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Decompositions of the Blaschke–Potapov Factors of the Truncated Hausdorff Matrix Moment Problem: The Case of an Even Number of Moments

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Abstract

We obtain multiplicative decompositions of the Blaschke–Potapov factors of the truncated Hausdorff matrix moment (THMM) problem in the case of an even number of moments. Our result develops the multiplicative representation of the resolvent matrix of the THMM problem in the case of even number of moments by I. Serikova in "*The multiplicative structure of resolvent matrix of the moment problem on the kompact interval (case of even numbers of moments)*," Vestnik Kharkov Univ. Ser. Mat. Prikl. Mat. i Mekh. no. 790 (2007). We show that every such Blaschke–Potapov factor can be represented as a product of tridiagonal block matrices containing Stieltjes matrix parameters (SMP) depending on a and b. These SMP in turn are a generalization of the Dyukarev's Stieltjes parameters introduced in "*Indeterminacy criteria for the Stieltjes matrix moment problem,*" Mathematical Notes, Vol. 75 (2004).

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1 Introduction

This work can be considered as the second part of the paper [1]. For completeness we introduce all the required objects which occasionally appear in [1].

Let p, q and n be positive integers. We will use $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}_0$ and $\mathbb{N}$ to denote the set of all complex numbers, the set of all real numbers, the set of all nonnegative integers, and the set of all positive integers, respectively. The notation $\mathbb{C}^{q\times q}$ stands for the set of all complex $q\times q$ matrices. For the null matrix that belongs to $\mathbb{C}^{p\times q}$ we will write $0_{p\times q}$. We denote by

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0_q and I_q the null and the identity matrices in $\mathbb{C}^{q\times q}$, respectively. In cases where the sizes of the null and the identity matrix are clear, we will omit the indices.

The main object of this work is the Blaschke–Potapov factors of the THMM in the case of an even number of moments which is stated as follows: Let a and b be real numbers with $a<b$, let $m\in\mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q\times q$ matrices. Describe the set $\mathbb{M}_{\geq}^q[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$ of all nonnegative Hermitian $q\times q$ measures σ defined on the σ-algebra of all Borel subsets of the interval $[a,b]$ such that

$$s_j=\int\limits_{[a,b]} t^j d\sigma(t) \tag{1.1}$$

holds true for each integer j with $0\leq j\leq 2n+1$.

The solvability of the problem $\mathbb{M}_{\geq}^q[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$ can be expressed by distinguished block Hankel matrices constructed from the given data:

$$K_{1,j}:=b\widetilde{H}_{0,j}-\widetilde{H}_{1,j},\quad K_{2,j}:=-a\widetilde{H}_{0,j}+\widetilde{H}_{1,j},\quad 0\leq j\leq n, \tag{1.2}$$

where

$$\widetilde{H}_{0,j}:=(s_{l+k})_{l,k=0}^{j},\quad \widetilde{H}_{1,j}:=(s_{l+k+1})_{l,k=0}^{j},\quad 0\leq j\leq n. \tag{1.3}$$

Namely in [6, Theorem 1.3], it is proved that there exists a solution of the problem $\mathbb{M}_{\geq}^q[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$ if and only if $K_{1,j}$ and $K_{2,j}$ are positive semidefinite.

The set of solutions for the THMM problem in the even case of number of moments is given with the help of the linear fractional transformation of the form; see [6, Theorem 6.12]:

$$s(z)=(\alpha^{(2n+1)}(z)\mathbf{p}(z)+\beta^{(2n+1)}(z)\mathbf{p}(z))(\gamma^{(2n+1)}(z)\mathbf{p}(z)+\delta^{(2n+1)}(z)\mathbf{p}(z))^{-1}. \tag{1.4}$$

The pair column$(\mathbf{p},\mathbf{q})$ satisfies certain properties; see [6, Definition 5.2].

Definition 1.1. The $2q\times 2q$ matrix–valued function

$$U^{(2n+1)}=\begin{pmatrix}\alpha^{(2n+1)} & \beta^{(2n+1)}\\ \gamma^{(2n+1)} & \delta^{(2n+1)}\end{pmatrix}$$

is called the resolvent matrix (RM) of the problem $\mathbb{M}_{\geq}^q[[a,b],\mathfrak{B}\cap[a,b];(s_j)_{j=0}^{2n+1}]$.

In [24], a representation of the RM of the THMM problem $U^{(2n+1)}$ in the case of an even number of moments with the help of Blaschke–Potapov factors $b^{(2j+1)}(z)$ was given:

$$U^{(2n+1)}(z)=b^{(1)}(z)\cdot b^{(3)}(z)\cdots b^{(2n+1)}(z). \tag{1.5}$$

Similar decompositions of RM of matrix interpolation problems were studied in [12], [13], [15], [17], [25] and [4].

1.1 Main results of the present work

The main results of this work are the following:

a) We find two multiplicative decompositions for each Blaschke–Potapov factor $b^{(2j+1)}$ of the THMM in the case of even number of moments via two families of Stieltjes parameters depending on the terminal point of the interval $[a,b]$. See Theorem 4.10.

b) As a consequence, in Corollary 4.11 two multiplicative representations of the RM, $U^{(2n+1)}(z)$, in terms of two families of Stieltjes parameters are given.

Note that we essentially use the orthogonal polynomials $P_{k,j}$, $\Gamma_{k,j}$ on $[a,b]$ and their second kind polynomials $Q_{k,j}$, $\Theta_{k,j}$; see Definitions A.2 and 2.2. See also [2].

Orthogonal matrix polynomials (OMP) were first considered by M. G. Krein in 1949 [22], [23]. Further research on OMP on the real line was conducted by I. V. Kovalishina [21], H. Dym [11], A. Durán [10], H. Dette [9], Damanik/Pushnitski/Simon [8] and the references therein. See also [16], [19], [20], and [18].

2 Notations and Preliminaries

Let $R_j : \mathbb{C} \to \mathbb{C}^{(j+1)q\times(j+1)q}$ be given by

$$R_{1,j}(z) := (I_{(j+1)q} - zT_j)^{-1}, \qquad j \geq 0, \tag{2.1}$$

with

$$T_0 := \quad 0_q, \quad T_j := \begin{pmatrix} 0_{q\times jq} & 0_q \\ I_{jq} & 0_{jq\times q} \end{pmatrix}, \qquad j \geq 1. \tag{2.2}$$

Let

$$v_1 := I_q, \;\; v_j := \begin{pmatrix} I_q \\ 0_{jq\times q} \end{pmatrix} = \begin{pmatrix} v_{1,j-1} \\ 0_q \end{pmatrix}, \quad \forall j \in \mathbb{N}. \tag{2.3}$$

Let $n \in \mathbb{N}_0$, and let $(s_j)_{j=0}^{2n+1}$ be a sequence of complex $q \times q$. Then we define the matrices

$$y_{[j,k]} := \begin{pmatrix} s_j \\ s_{j+1} \\ \dots \\ s_k \end{pmatrix}, \quad 0 \leq j \leq k, \tag{2.4}$$

$$\widetilde{u}_{1,0} := s_0, \quad \widetilde{u}_{2,0} := -s_0 \tag{2.5}$$

and

$$\widetilde{u}_{1,j} := y_{[0,j]} - b\begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix}, \quad \widetilde{u}_{2,j} := -y_{[0,j]} + a\begin{pmatrix} 0_q \\ y_{[0,j-1]} \end{pmatrix} \tag{2.6}$$

for every $1 \leq j \leq n-1$.

Furthermore, for $1 \leq j \leq n$, let

$$\widetilde{Y}_{1,j} := by_{[j,2j-1]} - y_{[j+1,2j]}, \quad \widetilde{Y}_{2,j} := -ay_{[j,2j-1]} + y_{[j+1,2j]}. \tag{2.7}$$

Definition 2.1. Let the block Hankel matrices $K_{1,j}$ and $K_{2,j}$ be defined by (1.2). The sequence $(s_k)_{k=0}^{2j+1}$ is called Hausdorff positive (resp. nonnegative) on $[a,b]$ if the block Hankel matrices $K_{1,j}$ and $K_{2,j}$ are both positive (resp. nonnegative) definite matrices.

In [6, Theorem 1.3], it was proven that the THMM problem in the case of an even number of moments is solvable if and only if the sequence $(s_k)_{k=0}^{2n+1}$ is Hausdorff nonnegative on $[a,b]$.

In the sequel we will consider only sequences which are Hausdorff positive on $[a,b]$.

Let $\widehat{K}_{1,j}$ (resp. $\widehat{K}_{2,j}$) denote the Schur complement of the block $bs_{2j}-s_{2j+1}$ (resp. $-as_{2j}+s_{2j+1}$) of the matrix $K_{1,j}$ (resp. $K_{2,j}$). Denote

$$\widehat{K}_{1,0} := bs_0 - s_1, \quad \widehat{K}_{1,j} := bs_{2j} - s_{2j+1} - \widetilde{Y}_{1,j}^* K_{1,j-1}^{-1} \widetilde{Y}_{1,j},\ 1 \le j \le n, \tag{2.8}$$

$$\widehat{K}_{2,0} := -as_0 + s_1, \quad \widehat{K}_{2,j} := -as_{2j} + s_{2j+1} - \widetilde{Y}_{2,j}^* K_{2,j-1}^{-1} \widetilde{Y}_{2,j},\ 0 \le j \le n. \tag{2.9}$$

The quantities (2.8) and (2.9) have been defined in [9] for $a=0$ and $b=1$.

Let us recall four families of matrix polynomials that were first studied in [26].

Definition 2.2. Let $K_{k,j}$, $\widetilde{u}_{k,j}$, $\widetilde{Y}_{k,j}$, for $k=1,2$, R_j and v_j be as in (1.2), (2.5), (2.6), (2.7), (2.1) and (2.3), respectively. Let $(s_k)_{k=0}^{2j+1}$ be a sequence which is Hausdorff positive on $[a,b]$. Let

$$\Gamma_{1,0}(z) := I_q, \Gamma_{2,0}(z) := I_q, \ \ \Theta_{1,0}(z) := s_0, \Theta_{2,0}(z) := -s_0 \tag{2.10}$$

for all $z \in \mathbb{C}$. For $k \in \{1,2\}$ and $1 \le j \le n$ define

$$\Gamma_{1,j}(b,z) := (-\widetilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) v_j, \tag{2.11}$$

$$\Gamma_{2,j}(a,z) := (-\widetilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) v_j, \tag{2.12}$$

$$\Theta_{1,j}(b,z) := (-\widetilde{Y}_{1,j}^* K_{1,j-1}^{-1}, I_q) R_j(z) \widetilde{u}_{1,j}, \tag{2.13}$$

$$\Theta_{2,j}(a,z) := (-\widetilde{Y}_{2,j}^* K_{2,j-1}^{-1}, I_q) R_j(z) \widetilde{u}_{2,j} \tag{2.14}$$

for all $z \in \mathbb{C}$.

As in the case of an odd number of moments, we will usually omit the dependence of the polynomials $\Gamma_{k,j}$ and $\Theta_{k,j}$ for $k=1,2$ on the parameters a and b.

In [26] it was proved that polynomials $\Gamma_{1,j}$ and $\Gamma_{2,j}$ are OMP on $[a,b]$ with respect to positive measures $(b-t)\sigma$ and $(t-a)\sigma$ on $[a,b]$, respectively. The polynomials $\Theta_{k,j}$ are known as the second kind polynomials to $\Gamma_{k,j}$.

Definition 2.3. [6, Formula (6.20)] Let $K_{k,j}$, $\widetilde{u}_{k,j}$, for $k=1,2$, R_j and v_j be as in (1.2), (2.5), (2.6), (2.1) and (2.3), respectively. Let $(s_k)_{k=0}^{2j+1}$ be a sequence which is Hausdorff positive on $[a,b]$. The $2q \times 2q$ matrix polynomial

$$U^{(2j+1)}(a,b,z) := \begin{pmatrix} \alpha^{(2j+1)}(a,b,z) & \beta^{(2j+1)}(a,z) \\ \gamma^{(2j+1)}(a,b,z) & \delta^{(2j+1)}(a,z) \end{pmatrix}, z \in \mathbb{C}, \quad 1 \le j \le n, \tag{2.15}$$

is called the RM of the THMM problem in the case of an even number of moments, where

$$\alpha^{(2j+1)}(a,b,z) := I_q - (z-a)\widetilde{u}_{2,j}^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j, \tag{2.16}$$

$$\beta^{(2j+1)}(a,z) := \widetilde{u}_{1,j}^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \widetilde{u}_{1,j}, \tag{2.17}$$

$$\gamma^{(2j+1)}(a,b,z) := -(b-z)(z-a) v_j^* R_j^*(\bar{z}) K_{1,j}^{-1} R_j(a) v_j, \tag{2.18}$$

$$\delta^{(2j+1)}(a,z) := I_q + (z-a) v_j^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \widetilde{u}_{1,j}. \tag{2.19}$$

Below, we usually omit the dependence on a and b.

By [3, Formula (51)] we have the equality

$$v_j^* R_j^*(a) K_{1,j}^{-1} R_j(a) v_j = P_{2,j}^*(a) Q_{2,j}^{*^{-1}}(a). \tag{2.20}$$

Definition 2.4. [6, Formula (6.2)] Let $(s_k)_{k=0}^{2j+1}$ be an even Hausdorff positive on $[a,b]$ sequence. The $2q \times 2q$ matrix polynomial

$$\widetilde{U}_1^{(2j+1)}(a,z) := \begin{pmatrix} \widetilde{\alpha}_1^{(2j+1)}(a,z) & \widetilde{\beta}_1^{(2j+1)}(a,z) \\ \widetilde{\gamma}_1^{(2j+1)}(a,z) & \widetilde{\delta}_1^{(2j+1)}(a,z) \end{pmatrix}, z \in \mathbb{C}, \quad 1 \le j \le n, \tag{2.21}$$

is called the first auxiliary matrix of the THMM problem in the case of an even number of moments, where

$$\widetilde{\alpha}_1^{(2j+1)}(a,z) := I_q - (z-a)\widetilde{u}_{2,j}^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) v_j, \tag{2.22}$$

$$\widetilde{\beta}_1^{(2j+1)}(a,z) := (z-a)\widetilde{u}_{2,j}^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \widetilde{u}_{2,j}, \tag{2.23}$$

$$\widetilde{\gamma}_1^{(2j+1)}(a,z) := -(z-a) v_j^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) v_j, \tag{2.24}$$

$$\widetilde{\delta}_1^{(2j+1)}(a,z) := I_q + (z-a) v_j^* R_j^*(\bar{z}) K_{2,j}^{-1} R_j(a) \widetilde{u}_{2,j}. \tag{2.25}$$

Below, we usually omit the dependence on a.

Let

$$N_{1,j} := (b-a) v_j^* R_j^*(a) K_{1,j}^{-1} R_j(a) v_j, \tag{2.26}$$

$$A_1^{(2j+1)} := \begin{pmatrix} I_q & 0_q \\ -N_{1,j} & I_q \end{pmatrix}, \tag{2.27}$$

In [6] the following equality was proved:

$$U^{(2j+1)} = \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \widetilde{U}_1^{(2j+1)} A_1^{2j+1} \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \tag{2.28}$$

3 Main Algebraic Identities

In this section we recall and introduce some important algebraic identities which will be used further.

If $K_{k,j}$ is a positive definite matrix, its inverse for $1 \le j \le n$ and $k = 1,2$ can be written in the form

$$K_{k,j}^{-1} = \begin{pmatrix} K_{k,j-1}^{-1} & 0_{jq\times q} \\ 0_{q\times jq} & 0_q \end{pmatrix} + \begin{pmatrix} -K_{k,j-1}^{-1}\widetilde{Y}_{k,j} \\ I_q \end{pmatrix} \widehat{K}_{k,j}^{-1}(-\widetilde{Y}_{k,j}^{*}K_{k,j-1}^{-1}, I_q). \tag{3.1}$$

Denote

$$u_j := -y_{[0,j]}, \tag{3.2}$$

$$\Xi_{2,j}^{K} := \begin{pmatrix} -K_{2,j-1}^{-1}\widetilde{Y}_{2,j} \\ I_q \end{pmatrix}, \quad \Xi_{1,j}^{K} := \begin{pmatrix} -K_{1,j-1}^{-1}\widetilde{Y}_{1,j} \\ I_q \end{pmatrix}. \tag{3.3}$$

For each positive integer n, let

$$L_{1,n} := \left(\delta_{j,k+1}I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}, \quad \text{and} \quad L_{2,n} := \left(\delta_{j,k}I_q\right)_{\substack{j=0,\dots,n \\ k=0,\dots,n-1}}, \tag{3.4}$$

where $\delta_{j,k}$ is the Kronecker symbol: $\delta_{j,k} := 1$ if $j = k$ and $\delta_{j,k} := 0$ if $j \neq k$.

Remark 3.1. Let v_j, $L_{2,j}$, u_j, R_j, $\widetilde{u}_{2,j}$, $H_{1,j}$, T_j, $K_{k,j}$, $\Xi_{k,j}^{K}$, $\widehat{K}_{k,j}$ for $k = 1,2$, and the polynomials $\Gamma_{1,j}$, $Q_{2,j}$ $P_{1,j}$, $\Theta_{2,j}$ be as in (2.3), (3.4), (3.2), (2.1), (2.6), (A.1), (2.2), (1.2), (3.3), (2.8), (2.9), Definition A.2 and Definition2.2, respectively. Then the following identities hold:

$$v_{j-1} - L_{2,j}^{*}v_j = 0, \tag{3.5}$$

$$u_j - R_j(a)\widetilde{u}_{2,j} = 0, \tag{3.6}$$

$$K_{1,j} - (b-a)H_{1,j} + K_{2,j} = 0, \tag{3.7}$$

$$K_{k,j}\Xi_{k,j}^{K} - \begin{pmatrix} 0_{jq\times q} \\ \widehat{K}_{k,j} \end{pmatrix} = 0, \quad k = 1,2, \tag{3.8}$$

$$T_jK_{2,j}\Xi_{2,j}^{K} = 0, \tag{3.9}$$

$$R_j^{-1}(a)H_{1,j} + v_ju_j^{*} - T_jK_{2,j} = 0, \tag{3.10}$$

$$\widehat{K}_{1,j} - \Gamma_{1,j}(a)Q_{2,j}^{*}(a) = 0, \tag{3.11}$$

$$\widehat{K}_{2,j} - \Theta_{2,j}(a)P_{1,j+1}^{*}(a) = 0, \tag{3.12}$$

$$R_j^{*^{-1}}(\bar{z})L_{2,j}.R_{j-1}^{*}(\bar{z}) - L_{2,j} = 0, \tag{3.13}$$

$$\Xi_{1,j}^{K} - \Xi_{2,j}^{K} - (b-a)L_{2,j}K_{1,j-1}^{-1}R_{j-1}(a)v_{j-1}\widetilde{u}_{2,j}^{*}R_j^{*}(a)\Xi_{2,j}^{K} = 0. \tag{3.14}$$

Proof. Identities (3.5), (3.6), (3.7) and (3.13) are verified by direct calculation. To prove equality (3.8) for $k = 1,2$ one uses the identities

$$K_{1,j} = \begin{pmatrix} K_{1,j-1} & \widetilde{Y}_{1,j} \\ \widetilde{Y}_{1,j}^{*} & bs_{2j} - s_{2j+1} \end{pmatrix}, \quad K_{2,j} = \begin{pmatrix} K_{2,j-1} & \widetilde{Y}_{2,j} \\ \widetilde{Y}_{2,j}^{*} & -as_{2j} + s_{2j+1} \end{pmatrix}, \tag{3.15}$$

(3.3), and (2.8). Identities (3.10), (3.13), (3.11) and (3.12) were proved in [2, Formula (6.25)] [3, Formula (A.15)], [3, Formula (A.16)] and [2, Formula (6.10)], respectively.

Now we prove (3.14). We have

$$\begin{aligned}
&\Xi^K_{1,j}-\Xi^K_{2,j}-(b-a)L_{2,j}K^{-1}_{1,j-1}L^*_{2,j}H_{1,j}R_{j-1}(a)v_{j-1}\widetilde{u}^*_{2,j}R^*_j(a)\Xi^K_{2,j}\\
&\quad=\Xi^K_{1,j}-\left(I+(b-a)L_{2,j}K^{-1}_{1,j-1}L^*_{2,j}H_{1,j}\right)\Xi^K_{2,j}\\
&\quad=\Xi^K_{1,j}-\left(\begin{pmatrix} K_{1,j-1} & 0\\ 0 & 0\end{pmatrix}K_{2,j}+\begin{pmatrix} 0 & -K^{-1}_{1,j-1}\widetilde{Y}_{1,j}\\ 0 & I_q\end{pmatrix}\right)\Xi^K_{2,j}\\
&\quad=0.
\end{aligned}$$

In the first equality we used (3.5), (3.6), (3.10), (3.9) and (3.13). In the second equality we used the obvious identity $I=K^{-1}_{1,j}K_{1,j}$, as well as (3.1), (3.8) for $k=1$ and (3.7). In the last equality we used (3.8) for $k=2$. □

Lemma 3.2. *Let $\widehat{K}_{k,j}$, $P_{k,j}$, $\Gamma_{k,j}$, $Q_{2,j}$ and $\Theta_{2,j}$ be as in (2.8), (2.9), Definitions A.2 and 2.2, respectively. Then the following identities hold:*

$$\widehat{K}_{2,j}+\widehat{K}_{1,j}+(b-a)\Theta_{2,j}(a)\Gamma^*_{1,j}(a)=0, \tag{3.16}$$

$$Q^{-1}_{2,j}(a)P_{2,j}(a)-Q^{-1}_{2,j-1}(a)P_{2,j-1}(a)-\Gamma^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a)=0, \tag{3.17}$$

$$\Theta_{2,j}(a)P^*_{1,j+1}(a)+\Gamma_{2,j}(a)Q^*_{2,j}(a)+(b-a)\Theta_{2,j}(a)P^*_{2,j}(a)=0, \tag{3.18}$$

$$\Gamma_{1,j}(a)-\Gamma_{2,j}(a)+(b-a)\Theta_{2,j}(a)Q^{-1}_{2,j-1}(a)P_{2,j-1}(a)=0. \tag{3.19}$$

Proof. The equality (3.16) was proved in [24]. The proof of (3.17) is by direct calculation. Use (2.20), (3.1), (2.11), (2.8), the second equality of (2.3) and the identity

$$R_j(z)=\left(\begin{array}{c|c} R_{j-1}(z) & 0_{(j-1)q\times q}\\ \hline (z^jI_q,z^{j-1}I_q,\ldots,zI_q) & I_q\end{array}\right).$$

Identity (3.18) was proved in [3, Formula (129)]. Now we prove (3.19); more precisely, we use its equivalent adjoint complex form:

$$\begin{aligned}
&\Gamma^*_{1,j}(a)-\Gamma^*_{2,j}(a)+(b-a)P^*_{2,j-1}(a)Q^{*^{-1}}_{2,j-1}(a)\Theta^*_{2,j}(a)\\
&\quad=v_jR^*_j(a)\left(\Xi^K_{1,j}-\Xi^K_{2,j}-(b-a)L_{2,j}K^{-1}_{1,j-1}R_{j-1}(a)v_{j-1}\widetilde{u}^*_{2,j}R^*_j(a)\Xi^K_{2,j}\right)\\
&\quad=0.
\end{aligned}$$

The first equality follows from (2.11), (2.12), (2.14), (2.20) and (3.5). In the last equality we used (3.14). □

4 Two Decompositions of the Blaschke–Potapov Factors in the Case of an Even Number of Moments

In this section we give a multiplicative representation of the Blaschke–Potapov factors $b^{(2j+1)}$ of the RM of the THMM.

Definition 4.1. Let $\widehat{K}_{k,j}$, $\Gamma_{1,j}$ and $\Theta_{2,j}$ be as in (2.8), (2.9), (2.11) and (2.14). Define

$$b^{(2j+1)}(z) := \begin{pmatrix} I_q + (z-a)\Theta^*_{2,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a) & \Theta^*_{2,j}(a)\widehat{K}^{-1}_{2,j}\Theta_{2,j}(a) \\ -(z-a)(b-z)\Gamma^*_{1,j}(a)\widehat{K}^{-1}_{1,j}\Gamma_{1,j}(a) & I_q + (z-a)\Gamma^*_{1,j}(a)\widehat{K}^{-1}_{2,j-1}\Theta_{2,j}(a) \end{pmatrix}, \tag{4.1}$$

for $0 \le j \le n$.

In [24] it was proved that the RM of the THMM problem can be represented in the form,

$$U^{(2j+1)}(z) = U^{(2j-1)}(z)b^{(2j+1)}(z), \quad 1 \le j \le n. \tag{4.2}$$

4.1 First Decomposition of the Blaschke–Potapov factors in the Case of an Even Number of Moments

Definition 4.2. Let $H_{1,j}$, $K_{2,j}$, R_j, v_j, $\widetilde{u}^*_{2,j}$ be defined by (A.2), (1.2), (2.1), (2.6), respectively. Denote by

$$\begin{aligned} M_0(a) &:= s_0^{-1}, \\ M_j(a) &:= v_j^* R_j^*(a) H_j^{-1} R_j(a) v_j - v^*_{j-1} R^*_{j-1}(a) H^{-1}_{1,j-1} R_{j-1}(a) v_{j-1}, \end{aligned} \tag{4.3}$$

$$\begin{aligned} L_0(a) &:= \widetilde{u}^*_{2,0} K^{-1}_{2,0} \widetilde{u}_{2,0}, \\ L_j(a) &:= \widetilde{u}^*_{2,j} R^*_j(a) K^{-1}_{2,j} R_j(a) \widetilde{u}_{2,j} - \widetilde{u}^*_{2,j-1} R^*_{j-1}(a) K^{-1}_{2,j-1} R_{j-1}(a) \widetilde{u}_{2,j-1}. \end{aligned} \tag{4.4}$$

These matrices are called Stieltjes matrix parameters of the THMM problem and first introduced in [1]. We usually shall omit the dependence on a.

Clearly the matrices M_j and L_j are positive definite. Note that for $a = 0$, the matrices M_j and L_j became the Stieltjes parameters introduced by Yu. Dyukarev in [14]. Below, we usually omit the dependence on a.

Remark 4.3. Let L_j, $\Theta_{2,j}$, $P_{1,j}$ and $\widehat{K}_{2,j}$ be as in (4.4), (2.14), (A.8) and (2.9). Then the following identity holds:

$$L_j(a) = \Theta^*_{2,j}(a)\widehat{K}^{-1}_{2,j}\Theta_{2,j}(a) = P^{-1}_{1,j+1}(a)\Theta_{2,j}(a). \tag{4.5}$$

Proof. The first equality of (4.5) is by direct calculation. The second equality of (4.5) is obtained using (3.12). □

Proposition 4.4. *Let $b^{(2j+1)}$, $A_1^{(2j+1)}$ and $c^{(2j+1)}$ be as in (4.1), (2.27) and (B.1). Then the following equality holds:*

$$b^{(2j+1)}(z) = \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix} A_1^{(2j-1)^{-1}} c^{(2j+1)} A_1^{(2j+1)} \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tag{4.6}$$

for $j \in \{1, \ldots, n\}$.

Proof. By using the notation

$$\begin{pmatrix} b_j^{11} & b_j^{12} \\ b_j^{21} & b_j^{22} \end{pmatrix} := b^{(2j+1)} \quad \text{and} \quad \begin{pmatrix} c_j^{11} & c_j^{12} \\ c_j^{21} & c_j^{22} \end{pmatrix} := c^{(2j+1)} \tag{4.7}$$

by (4.6), (4.7) and (2.27) it is sufficient to prove that

$$b_j^{11} - c_j^{11} + c_j^{12} N_{1,j} = 0, \tag{4.8}$$

$$-(z-a)N_{1,j-1} b_j^{11} - b_j^{21} - (z-a)c_j^{21} + (z-a)c_j^{22} N_{1,j} = 0, \tag{4.9}$$

$$-(z-a)N_{1,j} b_j^{12} + b_j^{22} - c_j^{22} = 0. \tag{4.10}$$

We prove (4.8). By employing (4.7), (4.1), (B.1) and (2.26) we have

$$\begin{aligned}
&b_j^{11} - c_j^{11} + c_j^{12} N_{1,j} \\
&=(z-a)\Theta_{2,j}^*(a)\left(\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a) - \widehat{K}_{2,j}^{-1}\Gamma_{2,j}(a) + (b-a)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a)P_{2,j}^*(a)Q_{2,j}^{*^{-1}}(a)\right) \\
&=(z-a)P_{1,j+1}^*(a)\Theta_{2,j}^{-1}(a)\left(\Theta_{2,j}(a)P_{1,j+1}^*(a) - \Gamma_{2,j}(a)Q_{2,j}^*(a) + (b-a)\Theta_{2,j}(a)P_{2,j}^*(a)\right)Q_{2,j}^{*^{-1}}(a) \\
&=0.
\end{aligned}$$

In the second equality we used (3.11) and (3.12). The last equality follows from (3.18).

Next we prove (4.9). By (4.7), (4.1), (B.1), (2.26) and (3.17) we have

$$\begin{aligned}
&(z-a)N_{1,j-1}b_j^{11} + b_j^{21} + (z-a)c_j^{21} - (z-a)c_j^{22}N_{1,j} \\
&\quad=(z-a)^2\left(\left(\Gamma_{1,j}^*(a) - (b-a)P_{2,j-1}^*(a)Q_{2,j-1}^*(a)\Theta_{2,j}^*(a)\right)\right. \\
&\qquad\left. + \Gamma_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\left(\Gamma_{2,j} + (b-a)\Theta_{2,j}(a)P_{2,j}^*(a)Q_{2,j}^{*^{-1}}(a)\right)\right) \\
&\quad=(z-a)^2\left(\Gamma_{1,j}^*(a) - \Gamma_{2,j}^*(a) - (b-a)P_{2,j-1}^*(a)Q_{2,j-1}^{*^{-1}}(a)\Theta_{2,j}^*(a)\right)Q_{2,j}^{*^{-1}} \\
&\quad=0.
\end{aligned}$$

Now we prove (4.10):

$$\begin{aligned}
&(z-a)N_{1,j-1}b_j^{12} - b_j^{22} - c_j^{22} \\
&\quad=(z-a)\left((b-a)P_{2,j}^*(a)Q_{2,j}^{*^{-1}}(a)\Theta_{2,j}^*(a) - \Gamma_{1,j}^*(a) + \Gamma_{2,j}^*(a)\right)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a) \\
&\quad=0.
\end{aligned}$$

□

4.2 Second Decomposition of the Blaschke–Potapov factors in the Case of an Even Number of Moments

Definition 4.5. Let v_j, R_j, $H_{1,j}$, $K_{2,j}$ and $\widetilde{u}_{2,j}$ be as in (2.3), (2.1), (A.1), (1.2) and (2.6). Denote

$$\begin{aligned}
M_0^{(2n+1)} &:= s_0^{-1}, \\
M_j^{(2n+1)}(a,b) &:= v_j^* R_j^*(a) H_{1,j}^{-1} R_j(a) v_j - (b-a)v_{j-1}^* R_{j-1}^*(a) K_{2,j-1}^{-1} R_{j-1}(a) v_{j-1}, \quad 1 \le j \le n.
\end{aligned} \tag{4.11}$$

These matrices are called the Stieltjes parameters of the THMM problem in the case of an even number of moments. Below, we shall omit the dependence on a and b.

Remark 4.6. The following identities hold:

$$v_j^* R_j^*(a) H_{1,j}^{-1} R_j(a) v_j = -\Gamma_{2,j}^*(a) \Theta_{2,j}^{*^{-1}}(a), \tag{4.12}$$

$$M_j^{(2n+1)} = -\Theta_{2,j}^{-1}(a)\Gamma_{2,j}(a) - (b-a) Q_{2,j-1}^{-1}(a) P_{2,j-1}(a) = \Theta_{2,j}^{-1}(a)\Gamma_{1,j}(a). \tag{4.13}$$

Proof. Identity (4.12) was proved in [3, Formula (33)]. The first equality of (4.13) follows from (4.12) and (2.20). The second equality of (4.13) is a consequence of (3.19). □

Definition 4.7. Let $\Theta_{2,j}$, $\Gamma_{1,j}$ and $\widehat{K}_{2,j}$ be as in (2.14), (2.11) and (2.9), respectively. Define

$$\begin{aligned} \widetilde{b}^{(1)}(z) &:= \begin{pmatrix} I_q - (z-a)\Theta_{2,0}^*(a)\widehat{K}_{2,0}^{-1}\Gamma_{2,0}(a) & (z-a)\Theta_{2,0}^*(a)\widehat{K}_{2,0}^{-1}\Theta_{2,0}(a) \\ -(z-a)\Gamma_{2,0}^*(a)\widehat{K}_{2,0}^{-1}\Gamma_{2,0}(a) & I_q + (z-a)\Gamma_{2,0}^*(a)\widehat{K}_{2,0}^{-1}\Theta_{2,0}(a) \end{pmatrix}, \\ \widetilde{b}^{(2j+1)}(z) &:= \begin{pmatrix} I_q - (z-a)\Theta_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Gamma_{1,j}(a) & (z-a)\Theta_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a) \\ -(z-a)\Gamma_{1,j}^*(a)\widehat{K}_{2,j}^{-1}\Gamma_{1,j}(a) & I_q + (z-a)\Gamma_{1,j}^*(a)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a) \end{pmatrix}, \end{aligned} \tag{4.14}$$

for $1 \le j \le n$.

Let

$$\gamma_0(a,b) := (a-b)\widehat{K}_{1,0}^{-1}, \quad \gamma_j(a,b) := (a-b)\Gamma_{1,j}^*(a)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a), \quad 1 \le j \le n \tag{4.15}$$

and

$$B_j(a,b) := \begin{pmatrix} I_q & 0_q \\ \gamma_j(a,b) & I_q \end{pmatrix}, \quad 0 \le j \le n \tag{4.16}$$

We will omit the dependence of the γ_j and B_j on a and b.

Lemma 4.8. *Let $b^{(2j+1)}$, $\widetilde{b}^{(2j+1)}$ and B_j be as in (4.1), (4.14) and (4.16), respectively. Then the following equality holds,*

$$b^{(2j+1)}(z) = \begin{pmatrix} \frac{1}{z-a} I_q & 0_q \\ 0_q & I_q \end{pmatrix} \widetilde{b}^{(2j+1)}(z) B_j \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}, \quad j \in \{0,\dots,n\}. \tag{4.17}$$

Proof. We use the notations (4.7) and

$$\begin{pmatrix} \widetilde{b}_j^{11} & \widetilde{b}_j^{12} \\ \widetilde{b}_j^{21} & \widetilde{b}_j^{22} \end{pmatrix} := \widetilde{b}^{(2j+1)}. \tag{4.18}$$

To prove (4.17) it is sufficient to verify the following two equalities:

$$b_j^{11} - \widetilde{b}_j^{11} - (z-a)\widetilde{b}_j^{12}\gamma_j = 0, \tag{4.19}$$

$$b_j^{21} - (z-a)\widetilde{b}_j^{21} - (z-a)\widetilde{b}_j^{22}\gamma_j = 0. \tag{4.20}$$

We consider the left–hand side of (4.19). For $j=0$ this equality readily follows by using the identity $K_{2,0}+K_{1,0}-(a-b)\Theta_{2,0}(a)=0$. For $1\le j\le n$, we have

$$\begin{aligned}
&b_j^{11}-\widetilde{b}_j^{11}-(z-a)\widetilde{b}_j^{12}\gamma_j\\
=&(z-a)\Theta^*_{2,j}(a)\widehat{K}_{1,j}^{-1}\left(\widehat{K}_{2,j}+\widehat{K}_{1,j}+(b-a)\Theta_{2,j}(a)\Gamma^*_{1,j}(a)\right)\widehat{K}_{1,j}^{-1}\Gamma_{1,j}(a)\\
=&0.
\end{aligned}$$

In the first equality we use (4.1), (4.7), (4.14), (4.18) and (4.16). The second equality is a consequence of (3.16). The equality (4.20) is proved in a similar way. □

Lemma 4.9. *Let $\widetilde{b}^{(2j+1)}$, M_j^{2n+1} and L_j be as in (4.14), (4.11) and (4.4), respectively. Then the following identity is valid for $0\le j\le n$:*

$$\widetilde{b}_j^{(2j+1)}(z)=\begin{pmatrix} I_q & 0_q\\ -M_j^{(2n+1)} & I_q\end{pmatrix}\begin{pmatrix} I_q & -(z-a)L_j\\ 0_q & I_q\end{pmatrix}\begin{pmatrix} I_q & 0_q\\ M_j^{(2n+1)} & I_q\end{pmatrix}. \tag{4.21}$$

Proof. Taking into account (4.18), equality (4.21) is equivalent to the following equalities,

$$\widetilde{b}_j^{11}=I_q-L_jM_j^{(2n+1)}, \tag{4.22}$$
$$\widetilde{b}_j^{12}=-(z-a)L_j, \tag{4.23}$$
$$\widetilde{b}_j^{21}=(z-a)M_j^{(2n+1)}L_jM_j^{(2n+1)}, \tag{4.24}$$
$$\widetilde{b}_j^{22}=I_q+(z-a)M_j^{(2n+1)}L_j. \tag{4.25}$$

The equality (4.23) is verified because of (4.18), (4.4) and (4.11). We prove (4.22):

$$\begin{aligned}
&\widetilde{b}_j^{11}-I_q+L_jM_j^{(2n+1)}\\
=&(z-a)\Theta^*_{2,j}(a)\widehat{K}_{2,j}^{-1}(-\Gamma_{1,j}(a)+\Theta_{2,j}(a)M_j^{(2n+1)})\\
=&0_q.
\end{aligned}$$

In the last equality we used the second equality of (4.13). The equalities (4.24) and (4.25) can also be proved by using the second equality of (4.13). □

The following is the main result of this work.

Theorem 4.10. *Let $b^{(2j+1)}$, A_1^{2j+1}, M_j, L_j, M_j^{2n+1} and B_j be as in (4.1), (2.27), (4.3), (4.4), (4.11) and (4.16). Then the following identities hold*
a)

$$\begin{aligned}
b^{(2j+1)}(z)=&\begin{pmatrix}\frac{1}{z-a}I_q & 0_q\\ 0_q & I_q\end{pmatrix}A_1^{(2j-1)^{-1}}\overleftarrow{\prod_{k=0}^{j}}\begin{pmatrix} I_q & 0_q\\ -M_k & I_q\end{pmatrix}\begin{pmatrix} I_q & (z-a)L_j\\ 0_q & I_q\end{pmatrix}\overrightarrow{\prod_{k=0}^{j}}\begin{pmatrix} I_q & 0_q\\ M_k & I_q\end{pmatrix}\\
&\cdot A_1^{(2j+1)}\begin{pmatrix}(z-a)I_q & 0_q\\ 0_q & I_q\end{pmatrix}
\end{aligned} \tag{4.26}$$

for $j \in \{1,\dots,n\}$.
b)

$$b^{(2j+1)}(z) = \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ -M_j^{(2n+1)} & I_q \end{pmatrix} \begin{pmatrix} I_q & -(z-a)L_j^{(2n+1)} \\ 0_q & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ M_j^{(2n+1)} & I_q \end{pmatrix} \cdot B_j \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tag{4.27}$$

for $1 \leq j \leq n$.

Proof. The equality (4.26) follows from (4.6) and (B.2). Equality (4.27) readily follows from (4.17) and (4.21). □

The next result readily follows from Theorem 4.10:

Corollary 4.11. *Under the same conditions as the previous theorem, the following two decompositions of the resolvent matrix* $U^{(2n+1)}$ *defined in (2.15) are valid:*

$$U^{(2n+1)}(z) = \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{j=0}^{n-1}} \left[\begin{pmatrix} I_q & 0_q \\ -M_j^{(2n+1)} & I_q \end{pmatrix} \begin{pmatrix} I_q & -(z-a)L_j^{(2n+1)} \\ 0_q & I_q \end{pmatrix} \cdot \begin{pmatrix} I_q & 0_q \\ M_j^{(2n+1)} & I_q \end{pmatrix} \begin{pmatrix} I_q & 0_q \\ (z-a)\gamma_j & I_q \end{pmatrix} \right] \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix} \tag{4.28}$$

and

$$U^{(2n+1)}(z) = \begin{pmatrix} \frac{1}{z-a}I_q & 0_q \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{j=0}^{n-1}} \left[\begin{pmatrix} I_q & 0_q \\ -M_j & I_q \end{pmatrix} \begin{pmatrix} I_q & -(z-a)L_j \\ 0_q & I_q \end{pmatrix} \cdot \begin{pmatrix} I_q & 0_q \\ M_j & I_q \end{pmatrix} \right] \begin{pmatrix} I_q & 0_q \\ -N_{1,j} & I_q \end{pmatrix} \begin{pmatrix} (z-a)I_q & 0_q \\ 0_q & I_q \end{pmatrix}. \tag{4.29}$$

A OMP on $[a,b]$: The Case of an Odd Number of Moments

In this appendix we recall the OMP on $[a,b]$ generated by a positive Hausdorff sequence $(s_k)_{k=0}^{2j+1}$ on $[a,b]$. Let $\widetilde{H}_{2,j} := (s_{l+k+2})_{l,k=0}^{j}$, $0 \leq j \leq n-1$, and let $H_{1,j}$ and $H_{2,j}$ be defined by (1.3). Denote

$$H_{1,j} := \widetilde{H}_{0,j}, \quad 0 \leq j \leq n, \tag{A.1}$$

$$H_{2,j} := -ab\widetilde{H}_{0,j} + (a+b)\widetilde{H}_{1,j} - \widetilde{H}_{2,j}, \quad 0 \leq j \leq n-1. \tag{A.2}$$

Let

$$u_{1,0} := 0_q, \quad u_{1,j} := \begin{pmatrix} 0_q \\ -y_{[0,j-1]} \end{pmatrix}, 1 \leq j \leq n, \tag{A.3}$$

$$u_{2,0} := -(a+b)s_0 + s_1, \quad u_{2,j} := \begin{pmatrix} -(a+b)s_0 + s_1 \\ -\widehat{y}_{[0,j-1]} \end{pmatrix}, 1 \leq j \leq n-1 \tag{A.4}$$

and let

$$\widehat{s}_j \quad := \quad -abs_j + (a+b)s_{j+1} - s_{j+2}, \quad 0 \le j \le 2n-2. \tag{A.5}$$

Define the following matrices:

$$Y_{1,j} := \begin{pmatrix} s_j \\ s_{j+1} \\ \vdots \\ s_{2j-1} \end{pmatrix}, 1 \le j \le n, \qquad Y_{2,j} := \begin{pmatrix} \widehat{s}_j \\ \widehat{s}_{j+1} \\ \vdots \\ \widehat{s}_{2j-1} \end{pmatrix}, 1 \le j \le n-1. \tag{A.6}$$

Definition A.1. Let the block Hankel matrices $H_{1,j}$ and $H_{2,j-1}$ be defined by (A.1) and (A.2). The sequence $(s_k)_{k=0}^{2j}$ is called Hausdorff positive (resp. nonnegative) on $[a,b]$ if the block Hankel matrices $H_{1,j}$ and $H_{2,j-1}$ are both positive (resp. nonnegative) definite matrices.

Definition A.2. Let $(s_k)_{k=0}^{2j}$ be an odd positive Hausdorff on $[a,b]$ sequence. Let

$$P_{1,0}(z) := I_q,\ \ P_{2,0}(z) := I_q,\ \ Q_{1,0}(z) := 0_q,\ \ Q_{2,0}(a,b,z) := -(u_{2,0} + zs_0), \tag{A.7}$$

$$P_{1,j}(z) := (-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_j(z) v_j, \quad 1 \le j \le n, \tag{A.8}$$

$$P_{2,j}(a,b,z) := (-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z) v_j, \quad 1 \le j \le n-1, \tag{A.9}$$

$$Q_{1,j}(z) := -(-Y_{1,j}^* H_{1,j-1}^{-1}, I_q) R_{1,j}(z) u_{1,j}, \quad 1 \le j \le n \tag{A.10}$$

and

$$Q_{2,j}(a,b,z) := -(-Y_{2,j}^* H_{2,j-1}^{-1}, I_q) R_j(z)(u_{2,j} + zv_j s_0), \quad 1 \le j \le n-1. \tag{A.11}$$

In [5] it was proved that polynomials $P_{k,j}$ for $k = 1,2$ in fact are OMP on $[a,b]$. In [2] some properties of second kind polynomials $Q_{k,j}$ and $\Theta_{k,j}$ for $k = 1,2$ were discussed. In [7] explicit interrelations between $P_{k,j}$, $\Gamma_{k,j}$ and their second kind polynomials were studied. See Definition 2.2.

B Blaschke–Potapov Factors of the Auxiliary Matrix in the Case of Even Number of Moments

Let $\widehat{K}_{2,j}$ and $\Theta_{2,j}$, $\Gamma_{2,j}$ be as in (2.21), (2.9) and Definition 2.2, respectively. Define

$$c^{(2j+1)}(z) := \begin{pmatrix} I_q - (z-a)\Theta_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Gamma_{2,j}(a) & (z-a)\Theta_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a) \\ -(z-a)\Gamma_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Gamma_{2,j}(a) & I_q + (z-a)\Gamma_{2,j}^*(a)\widehat{K}_{2,j}^{-1}\Theta_{2,j}(a) \end{pmatrix}, \tag{B.1}$$

for $0 \le j \le n$. The matrix function $c^{(2j+1)}$ is called the Blaschke–Potapov factor of the matrix $\widetilde{U}_1^{(2n+1)}$, defined by (2.21).

In [3, Corollary 2.4] a representation of the first auxiliary matrix $\widetilde{U}_1^{(2n+1)}(z)$ via Blaschke–Potapov factors, $c^{(2j+1)}(z)$, was proved:

$$\widetilde{U}_1^{(2n+1)}(z) = c^{(1)}(z) \cdots c^{(2n+1)}(z).$$

Proposition B.1. *[3, Theorem 3.2] Let $c^{(2j+1)}$ be defined by (B.1), and M_j, L_j be defined by (4.3) and (4.4), respectively. Then the identity*

$$c^{(2j+1)}(z) = \overleftarrow{\prod_{k=0}^{j}} \begin{pmatrix} I_q & 0_q \\ -M_k & I_q \end{pmatrix} \begin{pmatrix} I_q & (z-a)L_j \\ 0_q & I_q \end{pmatrix} \overrightarrow{\prod_{k=0}^{j}} \begin{pmatrix} I_q & 0_q \\ M_k & I_q \end{pmatrix} \tag{B.2}$$

holds for $0 \leq j \leq n$.

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