Communications in **M**athematical **A**nalysis
Volume **17**, Number 2, pp. 14–23 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Meixner Polynomials and Representations of the 3D Lorentz Group SO$(2,1)$

Natig M. Atakishiyev *
Instituto de Matemáticas, Unidad Cuernavaca
Universidad Nacional Autónoma de México
Cuernavaca 62251, Morelos, México

Aynura M. Jafarova †
Institute of Mathematics and Mechanics
Azerbaijan National Academy of Sciences
Baku AZ 1141, Azerbaijan

Elchin I. Jafarov ‡
Institute of Physics
Azerbaijan National Academy of Sciences
Baku AZ 1143, Azerbaijan

(Communicated by Vladimir Rabinovich)

Abstract

We argue that the Meixner polynomials of a discrete variable are actually "encoded" within appropriate infinite-dimensional irreducible unitary representations of the three-dimensional Lorentz group $SO(2,1)$. Hence discrete series of irreducible unitary representation spaces of the non-compact group $SO(2,1)$ can be naturally interpreted as discrete versions of the linear harmonic oscillator in standard non-relativistic quantum mechanics.

AMS Subject Classification: 33D45, 39A70, 47B39

Keywords: Meixner polynomials; irreducible unitary representations; Lorentz group; harmonic oscillator

1 Introduction

Classical orthogonal polynomials of a discrete variable (the Hahn, Meixner, Charlier, and Kravchuk polynomials) are extensively used in physics and mathematics [1]. In particular,

*E-mail address: natig@matcuer.unam.mx
†E-mail address: aynure.jafarova@gmail.com
‡E-mail address: e.jafarov@physics.ab.az

discrete quantum oscillator models, which are constructed in terms of these polynomials of the hypergeometric type, find a large number of applications in signal [2]-[4] and in optical image processing [5, 6]. On the other hand, it is well known that explicit forms of irreducible representations of the classical Lie groups are essentially built with the aid of the commutation relations for the generators of those groups, which form associated with them Lie algebras [7, 8]. Recently it has became clear that the commutation relations for the generators of the compact three-dimensional rotation group $SO(3)$ and difference equations for the Kravchuk polynomials of a discrete variable are actually closely interconnected: the simplest solutions of equations, that define explicit forms of irreducible unitary representations group $SO(3)$, are found in terms of the Kravchuk polynomials and their discrete orthogonality weight functions [9].

The present work is devoted to showing that the approach followed in [9] is general enough to inquire into the building of explicit representations for the non-compact three-dimensional Lorentz group $SO(2,1)$ in terms of the Meixner polynomials of a discrete variable. We contend that the discrete Meixner quantum oscillator model [10] is actually "encoded" within infinite-dimensional irreducible unitary representations of the Lorentz group $SO(2,1)$.

A summary of what the remaining sections of this paper contains is as follows. In section 2 we present some basic background facts about the Meixner polynomials and associated with them functions, which are then used in section 3 in order to find the explicit forms of irreducible representations of the three-dimensional Lorentz group $SO(2,1)$. Hence discrete series of irreducible representations of the group $SO(2,1)$ are naturally interpreted as discrete versions of the linear quantum harmonic oscillator in terms of the Meixner functions. Section 4 contains concluding remarks. Finally, the appendix concludes this work with a straightforward derivation of the defining identities for the raising and lowering operators, associated with the Meixner functions.

Throughout this exposition we employ standard notations of the theory of special functions (see, for example, [11]-[13]) and of non-relativistic quantum mechanics [14].

2 Meixner polynomials and functions

The Meixner polynomials in the variable x of degree $n = 0,1,2,...$, are a two-parameter family of Gauss hypergeometric polynomials, [1]

$$M_n(x;\beta,\gamma) := {}_2F_1\left(-n,-x;\beta;1-\gamma^{-1}\right), \tag{2.1}$$

for $\beta > 0$ and $0 < \gamma < 1$ [13]. Since Gauss's hypergeometric function ${}_2F_1(a,b;c;z)$ is symmetric with respect to the numerator parameters a and b, from the definition (2.1) it follows at once that the Meixner polynomials $M_n(x;\beta,\gamma)$ *are self-dual*:

$$M_n(m;\beta,\gamma) = M_m(n;\beta,\gamma), \qquad n,m = 0,1,2,\ldots. \tag{2.2}$$

[1]We remind the reader that Gauss hypergeometric polynomials are usually particular cases of the Gauss hypergeometric functions ${}_2F_1(a,b;c;z) := \sum_{k=0}^{\infty} \frac{(a)_k(b)_k}{(c)_k k!} z^k$, $(a)_k := \Gamma(a+k)/\Gamma(a)$, with one of the parameters a or b being equal to a nonnegative integer, $-n$. Since $(-n)_k = (-1)^k n!/(n-k)!$ when $k \le n$ and $(-n)_k = 0$ when $k > n$ by definition, this type of the Gauss hypergeometric polynomials of degree n in the variable z are explicitly given as ${}_2F_1(-n,b;c;z) := \sum_{k=0}^{n} \frac{(b)_k}{(c)_k k!} C_n^k z^k$, where C_n^k are the binomial coefficients.

Their *discrete orthogonality relation* is

$$\sum_{m=0}^{\infty} M_n(m;\beta,\gamma) M_{n'}(m;\beta,\gamma)\rho(m) \,=\, d_n\,\delta_{nn'} \tag{2.3}$$

with respect to the weight function and square norm

$$\rho(x) = \frac{(\beta)_x\,\gamma^x}{\Gamma(x+1)}\,, \qquad d_n = \frac{n!}{\gamma^n\,(\beta)_n\,(1-\gamma)^\beta}\,. \tag{2.4}$$

Note that combining (2.2) and (2.3) leads to *the completeness (or closure) relations* for the Meixner polynomials $M_n(x;\beta,\gamma)$ of the form

$$\sum_{m=0}^{\infty} M_m(n;\beta,\gamma) M_m(n';\beta,\gamma)\rho(m) \,=\, d_n\,\delta_{nn'}\,. \tag{2.5}$$

In other words, contrary to the continuous case when the two properties of the type (2.3) and (2.5) require separate proofs, here (2.3) and (2.5) are mutually interconnected because of the self-duality property (2.2).

The Meixner polynomials (2.1) satisfy the three-term recurrence relations [13]

$$\begin{aligned}(\gamma-1)\,x\,M_n(x;\beta,\gamma) \,&=\, \gamma\,(n+\beta)\,M_{n+1}(x;\beta,\gamma)\\ &-[n+\gamma(n+\beta)]\,M_n(x;\beta,\gamma) + n\,M_{n-1}(x;\beta,\gamma)\end{aligned} \tag{2.6}$$

and the difference equation in the real argument

$$\Big[\gamma\,(x+\beta)\,T_+ + x\,T_- - (1+\gamma)(x+\beta/2) + (1-\gamma)(n+\beta/2)\Big]\,M_n(x;\beta,\gamma) = 0\,, \tag{2.7}$$

where by definition $T_\pm f(x) := f(x\pm 1)$.

We recall that *forward shift and backward shift operators* for the Meixner polynomials $M_n(x;\beta,\gamma)$ are of the form

$$\beta\gamma\Big(T_+ - 1\Big)M_n(x;\beta,\gamma) \,=\, n\,(\gamma-1)\,M_{n-1}(x;\beta+1,\gamma) \tag{2.8}$$

and

$$\Big[\gamma(x+\beta-1) - x\,T_-\Big]M_n(x;\beta,\gamma) \,=\, \gamma(\beta-1)\,M_{n+1}(x;\beta-1,\gamma)\,, \tag{2.9}$$

respectively [13].

The orthonormalized *Meixner functions* are defined as

$$f_n(x;\beta,\gamma) \,:=\, (-1)^n\,\sqrt{\rho(x)/d_n}\;M_n(x;\beta,\gamma)\,. \tag{2.10}$$

Evidently, most properties of the Meixner functions follow from those of the Meixner polynomials. In particular, from (2.3) and (2.5) it is plain that

$$\sum_{m=0}^{\infty} f_n(m;\beta,\gamma)\,f_{n'}(m;\beta,\gamma) \,=\, \delta_{nn'} \tag{2.11}$$

and

$$\sum_{m=0}^{\infty} f_m(n;\beta,\gamma)\, f_m(n';\beta,\gamma) \;=\; \delta_{nn'}\,. \tag{2.12}$$

The three-term recurrence relation (2.6) is equivalent to

$$\Big[(1-\gamma)x-(1+\gamma)n-\beta\gamma\Big] f_n(x;\beta,\gamma) = \sqrt{\gamma}\Big[\kappa_{n+1} f_{n+1}(x;\beta,\gamma)+\kappa_n f_{n-1}(x;\beta,\gamma)\Big], \tag{2.13}$$

where $\kappa_n := \sqrt{n\,(n+\beta-1)}$. Also, from the equation (2.7) it follows that the Meixner functions (2.10) are eigenfunctions of the difference Meixner "Hamiltonian" operator

$$H^{(M)} \;=\; \frac{1}{1-\gamma}\Big\{(1+\gamma)\,(x+\beta/2) - \sqrt{\gamma}\Big[\mu(x+1)\,T_+ + \mu(x)\,T_-\Big]\Big\},$$

$$\mu(x) := \sqrt{x(x+\beta-1)}\,, \tag{2.14}$$

with eigenvalues

$$\lambda_n \;=\; n + \beta/2\,. \tag{2.15}$$

The factorization of this difference "Hamiltonian" $H^{(M)}$ leads to the raising and lowering operators, explicitly defined as (the details are given more completely in [10])

$$J_+ = \frac{1}{\sqrt{\gamma}}\Big[x+\beta/2-H^{(M)}\Big] - \mu(x+1)\,T_+\,, \tag{2.16}$$

$$J_- = \frac{1}{\sqrt{\gamma}}\Big[x+\beta/2-H^{(M)}\Big] - \mu(x)\,T_-\,. \tag{2.17}$$

A straightforward derivation of the defining identities

$$J_+\, f_n(x;\beta,\gamma) = \kappa_{n+1}\, f_{n+1}(x;\beta,\gamma)\,, \qquad J_-\, f_n(x;\beta,\gamma) = \kappa_n\, f_{n-1}(x;\beta,\gamma)\,, \tag{2.18}$$

where $\kappa_n := \sqrt{n\,(n+\beta-1)}$, is given in the appendix.

3 Irreducible representations of the group $SO(2,1)$

It is well known that in the study of representations of the non-compact three-dimensional Lorentz group $SO(2,1)$ (or its covering group $SU(1,1)$) one employs essentially the algebraic properties of the generators of this group K_j, $j = 0,1,2$ (see, for example, [15, 16]). These generators form the closed Lie algebra $so(2,1)$ (or $su(1,1) \simeq sp(2,\Re) \simeq sl(2,R)$) with the commutation relations

$$[K_2,K_1] = \mathrm{i}\,K_0\,, \qquad [K_2,K_0] = \mathrm{i}\,K_1\,, \qquad [K_0,K_1] = \mathrm{i}\,K_2\,, \tag{3.1}$$

where by definition $[A,B] := AB-BA$. Unitary irreducible representations of the algebra $so(2,1)$ are known to be characterized by eigenvalues of *the invariant* (that is, commuting with all three generators K_j) *Casimir operator*

$$C := K_0^2 - K_1^2 - K_2^2 \;=\; \nu\,(\nu-1)I\,, \tag{3.2}$$

where I is the identity operator. There are three types of irreducible unitary representations of the group $SO(2,1)$: principal (continuous), discrete and complementary series representations. In this work we will be concerned only with discrete series irreducible unitary representations of the group $SO(2,1)$.

It is important to observe from the outset that from the $so(2,1)$ commutation relations (3.1) it follows at once that the double commutator of the generators K_0 and K_1 is equal to

$$\Big[K_0,[K_0,K_1]\Big] = \mathrm{i}[K_0,K_2] = K_1 . \tag{3.3}$$

This circumstance is essentially exploited in this work in the following way.

Recall that quantum-mechanical analogue of classical Newton's equation $m\frac{dv}{dt} = -\frac{dU}{dx}$ for a linear harmonic oscillator is written in terms of the position operator x and the Hamiltonian $\hat{H} := p^2/2m + m\omega^2 x^2/2$ as [14, 17]

$$\Big[\hat{H},[\hat{H},x]\Big] = (\hbar\omega)^2\, x . \tag{3.4}$$

A comparison of (3.3) with (3.4) shows that, with proper normalizations, the generators K_1 and K_0 can be *also* interpreted as the position operator X and the Hamiltonian H of some *discrete model* of the linear quantum harmonic oscillator. Taking into account that the momentum operator $\hat{P}$ in quantum mechanics is defined as $\omega\hat{P} := \mathrm{i}[\hat{H},x]$, one concludes that the association

$$K_1 \Rightarrow X, \qquad K_2 \Rightarrow -P, \qquad K_0 \Rightarrow H, \tag{3.5}$$

enables one to interpret the commutation relations (3.1) of the Lie algebra $so(2,1)$ as a closed defining algebra for a triplet X, P and H with the commutations relations

$$[X,H] = \mathrm{i}\,P, \qquad [H,P] = \mathrm{i}\,X, \qquad [X,P] = \mathrm{i}\,H, \tag{3.6}$$

and, consequently,

$$\Big[H,\big[H,X\big]\Big] = X . \tag{3.7}$$

To establish what kind of an oscillator model emerges from this interpretation (3.5) of the commutation relations (3.1), one recalls the following. To find the explicit form of an irreducible representation of the Lorentz group $SO(2,1)$ it is more convenient to consider the linear combinations of the generators K_1 and K_2 in the form $K_\pm = \pm\mathrm{i}\,K_1 - K_2$. Indeed, from (3.1) it then follows that

$$[K_-,K_+] = 2\,K_0, \qquad [K_0,K_\pm] = \pm\,K_\pm, \tag{3.8}$$

which means that the operators $K_\pm$ are actually *step (or raising and lowering, respectively) operators* : if $\vec{f}_m$ is an eigenvector of the operator K_0, *i.e.,* $K_0\vec{f}_m = (m+\nu)\vec{f}_m$, then the vectors $K_\pm\vec{f}_m$ represent eigenvectors $\vec{f}_{m\pm1}$ of the same operator K_0. By using this property of the step operators $K_\pm$, one can prove, pure algebraically and, most importantly, without employing explicit forms (realizations) of the generators $K_\pm$ and K_0, that for *any irreducible representation* of $SO(2,1)$ the operators $K_\pm$ and K_0 define an orthogonal basis consisting of the normalized eigenvectors of K_0 by the equations

$$K_+\vec{f}^{(\nu)}_m = \kappa^{(\nu)}_{m+1}\,\vec{f}^{(\nu)}_{m+1}, \qquad K_-\vec{f}^{(\nu)}_m = \kappa^{(\nu)}_m\,\vec{f}^{(\nu)}_{m-1}, \qquad K_0\,\vec{f}^{(\nu)}_m = (m+\nu)\,\vec{f}^{(\nu)}_m , \tag{3.9}$$

where $m = 0,1,2,...$ and $\kappa_m^{(\nu)} := \sqrt{m(m-1+2\nu)}$.

In terms of the initial generators K_j, $j = 0,1,2$, the equations (3.9) in *the canonical basis* $\vec{f}_m^{(\nu)}$ can be written as

$$
\begin{aligned}
K_1 \vec{f}_m^{(l)} &= \frac{1}{2\mathrm{i}} \left[\kappa_{m+1}^{(\nu)} \vec{f}_{m+1}^{(\nu)} - \kappa_m^{(\nu)} \vec{f}_{m-1}^{(\nu)} \right], \\
K_2 \vec{f}_m^{(\nu)} &= -\frac{1}{2} \left[\kappa_{m+1}^{(\nu)} \vec{f}_{m+1}^{(\nu)} + \kappa_m^{(\nu)} \vec{f}_{m-1}^{(\nu)} \right], \\
K_0 \vec{f}_m^{(\nu)} &= (m+\nu) \vec{f}_m^{(\nu)}.
\end{aligned}
\tag{3.9′}
$$

The problem of finding the form of an irreducible representation of the Lorentz group $SO(2,1)$ thus reduces to that of solving the equations (3.9), that is, finding explicit forms for the triplet of operators K_j, satisfying the commutation relations (3.1) and exhibiting properties (3.9′). So the key assumption in this strategy of constructing irreducible representation spaces is that the generator K_0 does have eigenvectors, associated with the linear spectrum of K_0.

Now it is straightforward to verify that:

Proposition. One of the simplest solutions of equations (3.9′) can be constructed in terms of the Meixner functions $f_n(x;\beta,\gamma)$, defined by (2.10).

Proof. Since the functions $f_n(x;\beta,\gamma)$ are eigenvectors of the difference "Hamiltonian" operator $H^{(M)}$, one may associate the $H^{(M)}$ with the generator K_0; thus the third line in (3.9′) holds with $\nu = \beta/2$. As for the first two lines in (3.9′), one associates with the generators K_1 and K_2 the difference operators

$$
J_1 := \frac{1}{2\mathrm{i}} \Big(J_+ - J_- \Big), \qquad J_2 := -\frac{1}{2} \Big(J_+ + J_- \Big), \tag{3.10}
$$

respectively, where the raising J_+ and lowering J_- operators for the Meixner functions $f_n(x;\beta,\gamma)$ are defined by (2.16) and (2.17). Thus one actually arrives at the explicit form of a discrete version of the linear quantum harmonic oscillator in terms of the Meixner functions $f_n(x;\beta,\gamma)$. □

An important aspect to observe at this point is that the Meixner functions $f_n(x;\beta,\gamma)$ are the *discrete eigenvectors* of the generator K_0 in (3.9′). Notice that contrary to the case of continuous representations [2], this discrete basis does not depend explicitly on the group parameters.

In closing this section it should be pointed out that in the three-volume encyclopedic monograph by N.Ja.Vilenkin and A.U.Klimyk the Meixner polynomials had been attributed to matrix elements of irreducible representations of the Lie group $SU(1,1)$, treated "as functions of column index" (see page 346 in [16]). So the algebraic reasoning in this section reveals that those "matrix elements as functions of column index" are simply matrix elements in *the canonical basis*, provided that the generator K_0 in (3.9′) is selected as the difference "Hamiltonian" operator $H^{(M)}$. Thus it becomes transparent how the former ones emerge from the group-theoretical point of view.

[2]Recall that the representation $g \to T_g$ is called *continuous* if the elements of the matrix T_g are continuous functions of g (see, for example, [7]).

4 Concluding remarks

We have demonstrated above that the generators of the three-dimensional Lorentz group $SO(2,1)$ can be interpreted as a triplet of the operators $\{X,P,H\}$, which define a discrete (finite) model of *the linear quantum harmonic oscillator* in terms of the Meixner functions.

A final remark concerns the possibility of studying the group-theoretic properties of families of discrete polynomials that are *not* associated with some Lie algebra. A recent work by Kalnins, Miller Jr. and Post [19] discussed the generic three-parameter second-order superintegrable system $S9$ in two dimensions in detail (see also [20]). It turns out that this superintegrable model is closely interconnected with hypergeometric orthogonal polynomials from the Askey scheme [13]. In particular, various function space realizations of the quadratic Racah–Wilson algebra, which is the symmetry algebra behind the $S9$ superintegrable model, can be associated with *all* hypergeometric polynomials in the Askey scheme. These remarkable works [19, 20] thus reveal the group-theoretic context of such intricate orthogonal families as the Wilson and Racah polynomials that satisfy second-order difference equations with quadratic spectra.

5 Appendix

We prove here two identities

$$J_+ f_n(x;\beta,\gamma) = \kappa_{n+1} f_{n+1}(x;\beta,\gamma), \qquad J_- f_n(x;\beta,\gamma) = \kappa_n f_{n-1}(x;\beta,\gamma), \tag{A.1}$$

where $\kappa_n = \sqrt{n(n+\beta-1)}$ and the operators $J_\pm$ are explicitly defined as

$$J_+ = \frac{1}{\sqrt{\gamma}}\left[x+\beta/2-H^{(M)}\right]-\mu(x+1)\,T_+\,, \tag{A.2}$$

$$J_- = \frac{1}{\sqrt{\gamma}}\left[x+\beta/2-H^{(M)}\right]-\mu(x)\,T_-\,, \tag{A.3}$$

with $\mu(x) = \sqrt{x(x+\beta-1)}$.

We begin with the first identity in (A.1): since the Meixner functions $f_n(x;\beta,\gamma)$ are eigenfunctions of the difference operator $H^{(M)}$ with the eigenvalues $\lambda_n = n+\beta/2$ (see (2.14) and (2.15)), one obtains that

$$\begin{aligned}
J_+ f_n(x;\beta,\gamma) &= \left\{\frac{1}{\sqrt{\gamma}}\left[x+\beta/2-H^{(M)}\right]-\mu(x+1)\,T_+\right\} f_n(x;\beta,\gamma)\\
&= \left[\frac{1}{\sqrt{\gamma}}(x-n)-\mu(x+1)\,T_+\right] f_n(x;\beta,\gamma)\\
&= \frac{(-1)^n}{\sqrt{d_n}}\left[\frac{1}{\sqrt{\gamma}}(x-n)-\mu(x+1)\,T_+\right]\rho^{1/2}(x)\,M_n(x;\beta,\gamma)\\
&= \frac{(-1)^n}{\sqrt{d_n}}\left[\frac{1}{\sqrt{\gamma}}(x-n)\rho^{1/2}(x)\,M_n(x;\beta,\gamma)-\mu(x+1)\rho^{1/2}(x+1)\,M_n(x+1;\beta,\gamma)\right]
\end{aligned}$$

$$= \frac{(-1)^n}{\sqrt{\gamma\, d_n}}\,\rho^{1/2}(x)\Big[(x-n)\,M_n(x;\beta,\gamma) - \gamma\,(x+\beta)\,M_n(x+1;\beta,\gamma)\Big], \tag{A.4}$$

where at the last step we have used the evident identity

$$\rho(x+1) = \frac{\gamma\,(x+\beta)}{x+1}\,\rho(x) \tag{A.5}$$

for the orthogonality weight function $\rho(x)$, defined in (3.4).

The next step is to use first the forward shift operator (3.8) and then the three-term recurrence relations (3.6) for the Meixner polynomials $M_n(x;\beta,\gamma)$ in order to transform a sum of two terms inside of the square brackets on the last line of (A.4) into

$$\begin{aligned}
&\big[(1-\gamma)\,x - n - \beta\gamma\big]M_n(x;\beta,\gamma) + \frac{n}{\beta}\,(1-\gamma)\,(x+\beta)\,M_{n-1}(x;\beta+1,\gamma)\\
&= n\gamma\,M_n(x;\beta,\gamma) + \frac{n}{\beta}\,(1-\gamma)\,(x+\beta)\,M_{n-1}(x;\beta+1,\gamma)\\
&\quad - n\,M_{n-1}(x;\beta,\gamma) - \gamma\,(n+\beta)\,M_{n+1}(x;\beta,\gamma)\,.
\end{aligned} \tag{A.6}$$

But the sum of the first three terms in (A.6) is zero for it represents a particular case of the relation (*cf* formula (38) on p.103 in [18])

$${}_2F_1\,(a-1,b\,;c\,;z) - (1-z)\,{}_2F_1\,(a,b\,;c\,;z) - (1-b/c)\,z\,{}_2F_1\,(a,b\,;c+1\,;z) = 0$$

between contiguous Gauss hypergeometric series with the parameters $a = 1-n$, $b = -x$, $c = \beta$ and the variable $z = 1-\gamma^{-1}$. This means that only the last term in (A.6) survives and the identity (A.4) consequently reduces to

$$J_+\,f_n(x;\beta,\gamma) = \frac{(-1)^{n+1}}{\sqrt{\gamma\, d_n}}\,\rho^{1/2}(x)\,\gamma\,(n+\beta)\,M_{n+1}(x;\beta,\gamma) = \kappa_{n+1}\,f_{n+1}(x;\beta,\gamma)\,, \tag{A.7}$$

upon use of $\gamma(n+\beta)\,d_{n+1} = (n+1)\,d_n$. This concludes our proof of the first identity in (A.1).

Now it is straightforward to follow the above-employed pattern for proving the second identity in (A.1):

$$J_-\,f_n(x;\beta,\gamma) = \left\{\frac{1}{\sqrt{\gamma}}\left[x+\beta/2 - H^{(M)}\right] - \mu(x)\,T_-\right\} f_n(x;\beta,\gamma)$$

$$= \left[\frac{1}{\sqrt{\gamma}}\,(x-n) - \mu(x)\,T_-\right] f_n(x;\beta,\gamma) = \frac{(-1)^n}{\sqrt{d_n}}\left[\frac{1}{\sqrt{\gamma}}\,(x-n) - \mu(x)\,T_-\right]\rho^{1/2}(x)\,M_n(x;\beta,\gamma)$$

$$= \frac{(-1)^n}{\sqrt{d_n}}\left[\frac{1}{\sqrt{\gamma}}\,(x-n)\rho^{1/2}(x)\,M_n(x;\beta,\gamma) - \mu(x)\rho^{1/2}(x-1)\,M_n(x-1;\beta,\gamma)\right]$$

$$= \frac{(-1)^n}{\sqrt{\gamma\, d_n}}\,\rho^{1/2}(x)\Big[(x-n)\,M_n(x;\beta,\gamma) - x\,M_n(x-1;\beta,\gamma)\Big], \tag{A.8}$$

where at the last step we have used the identity (A.5) (with x shifted by -1, that is, $x \Rightarrow x-1$). But

$$(n-x)\,M_n(x;\beta,\gamma) + x\,M_n(x-1;\beta,\gamma) = n\,M_{n-1}(x;\beta,\gamma)\,, \tag{A.9}$$

which represents a particular case of the second relation of Gauss (*cf* formula (32) on p.103 in [18])

$$(a-b)\,{}_2F_1\,(a,b\,;c\,;z) - a\,{}_2F_1\,(a+1,b;c\,;z) + b\,{}_2F_1\,(a,b+1\,;c\,;z) = 0$$

between contiguous hypergeometric functions with the parameters $a=-n$, $b=-x$, $c=\beta$ and the variable $z=1-\gamma^{-1}$. Taking into account that $\gamma(n+\beta-1)\,d_n = n\,d_{n-1}$, we thus conclude our proof of the second identity in (A.1).

Acknowledgments

Discussions with M.K.Atakishiyeva and K.B.Wolf are gratefully acknowledged. We thank the referee for helpful comments on an earlier version of this paper. The participation of NMA in this work has been partially supported by the CONACyT project CB220692 “Óptica Matemática” and by the *Proyecto de Redes* PROMEP (México).

References

[1] A. F. Nikiforov, S. K. Suslov, and V. B. Uvarov, *Classical Orthogonal Polynomials of a Discrete Variable*, Springer-Verlag, Berlin Heidelberg 1991.

[2] N. M. Atakishiyev and K. B. Wolf, Approximation on a finite set of points through Kravchuk functions. *Revista Mexicana de Física* **40** (1994), pp 366-377.

[3] N. M. Atakishiyev and K. B. Wolf, Fractional Fourier–Kravchuk transform. *J. Opt. Soc. Amer. A*, **14** (1997), pp 1467-1477.

[4] N. M. Atakishiyev, L. E. Vicent, and K. B. Wolf, Continuous vs. discrete fractional Fourier transforms. *Journal of Computational and Applied Mathematics* **107** (1999), pp 73-95.

[5] N. M. Atakishiyev, G. S. Pogosyan, and K. B. Wolf, Finite models of the oscillator. *Physics of Particles and Nuclei* **36** (2005), pp 247-265.

[6] K. B. Wolf, N. M. Atakishiyev, L. E. Vicent, G. Krötzsch, and J. Rueda-Paz, Finite optical Hamiltonian systems. In: *Proceedings of the 22nd Congress of the International Commission for Optics: Light for the Development of the World*, Puebla, México, August 15-19, 2011, R. Rodríguez-Vera and R. Díaz-Uribe, Editors, Proceedings of SPIE, **8011** (2011), pp 1-8.

[7] I. M. Gel’fand, R. A. Minlos, and Z. Ya. Shapiro, *Representations of the rotation and Lorentz groups and their applications*, Macmillan, New York 1963.

[8] D. P. Želobenko, *Compact Lie Groups and their Representations*, American Mathematical Society, Providence, RI, 1973.

[9] M. K. Atakishiyeva, N. M. Atakishiyev, and K. B. Wolf, Kravchuk oscillator revisited. *Journal of Physics: Conference Series* **512** (2014), Art. No.012031, 8 pages.

[10] N. M. Atakishiyev, E. I. Jafarov, Sh. M. Nagiyev, and K. B. Wolf, Meixner oscillators. *Revista Mexicana de Física* **44** (1998), pp 235-244.

[11] G. Gasper and M. Rahman, *Basic Hypergeometric Functions*, Cambridge University Press, Cambridge 2004.

[12] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge 1999.

[13] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer-Verlag, Berlin Heidelberg 2010.

[14] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics (Non-relativistic Theory)*, Pergamon Press, Oxford 1991.

[15] V. Bargmann, Irreducible unitary representations of the Lorentz group. *Annals of Mathematics*, **48** (1947), pp 568-640.

[16] N. Ja. Vilenkin and A. U. Klimyk, *Representations of Lie Groups and Special Functions*, **I**, Kluwer Academic Publishers, Dordrecht, The Netherlands 1991.

[17] I. A. Malkin and V. I. Man'ko, *Dynamical Symmetries and Coherent States of Quantum Systems*, in Russian, Fizmatgiz, Moscow 1979.

[18] A. Erdélyi, W. Magnus, F. Oberhettinger, and F. G. Tricomi, *Higher Transcendental Functions*, **I**, McGraw-Hill, New York 1953.

[19] E. G. Kalnins, W. Miller Jr., and S. Post, Contractions of 2D 2nd order quantum superintegrable systems and the Askey scheme for hypergeometric orthogonal polynomials. *SIGMA*, **9** (2013), Art. No.057, 28 pages.

[20] E. G. Kalnins, W. Miller Jr., and S. Post, Wilson polynomials and the generic superintegrable system on the 2-sphere. *J. Phys. A: Math.Theor.* **40** (2007), pp 11525-11538.