(z) = 0$. Thus, it remains to calculate $\mathcal{A}(z)$ using (1.16). Let $\mathcal{M}_{1,z}$ be the matrix of the operator $B^\mu - zE_H = B - zE_H$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}$:

$$\mathcal{M}_{1,z} = (((B - zE_H)f_k, f_j)_H)_{j,k=0}^{R-1}.$$

Since f_k are linear combinations of elements x_k with known coefficients, then all elements $((B-zE_H)f_k, f_j)_H$ are expressed in terms of known $\gamma_{d;n,m}$. Then the inverse matrix $\mathcal{M}_{1,z}^{-1}$ will be the matrix of the operator $(B^\mu - zE_H)^{-1} = R_z^\mu$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}$. Denote the matrix standing in the first ρ rows and the first ρ columns of $\mathcal{M}_{1,z}^{-1}$ by $\mathcal{M}_{2,z}$. The matrix $\mathcal{M}_{2,z}$ is the matrix of the operator $P_{L_N}^H R_z^\mu P_{L_N}^H$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}_0$. Finally, the matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}_2$ is equal to $\mathcal{G}^* \mathcal{M}_{2,z} \mathcal{G}$. The unique solution is given by formula (1.15), and we stop the algorithm.

Case (B): $\omega < R$. In this case $D(B) \neq H$. Let us calculate the extremal extensions B^μ and B^M. By Theorem 2.2 these extensions have the form (2.10). Set $\mathfrak{A}_3 := \{f_j\}_{j=\omega}^{R-1}$. Notice that $\mathfrak{A}_3$ is an orthonormal basis in $\mathcal{R} = H \ominus \mathcal{D}$. Denote by $\mathcal{B}_1$, $\mathcal{B}_2$ the matrices of the operators B_1, B_2 with respect to the bases $\mathfrak{A}_1$, and $\mathfrak{A}_1$, $\mathfrak{A}_3$, respectively. Observe that the matrices $\mathcal{B}_1$, $\mathcal{B}_2$ can be calculated explicitly. Denote by $\mathcal{X}$, $\widetilde{\mathcal{X}}$ the matrices of the operators X, $\widetilde{X}$ with respect to the bases $\mathfrak{A}_3$ and $\mathfrak{A}_1$. By (2.12) we obtain that the following relation holds:

$$\mathcal{B}_2^* = \mathcal{B}_1 \mathcal{X} + \mathcal{X}, \quad \mathcal{B}_2^* = \mathcal{B}_1 \widetilde{\mathcal{X}} - \widetilde{\mathcal{X}}. \tag{3.3}$$

Relations (3.3) are equivalent to a linear system of simultaneous equations with respect to the unknown elements of the matrices $\mathcal{X}$ and $\widetilde{\mathcal{X}}$. We can choose an arbitrary solutions of these systems.

Let $\mathcal{D}_\mu$, $\mathcal{D}_M$ be the matrices of the operators D_μ, D_M with respect to the basis $\mathfrak{A}_3$. By (2.11) we conclude that

$$\mathcal{D}_\mu = \mathcal{X}^* \mathcal{B}_1 \mathcal{X} + \mathcal{X}^* \mathcal{X} - I_{R-\omega}, \quad \mathcal{D}_M = \widetilde{\mathcal{X}}^* \mathcal{B}_1 \widetilde{\mathcal{X}} - \widetilde{\mathcal{X}}^* \widetilde{\mathcal{X}} + I_{R-\omega}. \tag{3.4}$$

Denote by $\mathcal{B}^\mu$, $\mathcal{B}^M$ the matrices of the operators B^μ, B^M with respect to the basis $\mathfrak{A}$. By (2.10) we obtain that

$$\mathcal{B}^\mu = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2^* \\ \mathcal{B}_2 & \mathcal{D}_\mu \end{pmatrix}, \quad \mathcal{B}^M = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2^* \\ \mathcal{B}_2 & \mathcal{D}_M \end{pmatrix}. \tag{3.5}$$

If $B^\mu = B^M$, then by Theorem 1.1 the moment problem is determinate. In this case we denote by $\mathcal{M}_{3,z}$ the matrix of the operator $B^\mu - zE_H$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}$. By (3.5) we see that

$$\mathcal{M}_{3,z} = \mathcal{B}^\mu - zI_R.$$

The matrix of the operator $(B^\mu - zE_H)^{-1} = R_z^\mu$, $z \in \mathbb{R}_e$, with respect to the basis $\mathfrak{A}$ is equal to $(\mathcal{B}^\mu - zI_R)^{-1}$. Denote the matrix standing in the first ρ rows and the first ρ columns of $(\mathcal{B}^\mu - zI_R)^{-1}$ by $\mathcal{M}_{4,z}$. The matrix $\mathcal{M}_{4,z}$ is the matrix of the operator $P_{L_N}^H R_z^\mu P_{L_N}^H$, $z \in \mathbb{R}_e$, with respect to the basis $\mathfrak{A}_0$. Finally, the matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}_2$ is equal to $\mathcal{G}^* \mathcal{M}_{4,z} \mathcal{G}$. The unique solution, according to Remark 2.2 in [5], is given by formula (1.15), and we stop the algorithm.

We assume that $B^\mu \neq B^M$.

Case (i): $\det(\mathcal{D}_M - \mathcal{D}_\mu) \neq 0$. Here we have the completely indeterminate case and $B_e = B$, $\mathcal{R}_e = \mathcal{R}$. The matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$, with respect to the basis $\mathfrak{A}_2$ is constructed in the same way as (3.5).

Denote the matrix standing in the first ρ rows (the first ρ columns) of $(\mathcal{B}^\mu - zI_R)^{-1}$ by $\mathcal{M}_{5,z}$ (respectively by $\mathcal{M}_{6,z}$), $z \in \mathbb{R}_e$. The matrix $\mathcal{M}_{5,z}$ ($\mathcal{M}_{6,z}$) is the matrix of the operator $P^H_{L_N} R^\mu_z$ (respectively of $R^\mu_z P^H_{L_N}$), $z \in \mathbb{R}_e$ with respect to the bases $\mathfrak{A}_0$, $\mathfrak{A}$.

The matrix of the operator $C^{\frac{1}{2}}$ with respect to the basis $\mathfrak{A}$ is equal to $\left(\mathcal{B}^M - \mathcal{B}^\mu\right)^{\frac{1}{2}}$. Denote matrix standing in the last $R-\omega$ rows (the last $R-\omega$ columns) of $\left(\mathcal{B}^M - \mathcal{B}^\mu\right)^{\frac{1}{2}}$ by $\mathcal{M}_7$ (respectively by $\mathcal{M}_8$). The matrix $\mathcal{M}_7$ ($\mathcal{M}_8$) is the matrix of the operator $P^H_{\mathcal{R}_e} C^{\frac{1}{2}}$ (respectively of $C^{\frac{1}{2}} P^H_{\mathcal{R}_e}$) with respect to the bases $\mathfrak{A}_3$, $\mathfrak{A}$. Then the matrices of $\mathcal{B}(z)$ and $C(z)$, $z \in \mathbb{R}_e$ with respect to the bases $\mathfrak{A}_2$, $\mathfrak{A}_3$ are equal to

$$\mathcal{M}_7 \mathcal{M}_{6,z} \mathcal{G}, \tag{3.6}$$

and

$$\mathcal{G}^* \mathcal{M}_{5,z} \mathcal{M}_8, \tag{3.7}$$

respectively. Finally, the matrix of $\mathcal{D}(z)$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}_3$ is equal to

$$\mathcal{M}_7 (\mathcal{B}^\mu - zI_R)^{-1} \mathcal{M}_8. \tag{3.8}$$

Solutions can be found by relation (1.15) and we stop the algorithm.

Case (ii): $\det(\mathcal{D}_M - \mathcal{D}_\mu) = 0$. Here we do not have the completely indeterminate case, and $B_e \neq B$, $\mathcal{R}_e \neq \mathcal{R}$. Of course, the matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$ with respect to the basis $\mathbb{A}_2$ is constructed in the same way as above in the case (i).

Consider a linear bounded operator T in $\mathcal{R}$:

$$T := D_M - D_\mu.$$

Observe that

$$C = B^M - B^\mu = \begin{pmatrix} 0 & 0 \\ 0 & T \end{pmatrix}.$$

Therefore $H_a \oplus \operatorname{Ker} T \subseteq \operatorname{Ker} C$. The converse inclusion is also valid, and we have

$$\operatorname{Ker} C = H_a \oplus \operatorname{Ker} T.$$

Since $D(B_e) = \operatorname{Ker}(B^M - B^\mu) = \operatorname{Ker} C$, then

$$\mathcal{R}_e = H \ominus D(B_e) = H \ominus \operatorname{Ker} C = R(T).$$

On the other hand we have

$$R(T) = \operatorname{Lin}\{T f_j\}_{j=\omega}^{R-1}.$$

Observe that

$$g_j := T f_j = \sum_{k=\omega}^{R-1} \alpha_{j,k} f_k, \qquad \omega \leq j \leq R-1, \tag{3.9}$$

where

$$\alpha_{j,k} = (Tf_j, f_k)_H, \qquad \omega \le j,k \le R-1. \tag{3.10}$$

Notice that $\alpha_{j,k}$ are elements of the matrix $\mathcal{B}^M - \mathcal{B}^\mu$ standing in the last $R-\omega$ rows, and in the last $R-\omega$ columns.

In a sequence

$$g_\omega, g_{\omega+1}, \ldots, g_{R-1}, \tag{3.11}$$

there exists a non-zero element. In the opposite case, we would have $R(T) = \{0\}$, $\operatorname{Ker} T = \mathcal{R}$, $T = 0$, $B^M = B^\mu$, and this contradicts our assumptions.

Let us apply the Gram-Schmidt orthogonalization procedure to the sequence (3.11), by removing linearly dependent elements. We shall obtain an orthonormal basis $\mathfrak{A}_4 = \{y_j\}_{j=0}^{\delta-1}$, $1 \le \delta \le R-\omega$, in $\mathcal{R}_e$. The elements of the basis $\mathfrak{A}_4$ are linear combinations of f_j, $\omega \le j \le R-1$ with known coefficients.

Recall that the matrix of the operator $C^{\frac{1}{2}}$ with respect to the basis $\mathfrak{A}$ is equal to $\left(\mathcal{B}^M - \mathcal{B}^\mu\right)^{\frac{1}{2}}$. Thus, we know all the following values:

$$(C^{\frac{1}{2}} f_j, f_k)_H, \qquad 0 \le j,k \le R-1. \tag{3.12}$$

Denote by $\mathcal{M}_9$ ($\mathcal{M}_{10}$) the matrix of the operator $P^H_{\mathcal{R}_e} C^{\frac{1}{2}}$ (respectively of $C^{\frac{1}{2}} P^H_{\mathcal{R}_e}$), with respect to the bases $\mathfrak{A}_4$, $\mathfrak{A}$. Since elements of the basis $\mathfrak{A}_4$ are linear combinations of f_j, then these matrices are calculated using the values from (3.12).

Denote the matrix, standing in the first ρ rows (the first ρ columns) of $(\mathcal{B}^\mu - zI_R)^{-1}$ by $\mathcal{M}_{5,z}$ (respectively by $\mathcal{M}_{6,z}$), $z \in \mathbb{R}_e$. The matrix $\mathcal{M}_{5,z}$ ($\mathcal{M}_{6,z}$) is the matrix of the operator $P^H_{L_N} R^\mu_z$ (respectively of $R^\mu_z P^H_{L_N}$), $z \in \mathbb{R}_e$ with respect to the bases $\mathfrak{A}_0$, $\mathfrak{A}$. Then the matrices of $\mathcal{B}(z)$ and $C(z)$, $z \in \mathbb{R}_e$ with respect to the bases $\mathfrak{A}_2$, $\mathfrak{A}_4$ are equal to

$$\mathcal{M}_9 \mathcal{M}_{6,z} \mathcal{G}, \tag{3.13}$$

and

$$\mathcal{G}^* \mathcal{M}_{5,z} \mathcal{M}_{10}, \tag{3.14}$$

respectively. Finally, the matrix of $\mathcal{D}(z)$, $z \in \mathbb{R}_e$ with respect to the basis $\mathfrak{A}_4$ is equal to

$$\mathcal{M}_9 (\mathcal{B}^\mu - zI_R)^{-1} \mathcal{M}_{10}. \tag{3.15}$$

All solutions can be found by relation (1.15), and the algorithm is completed.

Notice that during the algorithm we checked whether $B^\mu = B^M$. By Theorem 2.5 in [5] this gives an explicit answer to the question of the determinateness of the moment problem (1.1).

Example 3.1. Consider the moment problem (1.1) with $\ell = 2$, $d = 1$; $a = 0$, $b = 1$; $S_0 = 1$, $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{3}$; $N = 1$. In this case we have

$$\Gamma_1 = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix},$$

$$\widetilde{\Gamma}_1 = S_0 - S_2 = \frac{1}{6}.$$

Thus, $\Gamma_1 > 0, \widetilde{\Gamma}_1 > 0$, and conditions (1.4) are satisfied. Therefore the moment problem has a solution.

Let $\{x_n\}_{n=0}^{1}$ be a sequence of elements in a Hilbert space H such that $\mathrm{Lin}\{x_n\}_{n=0}^{1} = H$, and

$$(x_0,x_0)_H = 1,\ (x_0,x_1)_H = (x_1,x_0)_H = \frac{1}{2},\ (x_1,x_1)_H = \frac{1}{3}.$$

Notice that $\mathcal{D} = H_a = \mathrm{Lin}\{x_n\}_{n=0}^{dN-1} = \mathrm{Lin}\{x_0\}$. We have

$$Ax_0 = x_1; \quad Bx_0 = (2A - E_H)x_0 = 2x_1 - x_0, \qquad D(A) = D(B) = \mathrm{Lin}\{x_0\}.$$

Notice that $\|x_0\|_H^2 = (x_0,x_0)_H = 1 \neq 0$. Let us apply the Gram-Schmidt orthogonalization procedure to a sequence x_0, x_1. Set

$$f_0 = \frac{x_0}{\|x_0\|_H} = x_0;$$

Observe that

$$\|x_1 - (x_1,f_0)_H f_0\|_H^2 = \|x_1 - (x_1,x_0)_H x_0\|_H^2 = \left\| x_1 - \frac{1}{2}x_0 \right\|_H^2$$
$$= \left(x_1 - \frac{1}{2}x_0, x_1 - \frac{1}{2}x_0 \right)_H = \frac{1}{12}.$$

Then

$$f_1 = \sqrt{12}\left(x_1 - \frac{1}{2}x_0 \right).$$

Therefore $\mathfrak{A} = \{f_0, f_1\}$, $R = 2$. Moreover, $\mathfrak{A}_0 = \{f_0\}$, $\rho = 1$; $\mathfrak{A}_1 = \mathfrak{A}_0$, $\omega = 1$; $\mathfrak{A}_2 = \{\vec{e}_0\}$, and

$$\mathcal{G} = (x_0, f_0)_H = (x_0,x_0)_H = 1.$$

We have case (B): $\omega < R$. Let calculate the extremal extensions B^μ and B^M. We set $\mathfrak{A}_3 = \{f_1\}$. Then

$$\mathcal{B}_1 = (B_1 f_0, f_0)_H = (Bx_0, x_0)_H = (2x_1 - x_0, x_0)_H = 0,$$
$$\mathcal{B}_2 = (B_2 f_0, f_1)_H = \left(Bx_0,\ \sqrt{12}\left(x_1 - \frac{1}{2}x_0 \right) \right)_H = \frac{1}{\sqrt{3}}.$$

By (3.3) we may write

$$\frac{1}{\sqrt{3}} = 0 + \mathcal{X}, \quad \frac{1}{\sqrt{3}} = 0 - \widetilde{\mathcal{X}},$$

and therefore $\mathcal{X} = \frac{1}{\sqrt{3}}$, $\widetilde{\mathcal{X}} = -\frac{1}{\sqrt{3}}$. By (3.4) we get

$$\mathcal{D}_\mu = -\frac{2}{3}, \quad \mathcal{D}_M = \frac{2}{3}.$$

By (3.5) we conclude that

$$\mathcal{B}^\mu = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & -\frac{2}{3} \end{pmatrix}, \quad \mathcal{B}^M = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{2}{3} \end{pmatrix}.$$

Since $\det(\mathcal{D}_M - \mathcal{D}_\mu) = \frac{4}{3} \neq 0$, we have case (i). Let calculate the matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$. Observe that

$$(\mathcal{B}^\mu - zI_2)^{-1} = \frac{1}{(z+1)(z-\frac{1}{3})}\begin{pmatrix} -\frac{2}{3}-z & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & -z \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{4,z} = \frac{1}{(z+1)(z-\frac{1}{3})}\left(-\frac{2}{3}-z\right), \qquad z \in \mathbb{R}_e.$$

The matrix of $\mathcal{A}(z)$ is equal to

$$\frac{1}{(z+1)(z-\frac{1}{3})}\left(-\frac{2}{3}-z\right), \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{5,z} = \frac{1}{(z+1)(z-\frac{1}{3})}\left(-\frac{2}{3}-z, -\frac{1}{\sqrt{3}}\right),$$

$$\mathcal{M}_{6,z} = \frac{1}{(z+1)(z-\frac{1}{3})}\begin{pmatrix} -\frac{2}{3}-z \\ -\frac{1}{\sqrt{3}} \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Notice that

$$\left(\mathcal{B}^M - \mathcal{B}^\mu\right)^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{2}{\sqrt{3}} \end{pmatrix}.$$

Set

$$\mathcal{M}_7 = \left(0, \frac{2}{\sqrt{3}}\right), \quad \mathcal{M}_8 = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{3}} \end{pmatrix}.$$

The matrices of $\mathcal{B}(z)$ and $\mathcal{C}(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_2$, $\mathfrak{A}_4$ both are equal to

$$-\frac{2}{3(z+1)(z-\frac{1}{3})}.$$

The matrix of $\mathcal{D}(z)$ with respect to $\mathfrak{A}_4$ is equal to

$$-\frac{4z}{3(z+1)(z-\frac{1}{3})}.$$

By (1.15) we may write that all solutions of the moment problem can be found from the following relation:

$$\int_{-1}^{1} \frac{1}{t-z} dM^T\left(\frac{t+1}{2}\right) = -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} - \frac{4}{9(z+1)^2(z-\frac{1}{3})^2}\widehat{k}(z)\left(1 - \frac{4z}{3(z+1)(z-\frac{1}{3})}\widehat{k}(z)\right)^{-1}, \qquad z \in \mathbb{R}_e, \tag{3.16}$$

where $\widehat{k}(z)$ is the matrix of an arbitrary function $k(z) \in R_{\mathcal{R}_e}[-1,1]$. Here the following definition is useful.

Let $r \in \mathbb{N}$. A $\mathbb{C}_{r\times r}$-valued function $\widehat{k}(z)$ belongs to a class $R_r[-1,1]$ iff

1) $\widehat{k}(z)$ is analytic in $z \in \mathbb{C}\backslash[-1,1]$ and

$$\frac{\operatorname{Im}\widehat{k}(z)}{\operatorname{Im} z} \leq 0, \qquad z \in \mathbb{C}: \ \operatorname{Im} z \neq 0;$$

2) For $z \in \mathbb{R}\backslash[-1,1]$, $\widehat{k}^*(z) = \widehat{k}(z)$, $\widehat{k}^*(z)\widehat{k}(z) \leq I_r$, $\widehat{k}(z) \geq 0$.

Proposition 3.1. *Let $\mathcal{R}$ be a finite-dimensional Hilbert space, $r = \dim \mathcal{R} \geq 1$. Let $\mathfrak{A}' = \{h_j\}_{j=0}^{r-1}$ be an orthonormal basis in $\mathcal{R}$. A $[\mathcal{R}]$-valued function $k(z)$, $z \in \mathbb{C}\backslash[-1,1]$ belongs to the class $R_{\mathcal{R}}[-1,1]$ if and only if the function $\widehat{k}(z)$, $z \in \mathbb{C}\backslash[-1,1]$, where $\widehat{k}(z)$ is the matrix of the operator $k(z)$ with respect to $\mathcal{A}'$, belongs to $R_r[-1,1]$.*

The proof is straightforward.

Let $\widehat{k}(z) = 0$. Then

$$\int_{-1}^{1} \frac{1}{t-z} dM^T\left(\frac{t+1}{2}\right) = -\frac{z+\frac{2}{3}}{(z+1)(z-\frac{1}{3})} = \frac{-\frac{1}{4}}{z+1} + \frac{-\frac{3}{4}}{z-\frac{1}{3}}, \qquad z \in \mathbb{R}_e,$$

and therefore

$$M^T\left(\frac{t+1}{2}\right) = \begin{cases} 0, & t \leq -1 \\ \frac{1}{4}, & -1 < t \leq \frac{1}{3} \\ 1 & t > \frac{1}{3} \end{cases}.$$

Let $u = \frac{t+1}{2}$. Then

$$M(u) = \begin{cases} 0, & u \leq 0 \\ \frac{1}{4}, & 0 < u \leq \frac{2}{3} \\ 1 & u > \frac{2}{3} \end{cases}.$$

Example 3.2. Consider the moment problem (1.1) with $\ell = 2$, $d = 1$; $a = -1$, $b = 1$; $S_0 = 1$, $S_1 = \frac{1}{2}$, $S_2 = \frac{1}{3}$; $N = 1$. In this case we get

$$\Gamma_1 = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix},$$

$$\widetilde{\Gamma}_1 = S_0 - S_2 = \frac{2}{3}.$$

So, $\Gamma_1 > 0$, $\widetilde{\Gamma}_1 > 0$, conditions (1.4) are satisfied, and the moment problem has a solution.

Let $\{x_n\}_{n=0}^{1}$ be a sequence of elements in a Hilbert space H such that $\operatorname{Lin}\{x_n\}_{n=0}^{1} = H$, and

$$(x_0, x_0)_H = 1, \ (x_0, x_1)_H = (x_1, x_0)_H = \frac{1}{2}, \ (x_1, x_1)_H = \frac{1}{3}.$$

Observe that $\mathcal{D} = H_a = \operatorname{Lin}\{x_n\}_{n=0}^{dN-1} = \operatorname{Lin}\{x_0\}$. We have

$$Ax_0 = x_1; \quad B = A, \qquad D(A) = D(B) = \operatorname{Lin}\{x_0\}.$$

The Gram-Schmidt orthogonalization procedure will be the same as in Example 3.2. Thus, we shall obtain $\mathfrak{A} = \{f_0, f_1\}$, $f_0 = x_0$, $f_1 = \sqrt{12}\left(x_1 - \frac{1}{2}x_0\right)$, $R = 2$. The bases $\mathfrak{A}_0$, $\mathfrak{A}_1$, $\mathfrak{A}_2$ will be the same, $\rho = 1$, $\omega = 1$, and

$$\mathcal{G} = (x_0, f_0)_H = 1.$$

We obtain case (B): $\omega < R$. Let find the extremal extensions B^{μ} and B^{M}. Set $\mathfrak{A}_3 = \{f_1\}$. Then

$$\mathcal{B}_1 = (B_1 f_0, f_0)_H = \frac{1}{2},$$
$$\mathcal{B}_2 = (B_2 f_0, f_1)_H = \frac{1}{2\sqrt{3}}.$$

By (3.3) we may write

$$\frac{1}{2\sqrt{3}} = \frac{1}{2}\mathcal{X} + \mathcal{X}; \quad \frac{1}{2\sqrt{3}} = \frac{1}{2}\widetilde{\mathcal{X}} - \widetilde{\mathcal{X}},$$

and therefore $\mathcal{X} = \frac{1}{3\sqrt{3}}$, $\widetilde{\mathcal{X}} = -\frac{1}{\sqrt{3}}$. By (3.4) we get

$$\mathcal{D}_{\mu} = -\frac{17}{18}, \quad \mathcal{D}_M = \frac{5}{6}.$$

By (3.5) we conclude that

$$\mathcal{B}^{\mu} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & -\frac{17}{18} \end{pmatrix}, \quad \mathcal{B}^{M} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2\sqrt{3}} \\ \frac{1}{2\sqrt{3}} & \frac{5}{6} \end{pmatrix}.$$

Since $\det(\mathcal{D}_M - \mathcal{D}_{\mu}) = \frac{16}{9} \neq 0$, then we have case (i). Let us calculate the matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$. Notice that

$$(\mathcal{B}^{\mu} - zI_2)^{-1} = \frac{1}{(z+1)(z-\frac{5}{9})} \begin{pmatrix} -z - \frac{17}{18} & -\frac{1}{2\sqrt{3}} \\ -\frac{1}{2\sqrt{3}} & \frac{1}{2} - z \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{4,z} = \frac{-z - \frac{17}{18}}{(z+1)(z-\frac{5}{9})}, \qquad z \in \mathbb{R}_e.$$

The matrix of $\mathcal{A}(z)$ is equal to

$$\frac{-z - \frac{17}{18}}{(z+1)(z-\frac{5}{9})}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{5,z} = \frac{1}{(z+1)(z-\frac{5}{9})} \left(-z - \frac{17}{18}, -\frac{1}{2\sqrt{3}} \right),$$
$$\mathcal{M}_{6,z} = \frac{1}{(z+1)(z-\frac{5}{9})} \begin{pmatrix} -z - \frac{17}{18} \\ -\frac{1}{2\sqrt{3}} \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Notice that

$$\left(\mathcal{B}^{M} - \mathcal{B}^{\mu}\right)^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 \\ 0 & \frac{4}{3} \end{pmatrix}.$$

Set

$$\mathcal{M}_7 = \left(0, \frac{4}{3}\right), \quad \mathcal{M}_8 = \begin{pmatrix} 0 \\ \frac{4}{3} \end{pmatrix}.$$

The matrices of $\mathcal{B}(z)$ and $C(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_2$, $\mathfrak{A}_4$ both are equal to

$$-\frac{2}{3\sqrt{3}(z+1)(z-\frac{5}{9})}.$$

The matrix of $\mathcal{D}(z)$ with respect to $\mathfrak{A}_4$ is equal to

$$\frac{16(\frac{1}{2}-z)}{9(z+1)(z-\frac{5}{9})}.$$

By (1.15) we obtain that all solutions of the moment problem can be found from the following relation:

$$\int_{-1}^{1} \frac{1}{t-z} dM^T(t) = \frac{-z-\frac{17}{18}}{(z+1)(z-\frac{5}{9})} - \frac{4}{27(z+1)^2(z-\frac{5}{9})^2}\widehat{k}(z)\left(1+\frac{16(\frac{1}{2}-z)}{9(z+1)(z-\frac{5}{9})}\widehat{k}(z)\right)^{-1}, \quad z \in \mathbb{R}_e, \tag{3.17}$$

where $\widehat{k}(z) \in R_r[-1,1]$.

Let $\widehat{k}(z) = 0$. Then

$$\int_{-1}^{1} \frac{1}{t-z} dM(t) = \frac{-z-\frac{17}{18}}{(z+1)(z-\frac{5}{9})} = \frac{\frac{1}{28}}{-1-z} + \frac{\frac{27}{28}}{\frac{5}{9}-z}, \quad z \in \mathbb{R}_e,$$

and therefore

$$M(t) = \begin{cases} 0, & t \le -1 \\ \frac{1}{28}, & -1 < t \le \frac{5}{9} \\ 1 & t > \frac{5}{9} \end{cases}.$$

Let $\widehat{k}(z) = 1$. Then

$$\int_{-1}^{1} \frac{1}{t-z} dM(t) = \frac{\frac{1}{4}}{1-z} + \frac{\frac{3}{4}}{\frac{3}{9}-z}, \qquad z \in \mathbb{R}_e,$$

and therefore

$$M(t) = \begin{cases} 0, & t \le \frac{1}{3} \\ \frac{3}{4}, & \frac{1}{3} < t < 1 \\ 1 & t \ge 1 \end{cases}.$$

Example 3.3. Consider the moment problem (1.1) with $\ell = 2$, $d = 1$; $a = -1$, $b = 1$; $S_0 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$, $N = 2$. In this case we have

$$\Gamma_1 = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & \frac{2}{3} \\ 0 & 2 & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{2}{3} \end{pmatrix} = (\gamma_{d;n,m})_{n,m=0}^{3},$$

$$\widetilde{\Gamma}_1 = S_0 - S_2 = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}.$$

Since $\Gamma_1 \geq 0$, $\widetilde{\Gamma}_1 > 0$, the moment problem has a solution.

Consider a Hilbert space H and a sequence of elements $\{x_n\}_{n=0}^3$ in H such that $\mathrm{Lin}\{x_n\}_{n=0}^3 = H$, and

$$(x_n, x_m)_H = \gamma_{d;n,m}, \qquad 0 \leq n,m \leq 3.$$

Observe that $\mathcal{D} = H_a = \mathrm{Lin}\{x_n\}_{n=0}^{dN-1} = \mathrm{Lin}\{x_0, x_1\}$. We have

$$Ax_0 = x_2, Ax_1 = x_3; \quad B = A.$$

Notice that $\|x_0\|_H^2 = (x_0, x_0)_H = 2 \neq 0$. Apply the Gram-Schmidt orthogonalization procedure to a sequence x_0, x_1, x_2, x_3. We get $\mathfrak{A} = \{f_j\}_{j=0}^3$, $f_0 = \frac{1}{\sqrt{2}}x_0$, $f_1 = \frac{1}{\sqrt{2}}x_1$, $f_2 = \frac{1}{2}(3x_2 - x_1)$, $f_3 = \frac{1}{2}(3x_3 - x_0)$, $R = 4$. Moreover, $\mathfrak{A}_0 = \{f_j\}_{j=0}^1$, $\rho = 2$; $\mathfrak{A}_1 = \mathfrak{A}_0$, $\omega = 2$; $\mathfrak{A}_2 = \{\vec{e}_0, \vec{e}_1\}$, and

$$\mathcal{G} = \begin{pmatrix} (x_0, f_0)_H & (x_1, f_0)_H \\ (x_0, f_1)_H & (x_1, f_1)_H \end{pmatrix} = \sqrt{2} I_2.$$

We have case (B): $\omega < R$. Let us calculate B^μ and B^M. Set $\mathfrak{A}_3 = \{f_j\}_{j=2}^3$. Then

$$\mathcal{B}_1 = ((B_1 f_k, f_l)_H)_{l,k=0}^1 = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathcal{B}_2 = ((B_2 f_k, f_l)_H)_{l=2,3,\ k=0,1} = \frac{\sqrt{2}}{3} I_2.$$

By (3.3) we may write

$$\frac{\sqrt{2}}{3} I_2 = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{X} + \mathcal{X}, \quad \frac{\sqrt{2}}{3} I_2 = \frac{1}{3}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{\mathcal{X}} - \widetilde{\mathcal{X}},$$

and therefore

$$\mathcal{X} = \frac{\sqrt{2}}{8}\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}, \quad \widetilde{\mathcal{X}} = \frac{\sqrt{2}}{8}\begin{pmatrix} -3 & -1 \\ -1 & -3 \end{pmatrix}.$$

By (3.4) we obtain that

$$\mathcal{D}_\mu = \begin{pmatrix} -\frac{3}{4} & -\frac{1}{12} \\ -\frac{1}{12} & -\frac{3}{4} \end{pmatrix}, \quad \mathcal{D}_M = \begin{pmatrix} \frac{3}{4} & -\frac{1}{12} \\ -\frac{1}{12} & \frac{3}{4} \end{pmatrix}.$$

By (3.5) we get

$$\mathcal{B}^\mu = \begin{pmatrix} 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & 0 & -\frac{3}{4} & -\frac{1}{12} \\ 0 & \frac{\sqrt{2}}{3} & -\frac{1}{12} & -\frac{3}{4} \end{pmatrix}, \quad \mathcal{B}^M = \begin{pmatrix} 0 & \frac{1}{3} & \frac{\sqrt{2}}{3} & 0 \\ \frac{1}{3} & 0 & 0 & \frac{\sqrt{2}}{3} \\ \frac{\sqrt{2}}{3} & 0 & \frac{3}{4} & -\frac{1}{12} \\ 0 & \frac{\sqrt{2}}{3} & -\frac{1}{12} & \frac{3}{4} \end{pmatrix}.$$

Since $\det(\mathcal{D}_M - \mathcal{D}_\mu) \neq 0$, we have case (i). Observe that

$$(\mathcal{B}^\mu - zI_4)^{-1} = \frac{1}{z(z+1)(z-\frac{1}{2})} \cdot \begin{pmatrix} \frac{1}{6}(1-3z-6z^2) & \frac{1}{6}(-1-2z) & \frac{1-4z}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} \\ \frac{1}{6}(-1-2z) & \frac{1}{6}(1-3z-6z^2) & -\frac{1}{6\sqrt{2}} & \frac{1-4z}{6\sqrt{2}} \\ \frac{1-4z}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} & \frac{1}{12}+\frac{z}{4}-z^2 & \frac{1}{12}(-1+z) \\ -\frac{1}{6\sqrt{2}} & \frac{1-4z}{6\sqrt{2}} & \frac{1}{12}(-1+z) & \frac{1}{12}+\frac{z}{4}-z^2 \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{4,z} = \frac{1}{6z(z+1)(z-\frac{1}{2})}\begin{pmatrix} 1-3z-6z^2 & -1-2z \\ -1-2z & 1-3z-6z^2 \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

The matrix of $\mathcal{A}(z)$ is equal to

$$\frac{1}{3z(z+1)(z-\frac{1}{2})}\begin{pmatrix} 1-3z-6z^2 & -1-2z \\ -1-2z & 1-3z-6z^2 \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{5,z} = \frac{1}{z(z+1)(z-\frac{1}{2})}\begin{pmatrix} \frac{1}{6}(1-3z-6z^2) & \frac{1}{6}(-1-2z) & \frac{1-4z}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} \\ \frac{1}{6}(-1-2z) & \frac{1}{6}(1-3z-6z^2) & -\frac{1}{6\sqrt{2}} & \frac{1-4z}{6\sqrt{2}} \end{pmatrix},$$

$$\mathcal{M}_{6,z} = \frac{1}{z(z+1)(z-\frac{1}{2})}\begin{pmatrix} \frac{1}{6}(1-3z-6z^2) & \frac{1}{6}(-1-2z) \\ \frac{1}{6}(-1-2z) & \frac{1}{6}(1-3z-6z^2) \\ \frac{1-4z}{6\sqrt{2}} & -\frac{1}{6\sqrt{2}} \\ -\frac{1}{6\sqrt{2}} & \frac{1-4z}{6\sqrt{2}} \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Notice that

$$\left(\mathcal{B}^M - \mathcal{B}^\mu\right)^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix}.$$

Set

$$\mathcal{M}_7 = \begin{pmatrix} 0 & 0 & \sqrt{\frac{3}{2}} & 0 \\ 0 & 0 & 0 & \sqrt{\frac{3}{2}} \end{pmatrix}, \quad \mathcal{M}_8 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ \sqrt{\frac{3}{2}} & 0 \\ 0 & \sqrt{\frac{3}{2}} \end{pmatrix}.$$

The matrices of $\mathcal{B}(z)$ and $C(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_2$, $\mathfrak{A}_4$ both are equal to

$$\frac{\sqrt{\frac{3}{2}}}{6z(z+1)(z-\frac{1}{2})}\begin{pmatrix} 1-4z & -1 \\ -1 & 1-4z \end{pmatrix}.$$

The matrix of $\mathcal{D}(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_4$ is equal to

$$\frac{3}{2z(z+1)(z-\frac{1}{2})}\begin{pmatrix} \frac{1}{12}+\frac{z}{4}-z^2 & \frac{1}{12}(-1+z) \\ \frac{1}{12}(-1+z) & \frac{1}{12}+\frac{z}{4}-z^2 \end{pmatrix}.$$

Using (1.15) we may write that all solutions of the moment problem can be found from the following relation:

$$\int_{-1}^{1}\frac{1}{t-z}dM^T(t) = \frac{1}{3z(z+1)(z-\frac{1}{2})}\begin{pmatrix} 1-3z-6z^2 & -1-2z \\ -1-2z & 1-3z-6z^2 \end{pmatrix}$$
$$- \frac{1}{24z^2(z+1)^2(z-\frac{1}{2})^2}\begin{pmatrix} 1-4z & -1 \\ -1 & 1-4z \end{pmatrix}\widehat{k}(z)$$
$$\cdot\left(I_2 - \frac{3}{2z(z+1)(z-\frac{1}{2})}\begin{pmatrix} \frac{1}{12}+\frac{z}{4}-z^2 & \frac{1}{12}(-1+z) \\ \frac{1}{12}(-1+z) & \frac{1}{12}+\frac{z}{4}-z^2 \end{pmatrix}\widehat{k}(z)\right)^{-1}\begin{pmatrix} 1-4z & -1 \\ -1 & 1-4z \end{pmatrix}, \tag{3.18}$$

where $z \in \mathbb{R}_e$, $\widehat{k}(z) \in R_2[-1,1]$.

Let $\widehat{k}(z) = 0$. Then

$$\int_{-1}^{1}\frac{1}{t-z}dM^T(t) = \frac{1}{3z(z+1)(z-\frac{1}{2})}\begin{pmatrix} 1-3z-6z^2 & -1-2z \\ -1-2z & 1-3z-6z^2 \end{pmatrix}$$
$$= \frac{1}{3}\begin{pmatrix} \frac{2}{-z}+\frac{\frac{4}{3}}{-1-z}+\frac{\frac{8}{3}}{\frac{1}{2}-z} & \frac{-2}{-z}+\frac{-\frac{2}{3}}{-1-z}+\frac{\frac{8}{3}}{\frac{1}{2}-z} \\ \frac{-2}{-z}+\frac{-\frac{2}{3}}{-1-z}+\frac{\frac{8}{3}}{\frac{1}{2}-z} & \frac{2}{-z}+\frac{\frac{4}{3}}{-1-z}+\frac{\frac{8}{3}}{\frac{1}{2}-z} \end{pmatrix}, \qquad z \in \mathbb{R}_e,$$

and therefore

$$m_{0,0}(t) = m_{1,1}(t) = \begin{cases} 0, & t \le -1 \\ \frac{4}{9}, & -1 < t \le 0 \\ \frac{10}{9}, & 0 < t \le \frac{1}{2} \\ 2 & t > \frac{1}{2} \end{cases},$$

$$m_{0,1}(t) = m_{1,0}(t) = \begin{cases} 0, & t \le -1 \\ -\frac{2}{9}, & -1 < t \le 0 \\ -\frac{8}{9}, & 0 < t \le \frac{1}{2} \\ 0 & t > \frac{1}{2} \end{cases}.$$

Example 3.4. Consider the moment problem (1.1) with $\ell = 2$, $d = 1$; $a = -1$, $b = 1$; $S_0 = \begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix}$, $S_1 = \begin{pmatrix} 0 & \frac{2}{3} \\ \frac{2}{3} & 0 \end{pmatrix}$, $S_2 = \begin{pmatrix} \frac{2}{3} & 0 \\ 0 & \frac{2}{5} \end{pmatrix}$, $N = 2$. In this case we have

$$\Gamma_1 = \begin{pmatrix} S_0 & S_1 \\ S_1 & S_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 & \frac{2}{3} \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ 0 & \frac{2}{3} & \frac{2}{3} & 0 \\ \frac{2}{3} & 0 & 0 & \frac{2}{5} \end{pmatrix} = (\gamma_{d;n,m})_{n,m=0}^{3},$$

$$\widetilde{\Gamma}_1 = S_0 - S_2 = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{15} \end{pmatrix}.$$

Since $\Gamma_1 \ge 0$, $\widetilde{\Gamma}_1 \ge 0$, the moment problem has a solution.

Let $\{x_n\}_{n=0}^{3}$ be a sequence of elements in a Hilbert space H such that $\mathrm{Lin}\{x_n\}_{n=0}^{3} = H$, and

$$(x_n, x_m)_H = \gamma_{d;n,m}, \qquad 0 \le n, m \le 3.$$

Observe that $\mathcal{D} = H_a = \mathrm{Lin}\{x_0, x_1\}$. We have

$$Ax_0 = x_2, Ax_1 = x_3; \quad B = A.$$

Notice that $\|x_0\|_H^2 = (x_0, x_0)_H = 2 \neq 0$. Let apply the Gram-Schmidt orthogonalization procedure to a sequence x_0, x_1, x_2, x_3. We obtain $\mathfrak{A} = \{f_j\}_{j=0}^2$, $f_0 = \frac{1}{\sqrt{2}} x_0$, $f_1 = \sqrt{\frac{3}{2}} x_1$, $f_2 = \sqrt{\frac{45}{8}}(x_3 - \frac{1}{3} x_0)$, $R = 3$. Moreover, $\mathfrak{A}_0 = \{f_j\}_{j=0}^1, \rho = 2$; $\mathfrak{A}_1 = \mathfrak{A}_0$, $\omega = 2$; $\mathfrak{A}_2 = \{\vec{e}_0, \vec{e}_1\}$, and

$$\mathcal{G} = \sqrt{2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sqrt{3}} \end{pmatrix}.$$

We have case (B): $\omega < R$. Let us calculate the extremal extensions B^μ and B^M. Set $\mathfrak{A}_3 = \{f_2\}$. Then

$$\mathcal{B}_1 = ((B_1 f_j, f_k)_H)_{k,j=0}^1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\mathcal{B}_2 = ((B_2 f_j, f_k)_H)_{k=2,\ j=0,1} = \frac{2}{\sqrt{15}}(0, 1).$$

By (3.3) we may write

$$\frac{2}{\sqrt{15}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \mathcal{X} + \mathcal{X}, \quad \frac{2}{\sqrt{15}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \widetilde{\mathcal{X}} - \widetilde{\mathcal{X}},$$

and therefore

$$\mathcal{X} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ \sqrt{3} \end{pmatrix}, \quad \widetilde{\mathcal{X}} = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ -\sqrt{3} \end{pmatrix}.$$

By (3.4) we obtain that

$$\mathcal{D}_\mu = -\frac{3}{5}, \quad \mathcal{D}_M = \frac{3}{5}.$$

By (3.5) we get

$$\mathcal{B}^\mu = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & -\frac{3}{5} \end{pmatrix}, \quad \mathcal{B}^M = \begin{pmatrix} 0 & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{15}} \\ 0 & \frac{2}{\sqrt{15}} & \frac{3}{5} \end{pmatrix}.$$

Since $\det(\mathcal{D}_M - \mathcal{D}_\mu) \neq 0$, we have case (i). Notice that

$$(\mathcal{B}^\mu - zI_3)^{-1} = \frac{1}{(z+1)(-z^2 + \frac{2}{5}z + \frac{1}{5})} \begin{pmatrix} z^2 + \frac{3}{5}z - \frac{4}{15} & \frac{1}{\sqrt{3}}z + \frac{\sqrt{3}}{5} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{3}}z + \frac{\sqrt{3}}{5} & z^2 + \frac{3}{5}z & \frac{2}{\sqrt{15}}z \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{15}}z & z^2 - \frac{1}{3} \end{pmatrix}, \quad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{4,z} = \frac{1}{(z+1)(-z^2 + \frac{2}{5}z + \frac{1}{5})} \begin{pmatrix} z^2 + \frac{3}{5}z - \frac{4}{15} & \frac{1}{\sqrt{3}}z + \frac{\sqrt{3}}{5} \\ \frac{1}{\sqrt{3}}z + \frac{\sqrt{3}}{5} & z^2 + \frac{3}{5}z \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

The matrix of $\mathcal{A}(z)$, $z \in \mathbb{R}_e$, with respect to $\mathfrak{A}_2$ is equal to

$$\frac{2}{(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}\begin{pmatrix} z^2+\frac{3}{5}z-\frac{4}{15} & \frac{1}{3}z+\frac{1}{5} \\ \frac{1}{3}z+\frac{1}{5} & \frac{1}{3}(z^2+\frac{3}{5}z) \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Set

$$\mathcal{M}_{5,z} = \frac{1}{(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}\begin{pmatrix} z^2+\frac{3}{5}z-\frac{4}{15} & \frac{1}{\sqrt{3}}z+\frac{\sqrt{3}}{5} & \frac{2}{3\sqrt{5}} \\ \frac{1}{\sqrt{3}}z+\frac{\sqrt{3}}{5} & z^2+\frac{3}{5}z & \frac{2}{\sqrt{15}}z \end{pmatrix},$$

$$\mathcal{M}_{6,z} = \frac{1}{(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}\begin{pmatrix} z^2+\frac{3}{5}z-\frac{4}{15} & \frac{1}{\sqrt{3}}z+\frac{\sqrt{3}}{5} \\ \frac{1}{\sqrt{3}}z+\frac{\sqrt{3}}{5} & z^2+\frac{3}{5}z \\ \frac{2}{3\sqrt{5}} & \frac{2}{\sqrt{15}} \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

Notice that

$$\left(\mathcal{B}^M-\mathcal{B}^\mu\right)^{\frac{1}{2}} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \sqrt{\frac{6}{5}} \end{pmatrix}.$$

Set

$$\mathcal{M}_7 = \left(0, 0, \sqrt{\frac{6}{5}}\right), \quad \mathcal{M}_8 = \begin{pmatrix} 0 \\ 0 \\ \sqrt{\frac{6}{5}} \end{pmatrix}.$$

The matrices of $\mathcal{B}(z)$ and $C(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_2$, $\mathfrak{A}_4$ are equal to

$$\frac{4}{5\sqrt{3}(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}(1, z),$$

$$\frac{4}{5\sqrt{3}(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}\begin{pmatrix} 1 \\ z \end{pmatrix},$$

respectively. The matrix of $\mathcal{D}(z)$, $z \in \mathcal{R}_e$, with respect to $\mathfrak{A}_4$ is equal to

$$\frac{6(z^2-\frac{1}{3})}{5(z+1)(-z^2+\frac{2}{5}+\frac{1}{5})}.$$

Using (1.15) we may write that all solutions of the moment problem can be found from the following relation:

$$\int_{-1}^{1}\frac{1}{t-z}dM^T(t) = \frac{2}{(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}\begin{pmatrix} z^2+\frac{3}{5}z-\frac{4}{15} & \frac{1}{3}z+\frac{1}{5} \\ \frac{1}{3}z+\frac{1}{5} & \frac{1}{3}(z^2+\frac{3}{5}z) \end{pmatrix}$$

$$-\frac{16}{75(z+1)^2(-z^2+\frac{2}{5}z+\frac{1}{5})^2}\begin{pmatrix} 1 \\ z \end{pmatrix}\widehat{k}(z)$$

$$\left(1+\frac{6(z^2-\frac{1}{3})}{5(z+1)(-z^2+\frac{2}{5}+\frac{1}{5})}\widehat{k}(z)\right)^{-1}(1, z), \tag{3.19}$$

where $z \in \mathbb{R}_e$, $\widehat{k}(z) \in R_1[-1,1]$.

Let $\widehat{k}(z) = 0$. Then

$$\int_{-1}^{1} \frac{1}{t-z} dM^T(t) = \frac{2}{(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})} \begin{pmatrix} z^2+\frac{3}{5}z-\frac{4}{15} & \frac{1}{3}z+\frac{1}{5} \\ \frac{1}{3}z+\frac{1}{5} & \frac{1}{3}(z^2+\frac{3}{5}z) \end{pmatrix}, \qquad z \in \mathbb{R}_e.$$

We may write

$$\int_{-1}^{1} \frac{dm_{0,0}(t)}{t-z} = \frac{\frac{2}{9}}{-1-z} + \frac{\frac{8}{9}-\frac{\sqrt{6}}{18}}{\frac{1}{5}+\frac{\sqrt{6}}{5}-z} + \frac{\frac{8}{9}+\frac{\sqrt{6}}{18}}{\frac{1}{5}-\frac{\sqrt{6}}{5}-z}, \qquad z \in \mathbb{R}_e,$$

and therefore

$$m_{0,0}(t) = \begin{cases} 0, & t \le -1 \\ \frac{2}{9}, & -1 < t \le \frac{1}{5}-\frac{\sqrt{6}}{5} \\ \frac{10}{9}+\frac{\sqrt{6}}{18}, & \frac{1}{5}-\frac{\sqrt{6}}{5} < t \le \frac{1}{5}+\frac{\sqrt{6}}{5} \\ 2 & t > \frac{1}{5}+\frac{\sqrt{6}}{5} \end{cases}.$$

We may also write

$$\int_{-1}^{1} \frac{dm_{0,1}(t)}{t-z} = \frac{-\frac{2}{9}}{-z-1} + \frac{\frac{1}{\sqrt{6}}+\frac{1}{9}}{-z+\frac{1}{5}+\frac{\sqrt{6}}{5}} + \frac{-\frac{1}{\sqrt{6}}+\frac{1}{9}}{-z+\frac{1}{5}-\frac{\sqrt{6}}{5}}, \qquad z \in \mathbb{R}_e,$$

and therefore

$$m_{0,1}(t) = \begin{cases} 0, & t \le -1 \\ -\frac{2}{9}, & -1 < t \le \frac{1}{5}-\frac{\sqrt{6}}{5} \\ -\frac{1}{9}-\frac{1}{\sqrt{6}}, & \frac{1}{5}-\frac{\sqrt{6}}{5} < t \le \frac{1}{5}+\frac{\sqrt{6}}{5} \\ 0 & t > \frac{1}{5}+\frac{\sqrt{6}}{5} \end{cases}.$$

Finally, we have

$$\int_{-1}^{1} \frac{1}{t-z} dm_{1,1}(t) = \frac{2}{3(z+1)(-z^2+\frac{2}{5}z+\frac{1}{5})}(z^2+\frac{3}{5}z) = \frac{\frac{2}{9}}{-1-z} + \frac{\frac{2}{9}+\frac{1}{3\sqrt{6}}}{\frac{1}{5}+\frac{\sqrt{6}}{5}-z} + \frac{\frac{2}{9}-\frac{1}{3\sqrt{6}}}{\frac{1}{5}-\frac{\sqrt{6}}{5}-z}, \quad z \in \mathbb{R}_e.$$

Therefore

$$m_{1,1}(t) = \begin{cases} 0, & t \le -1 \\ \frac{2}{9}, & -1 < t \le \frac{1}{5}-\frac{\sqrt{6}}{5} \\ \frac{4}{9}-\frac{1}{3\sqrt{6}}, & \frac{1}{5}-\frac{\sqrt{6}}{5} < t \le \frac{1}{5}+\frac{\sqrt{6}}{5} \\ \frac{2}{3} & t > \frac{1}{5}+\frac{\sqrt{6}}{5} \end{cases}.$$

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