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On Hamiltonian Theory for Rotating Charge Coupled to the Maxwell Field

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Abstract

It is known that the Maxwell-Lorentz equations with Abraham's rotating extended electron can be derived from the the Hamilton least action principle applying the variational Poincaré equations on the Lie group SO(3). We prove that, rewritten in the Euler angles, these equations imply the standard Euler-Lagrange and Hamiltonian equations.

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1 Introduction

In [6] it is shown that the Maxwell-Lorentz system describing the motion of rotating extended charge (Abraham model [3, 4]) in electromagnetic field can be derived from the Hamilton least action principle via Poincaré equations [2, 8] on the Lie group SO(3). In the

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present paper we express the Maxwell-Lorentz system and the corresponding Lagrangian in Euler angles and show that in these variables the system imply the standard Euler-Lagrange equations. Further, we make the Legéndre transform and obtain the system of canonical equations for the corresponding Hamiltonian.

2 Euler-Lagrange form of Maxwell-Lorentz equations

The Maxwell field consists of the electric field $E(x,t)$ and the magnetic field $B(x,t)$, $x \in \mathbb{R}^3$, $t \in \mathbb{R}$ generated by a motion of a rotating charge. The external fields E^{ext} and B^{ext} are also generated by the corresponding external charges and currents. Let the rotating charge be centered at the position q with the velocity $\dot{q}$. For simplicity we assume that the mass distribution, $m\rho(x)$, and the charge distribution, $e\rho(x)$, are proportional to each other. Here m is the total mass, e is the total charge, and we use a system of units such that $m = 1$, $e = 1$. The coupling function $\rho(x)$ is a sufficiently smooth radially symmetric function of fast decay as $|x| \to \infty$,

$$\rho(x) = \rho_r(|x|). \tag{C}$$

2.1 Angular velocity

Let us denote by $\omega(t) \in \mathbb{R}^3$ the angular velocity "in space" (in the terminology of [2]) of the charge. Namely, let us fix a "center" point O of the rigid body which is the charge support. Then the trajectory of each fixed point of the body is described by

$$x(t) = q(t) + R(t)(x(0) - q(0)),$$

where $q(t)$ is the position of O at the time t, and $R(t) \in SO(3)$. Respectively, the velocity reads

$$\dot{x}(t) = \dot{q}(t) + \dot{R}(t)(x(0) - q(0)) = \dot{q}(t) + \dot{R}(t)R^{-1}(t)(x(t) - q(t)) = \dot{q}(t) + \omega(t) \times (x(t) - q(t)), \tag{2.1}$$

where $\omega(t) \in \mathbb{R}^3$ corresponds to the skew-symmetric matrix $\dot{R}(t)R^{-1}(t)$ by the rule

$$\dot{R}(t)R^{-1}(t) = \mathcal{J}\omega(t) := \begin{pmatrix} 0 & -\omega_3(t) & \omega_2(t) \\ \omega_3(t) & 0 & -\omega_1(t) \\ -\omega_2(t) & \omega_1(t) & 0 \end{pmatrix}. \tag{2.2}$$

We assume that x and q refer to a certain Euclidean coordinate system in $\mathbb{R}^3$, and the vector product $\times$ is defined in this system by standard formulas. The identification (2.2) of a skew-symmetric matrix and the corresponding angular velocity vector is true in any Euclidean coordinate system of the same orientation as the initial one.

2.2 Dynamical equations

Then the system of Maxwell-Lorentz equations with spin reads, see [9]

$$\dot{E} = \nabla \times B - (\dot{q} + \omega \times (x - q))\rho(x - q), \quad \dot{B} = -\nabla \times E, \tag{2.3}$$

$$\nabla \cdot E(x,t) = \rho(x - q(t)), \;\; \nabla \cdot B(x,t) = 0, \tag{2.4}$$

$$\ddot{q} = \int [E + E^{ext} + (\dot{q} + \omega \times (x - q)) \times (B + B^{ext})]\rho(x - q)dx, \tag{2.5}$$

$$I\,\dot{\omega} = \int (x - q) \times [E + E^{ext} + (\dot{q} + \omega \times (x - q)) \times (B + B^{ext})]\rho(x - q)dx, \tag{2.6}$$

where I is the moment of inertia defined by

$$I = \frac{2}{3}\int x^2 \rho(x)dx. \tag{2.7}$$

Here the equations (2.3) are Maxwell equations with the corresponding charge density and current, equations (2.4) are constraints. The back reaction of the field onto the particle is given through the Lorentz force equation (2.5), and the Lorentz torque equation (2.6) deals with rotational degrees of freedom.

2.3 Lagrangian functional and variational principle

First let us introduce *electromagnetic potentials* $\mathcal{A} = (A_0, A)$, $\mathcal{A}^{ext} = (A_0^{ext}, A^{ext})$:

$$B = \nabla \times A, \;\; E = -\nabla A_0 - \dot{A}. \tag{2.8}$$

$$B^{ext} = \nabla \times A^{ext}, \;\; E^{ext} = -\nabla A_0^{ext} - \dot{A}^{ext}. \tag{2.9}$$

Next we define the Lagrangian

$$\begin{aligned} L(\mathcal{A}, q, R, \dot{\mathcal{A}}, \dot{q}, \dot{R}) \;\; &= \frac{1}{2}\int \left(E^2 - B^2\right)dx + \frac{1}{2}\dot{q}^2 + \frac{1}{2}I\omega^2 - \int [A_0 + A_0^{ext}]\rho(x - q)dx + \\ &\int (\dot{q} + \omega \times (x - q)) \cdot [A + A^{ext}]\rho(x - q)dx, \end{aligned} \tag{2.10}$$

where E, B are expressed in terms of $\mathcal{A}$, $\dot{\mathcal{A}}$ by (2.8), and $\omega = \mathcal{J}^{-1}\dot{R}R^{-1}$ by (2.2).

The last two integrals represent the interaction term

$$\int [(A_0 + A_0^{ext}]\rho - j[A + A^{ext}]dx$$

in view of (2.1). The corresponding action functional has the form

$$S = S(\mathcal{A}, q, R) := \int_{t_1}^{t_2} L(\mathcal{A}(t), q(t), R(t), \dot{\mathcal{A}}(t), \dot{q}(t), \dot{R}(t))dt \tag{2.11}$$

Then the Hamilton least action principle reads

$$\delta S(\mathcal{A}, q, R) = 0, \tag{2.12}$$

where the variation is taken over $\mathcal{A}(t), q(t)$, $R(t)$ with the boundary conditions

$$(\delta\mathcal{A}, \delta q, \delta R)|_{t=t_1} = (\delta\mathcal{A}, \delta q, \delta R)|_{t=t_2} = 0. \tag{2.13}$$

We assume that all the involved functions and fields are sufficiently smooth and have (with all the necessary derivatives) a sufficient decay as $|x| \to \infty$ so that partial integrations below could be possible.

It is known [5, 7] that the variational equation

$$\frac{\delta S}{\delta \mathcal{A}} = 0$$

is equivalent to

$$\frac{d}{dt}\frac{\delta L}{\delta \dot{\mathcal{A}}} = L_{\mathcal{A}}, \tag{2.14}$$

and these Euler-Lagrange equations are equivalent to the Maxwell equations (2.3) with the constraints (2.4). Also the variational equation

$$\frac{\delta S}{\delta q} = 0$$

is equivalent to

$$\frac{d}{dt} L_{\dot{q}} = L_q, \tag{2.15}$$

and (see [2, 8]) the equation

$$\frac{\delta S}{\delta R} = 0$$

is equivalent to the Poincaré equations

$$\frac{d}{dt}\frac{\partial \hat{L}}{\partial \omega_k} = \sum_{ij} c_{ik}^{j} \omega_i \frac{\partial \hat{L}}{\partial \omega_j} + v_k(\hat{L}), \ \ k = 1,2,3. \tag{2.16}$$

Here $v_k(t)$ are right-invariant vector fields on SO(3) formed by right translations: $v_k(R) = \tilde{e}_k R$, $R \in \mathrm{SO}(3)$, $\tilde{e}_k = \mathcal{J} e_k$, e_k, $k = 1,2,3$ being the standard basis in $\mathbb{R}^3$; the coefficients c_{ik}^{j} arise from the commutation relations $[v_i, v_j] = \sum c_{ij}^{k} v_k$, $v_k(\hat{L})$ is the derivative of $\hat{L}$ with respect to the vector field v_k, $\hat{L}$ is L with $\omega(t)$ expressed in the coordinates $(\omega_1(t), \omega_2(t), \omega_3(t))$.

In [6] it is shown that the Euler-Lagrange equations (2.15) are equivalent to the Lorentz force equation (2.5), and, finally, the Poincaré equations (2.16) are equivalent to the Lorentz torque equation (2.6).

Below, we express ω and the Lagrangian in terms of Euler angles and prove that (2.6) implies the standard Euler-Lagrange equations in these variables.

2.4 Euler angles and angular velocity in the body

Let us introduce the Euler angles $\Phi = (\varphi, \psi, \theta)$ with respect to the coordinate frame with the origin at the point $q(t)$; the frame remains at every moment parallel to that at $q(0)$. The angular velocity in the body Ω is connected to ω through $\omega(t) = R(t)\Omega(t)$.

Remark 2.1. In manuals on the dynamics of rigid body the angular velocity in the body is usually denoted by ω, and in the space, Ω, see e.g. [1, 2]. We use the opposite notations to correspond to manuals on electrodynamics, in particular the fundamental monograph [9].

First recall that the rotation $R(t)$ is represented, in Euler angles, as the result of the three consequent rotations in the angles (φ,ψ,θ): $R(t) = R_\psi R_\theta R_\varphi$ (ψ,θ,φ depend on t), where

$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}, R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix}, R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

$$R(t) = \begin{pmatrix} \cos\varphi\cos\psi - \sin\varphi\sin\psi\cos\theta & -\sin\varphi\cos\psi - \cos\varphi\sin\psi\cos\theta & \sin\psi\sin\theta \\ \cos\varphi\sin\psi + \sin\varphi\cos\psi\cos\theta & -\sin\varphi\sin\psi + \cos\varphi\cos\psi\cos\theta & -\cos\psi\sin\theta \\ \sin\varphi\sin\theta & \cos\varphi\sin\theta & \cos\theta \end{pmatrix}.$$

Then one finds $J\omega = \dot{R}R^{-1}$ and thus ω, the final result is:

$$\omega = C\dot{\Phi}, \;\; C = \begin{pmatrix} \sin\psi\sin\theta & 0 & \cos\psi \\ -\cos\psi\sin\theta & 0 & \sin\psi \\ \cos\theta & 1 & 0 \end{pmatrix}, \;\; \dot{\Phi} = \begin{pmatrix} \dot{\varphi} \\ \dot{\psi} \\ \dot{\theta} \end{pmatrix}. \tag{2.17}$$

Let us express the Lagrangian in terms of Φ, $\mathcal{L}(\mathcal{A},q,\Phi,\dot{\mathcal{A}},\dot{q},\dot{\Phi}) = L(\mathcal{A},q,R,\dot{\mathcal{A}},\dot{q},\dot{R})$ i.e. substitute (2.17) to (2.10). Our main result is the following theorem.

Theorem 2.2. *The standard Euler-Lagrange equation*

$$\frac{d}{dt}\mathcal{L}_{\dot{\Phi}} = \mathcal{L}_\Phi \tag{2.18}$$

for the Lagrangian $\mathcal{L}$ in the variables $(\Phi,\dot{\Phi})$ equals the Lorentz torque equation (2.6) (with ω is expressed through (2.17)) multiplied by the matrix C^*.

Proof. Note that the term A^{ext} in (2.10) is additive and we omit it in the further computations as well as the terms E^{ext} and B^{ext} in (2.6). Let us change variables in the last integral of (2.10) and obtain

$$\int (\dot{q} + \omega\times x)\cdot A(x+q)\rho(x)dx.$$

Further, for simplicity of notations let us omit the argument $x+q$ and write A, $\dot{A}$ for $A(x+q)$, $\dot{A}(x+q)$ but remember that $\frac{d}{dt}A = \dot{A} + \dot{q}\cdot\nabla A$.

Step i) Let us write out the terms of the lagrangian involving Φ, $\dot{\Phi}$:

$$\mathcal{L} = \frac{I}{2}C\dot{\Phi}\cdot C\dot{\Phi} + \int(\dot{q} + C\dot{\Phi}\times x)\cdot A\rho\, dx + \cdots = \frac{I}{2}C\dot{\Phi}\cdot C\dot{\Phi} + \int(\dot{q}\cdot A + C\dot{\Phi}\cdot(x\times A)\rho\, dx + \ldots$$

Denote by C^j the j-th column of the matrix C. Then

$$\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_j} = IC^j\cdot C\dot{\Phi} + \int C^j\cdot(x\times A)\rho\, dx,$$

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{\Phi}_j} = I(\dot{C}^j\cdot C\dot{\Phi} + C^j(\dot{C}\dot{\Phi} + C\ddot{\Phi})) + \int[\dot{C}^j\cdot(x\times A) + C^j\cdot(x\times(\dot{A} + \dot{q}\cdot\nabla A))]\rho\, dx,$$

$$\frac{\partial\mathcal{L}}{\partial\Phi_j} = I(\frac{\partial C}{\partial\Phi_j}\dot{\Phi})\cdot C\dot{\Phi} + \int(\frac{\partial C}{\partial\Phi_j}\dot{\Phi})\cdot(x\times A)\rho\, dx,$$

and thus, the j-th Euler-Lagrange equation reads

$$I[\dot{C}^j\cdot C\dot{\Phi}+C^j(\dot{C}\dot{\Phi}+C\ddot{\Phi})-(\frac{\partial C}{\partial\Phi_j}\dot{\Phi})\cdot C\dot{\Phi}]=\int[(\frac{\partial C}{\partial\Phi_j}\dot{\Phi}-\dot{C}^j)\cdot(x\times A)-C^j\cdot(x\times(\dot{A}+\dot{q}\cdot\nabla A))]\rho\,dx. \tag{2.19}$$

Lemma 2.3. *The LHS of the Euler-Lagrange equations equals* $IC^*\omega$

The proof is a straightforward computation in view of (2.17).

Step ii) Now let us proceed to the equation (2.6). Introduce $E=-\nabla\cdot A_0-\dot{A}$, $B=\nabla\times A$, make the mentioned change of the variables and obtain that (2.6) reads

$$I\dot{\omega}=\int x\times(-\nabla\cdot A_0-\dot{A})\rho\,dx+\int x\times(\dot{q}\times(\nabla\times A))\rho\,dx+\int x\times((\omega\times x)\times(\nabla\times A))\rho\,dx.$$

Introduce the differentiations $\nabla_\alpha=(\nabla_{\alpha_1},\nabla_{\alpha_3},\nabla_{\alpha_3})$ in the angles around the coordinate axes; one has $\nabla_\alpha=x\times\nabla$. Then, after partial integration the first integral becomes

$$-\int(x\times\dot{A})\rho\,dx+\int A_0\nabla_\alpha\rho\,dx=-\int(x\times\dot{A})\rho\,dx,$$

since $\nabla_\alpha\rho=0$ in view of spherical symmetry of ρ. Similarly, the second integral transforms to

$$-\int[x\times(\dot{q}\cdot\nabla)A+(\dot{q}\cdot\nabla)\nabla_\alpha]\rho\,dx=-\int(x\times(\dot{q}\cdot\nabla)A)\rho\,dx.$$

For the third integral we apply the identity

$$x\times[(\omega\times x)\times(\nabla\times A)]=(\omega\times x)(x\cdot(\nabla\times A))$$

and after partial integration obtain

$$\int[x(\omega\cdot A)-(\omega\cdot x)A]\rho\,dx-\int(\omega\times x)(A\cdot\nabla_\alpha)\rho\,dx=\int[x(\omega\cdot A)-(\omega\cdot x)A]\rho\,dx.$$

Finally, (2.6) reads

$$I\dot{\omega}=\mathcal{T}:=-\int(x\times\dot{A})\rho\,dx-\int(x\times(\dot{q}\cdot\nabla)A)\rho\,dx+\int[x(\omega\cdot A)-(\omega\cdot x)A]\rho\,dx. \tag{2.20}$$

Step iii) Let us introduce the vector of the RHS of the Euler-Lagrange equations:

$$\mathcal{J}=\begin{pmatrix}\int[(\frac{\partial C}{\partial\Phi_1}\dot{\Phi}-\dot{C}^1)\cdot(x\times A)-C^1\cdot(x\times(\dot{A}+\dot{q}\cdot\nabla A))]\rho\,dx\\ \\ \int[(\frac{\partial C}{\partial\Phi_2}\dot{\Phi}-\dot{C}^2)\cdot(x\times A)-C^2\cdot(x\times(\dot{A}+\dot{q}\cdot\nabla A))]\rho\,dx\\ \\ \int[(\frac{\partial C}{\partial\Phi_3}\dot{\Phi}-\dot{C}^3)\cdot(x\times A)-C^3\cdot(x\times(\dot{A}+\dot{q}\cdot\nabla A))]\rho\,dx\end{pmatrix}.$$

To complete the proof of the theorem it remains to prove the following statement:

Lemma 2.4. *One has*

$$C^*\mathcal{T}=\mathcal{J}. \tag{2.21}$$

For the proof let us recall that $\omega=C\dot{\Phi}$ and note that the matrix C^* does not depend on x and applies directly to the integrands in $\mathcal{T}$. Then (2.21) follows by a straightforward computation. The proof of the theorem is complete. □

3 Hamiltonian Form of Dynamical Equations

According to the general formalism the Euler-Lagrange equations are equivalent to the Hamiltonian equations. Thus, the corresponding Hamiltonian equation should coincide with the Lorentz torque equation multiplied by the matrix C^*.

A difficulty arises from the fact that the Legéndre transform is not completely invertible. In detail, let us apply the Legéndre transform to the Lagrangian

$$\mathcal{L}(\mathcal{A},q,\Phi;\dot{\mathcal{A}},\dot{q},\dot{\Phi}) = \frac{1}{2}\int(E^2-B^2)dx+\frac{1}{2}\dot{q}^2+\frac{I}{2}\omega^2$$

$$-\int A_0(x+q)\rho\,dx+\int(\dot{q}+\omega\times x)A(x+q)\rho\,dx,$$

where $\mathcal{A}=(A_0,A)$, $\Phi=(\varphi,\psi,\theta)$, $\dot{\Phi}=(\dot{\varphi},\dot{\psi},\dot{\theta})$, and it is substituted $E=-\nabla A_0-\dot{A}$, $B=\nabla\times A$, $\omega=C\dot{\Phi}$. For canonical momenta one has

$$P_{\mathcal{A}}=\frac{\delta\mathcal{L}}{\delta\dot{\mathcal{A}}}=(0,\nabla A_0+\dot{A})=(0,-E),\quad p=\frac{\partial\mathcal{L}}{\partial\dot{q}}=\dot{q}+\int A(x+q)\rho\,dx,$$

$$P_\Phi=\frac{\partial\mathcal{L}}{\partial\dot{\Phi}}=\frac{\partial}{\partial\dot{\Phi}}(\frac{I}{2}(C\dot{\Phi})^2+C\dot{\Phi}\cdot M),$$

where $M=M(A,q):=\int(x\times A(x+q))\rho\,dx$. Since

$$\frac{\partial}{\partial\dot{\Phi}}\frac{I}{2}C\dot{\Phi}\cdot C\dot{\Phi}=\frac{I}{2}(2C^*C\dot{\Phi})=IC^*C\dot{\Phi},\ \ \frac{\partial}{\partial\dot{\Phi}}C\dot{\Phi}\cdot M=C^*M,$$

we obtain

$$P_\Phi=IC^*C\dot{\Phi}+C^*M.$$

To invert the Legéndre transform let us note that 1) $\mathcal{L}$ does not depend on $\dot{A}_0$; 2) $\dot{q}=p-\int A(x+q)\rho\,dx$; and 3)

$$\dot{\Phi}=\frac{1}{I}(C^*C)^{-1}[P_\Phi-C^*M]=\frac{1}{I}C^{-1}(C^*)^{-1}[P_\Phi-C^*M]=\frac{1}{I}[(C^*C)^{-1}P_\Phi-C^{-1}M].$$

So, the invertibility is up to the invertibility of the matrix C^*C, the inverse matrix $(C^*C)^{-1}$ has a singularity at $\sin\theta=0$. Further,

$$\omega=C\dot{\Phi}=\frac{1}{I}[(C^*)^{-1}P_\Phi-M],$$

here again the matrix $(C^*)^{-1}$ is singular at $\sin\theta=0$.

In the region $\sin\theta\neq0$ one can obtain the Hamiltonian

$$\mathcal{H}(\mathcal{A},q,\Phi;P_{\mathcal{A}},p,P_\Phi):=(P_{\mathcal{A}}\cdot\dot{\mathcal{A}}+p\cdot\dot{q}+P_\Phi\cdot\dot{\Phi}-\mathcal{L})|_{\dot{\mathcal{A}}\to P_{\mathcal{A}},\dot{q}\to p,\dot{\Phi}\to P_\Phi}$$

by a straightforward computation.

Lemma 3.1. *The Hamiltonian reads*

$$\mathcal{H}(\mathcal{A},q,\Phi;P_{\mathcal{A}},p,P_\Phi)=\frac{1}{2}\int(E^2+B^2)dx+\frac{1}{2}\dot{q}^2+\frac{I}{2}\omega^2+\int E\cdot\nabla A_0dx+\int A_0(x+q)\rho dx, \tag{3.1}$$

where it is inserted $E=-\nabla A_0-\dot{A}$, $B=\nabla\times A$, $\dot{q}=p-\int A(x+q)\rho dx$, $\omega=(1/I)[(C^*)^{-1}P_\Phi-M]$.

Now, due to the explicit form of the Hamiltonian, we can overcome the difficulty arising from the singularity. First let us note that the final form of the Hamilton equation

$$\dot{P}_\Phi=-\frac{\partial\mathcal{H}}{\partial\Phi} \tag{3.2}$$

does not involve singularities. Indeed, since $P_\Phi=C^*(I\omega+M)$, the LHS of (3.2) reads $\dot{P}_\Phi=\dot{C}^*(I\omega+M)+C^*(I\dot{\omega}+\dot{M})$. Recall that $M=\int x\times A(x+q)\rho dx$, then

$$\dot{M}=\int x\times(\dot{A}(x+q)+(\dot{q}\cdot\nabla)A(x+q))\rho dx.$$

For the RHS of (3.2) we have

$$\frac{\partial\mathcal{H}}{\partial\Phi}=\frac{\partial}{\partial\Phi}(\frac{I}{2}\omega^2)=\frac{\partial}{\partial\Phi}\frac{I}{2}\left(\frac{1}{I}((C^*)^{-1}P_\Phi-M)\right)^2=\frac{\partial}{\partial\Phi}\frac{1}{2I}(DP_\Phi-M)^2,\ \text{where } D:=(C^*)^{-1}.$$

Further,

$$\frac{\partial}{\partial\Phi}\frac{1}{2I}(DP_\Phi-M)^2=\frac{1}{2I}2(DP_\Phi-M)\cdot(\frac{\partial}{\partial_\Phi}D)P_\Phi=\omega\cdot(\frac{\partial}{\partial_\Phi}D)P_\Phi$$

Since $DC^*=E$ and $\frac{\partial}{\partial\Phi}(DC^*)=0$, one has

$$\frac{\partial}{\partial\Phi}D=-(C^*)^{-1}(\frac{\partial}{\partial\Phi}C^*)(C^*)^{-1},$$

that is three matrix equalities. Then

$$\omega\cdot(\frac{\partial}{\partial\Phi}D)P_\Phi=-\omega\cdot\left[(C^*)^{-1}(\frac{\partial}{\partial\Phi}C^*)(C^*)^{-1}\right]C^*(I\omega+M)=$$

$$-((C^*)^{-1})^*\omega\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M)=-C^{-1}\omega\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M)=-\dot{\Phi}\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M).$$

Finally, the Hamiltonian equation (3.2) reads

$$\dot{C}^*(I\omega+M)+C^*(I\dot{\omega}+\dot{M})=\dot{\Phi}\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M). \tag{3.3}$$

Let us rewrite it as

$$\dot{C}^*(I\omega+M)+C^*I\dot{\omega}=-C^*\int x\times(\dot{A}(x+q)+(\dot{q}\cdot\nabla)A(x+q))\rho dx+\dot{\Phi}\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M). \tag{3.4}$$

On the other hand, the Lorentz torque equation (2.6) multiplied by the matrix C^* reads

$$C^*I\dot{\omega}=-C^*\int x\times\dot{A}(x+q)\rho dx+C^*\int x\times(\dot{q}\times(\nabla\times A(x+q)))\rho dx$$

$$+C^* \int x \times ((\omega \times x) \times (\nabla \times A(x+q)))\rho\, dx. \tag{3.5}$$

To see that (3.4) coincides with (3.5) it remains to check that

$$-\int x \times ((\dot{q}\cdot\nabla)A(x+q))\rho\, dx = \int x \times (\dot{q} \times (\nabla \times A(x+q)))\rho\, dx$$

and

$$\dot{C}^*(I\omega + M) = \dot{\Phi}\cdot(\frac{\partial}{\partial\Phi}C^*)(I\omega+M) - C^* \int x \times ((\omega \times x) \times (\nabla \times A(x+q)))\rho\, dx.$$

This is checked by a straightforward computation involving partial integration and taking the spherical symmetry of ρ into account.

Thus, we come to our final result.

Theorem 3.2. *Let the Hamiltonian be given by (3.1). Then the Hamilton equation* $\dot{P}_\Phi = -\frac{\partial\mathcal{H}}{\partial\Phi}$ *in the form (3.3) coincides with the Lorentz torque equation (2.6) multiplied by the matrix* C^*.

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