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Soliton Dynamics in an Extended Nonlinear Schrödinger Equation with Inhomogeneous Dispersion and Self-phase Modulation

N. V. Aseeva*
Department of Information systems and technologies
National Research university Higher School of Economics
Nizhny Novgorod 603155, Russia

E. M. Gromov†
Department of Mathematics
National Research university Higher School of Economics
Nizhny Novgorod 603155, Russia

B. A. Malomed‡
Department of Physical Electronics, School of Electrical Engineering, Faculty of Engineering
Tel Aviv University
Tel Aviv, 69978,Izrael

V. V. Tyutin§
Department of Mathematics
National Research university Higher School of Economics
Nizhny Novgorod 603155, Russia

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Abstract

Evolution of solitons is addressed in the framework of an extended nonlinear Schrödinger equation (NLSE), including a pseudo-stimulated-Raman-scattering (pseudo-SRS) term, i.e., a spatial-domain counterpart of the SRS term which is well known as an ingredient of the temporal-domain NLSE in optics. In the present context, it is induced by the underlying interaction of the high-frequency envelope wave with a damped low-frequency wave mode. Also included are spatial inhomogeneity of both the second-

*E-mail address: naseeva@hse.ru
†E-mail address: egromov@hse.ru
‡E-mail address: malomed@post.tau.ac.il
§E-mail address: vtyutin@hse.ru

order dispersion (SOD) and self-phase modulation (SPM). It is shown that the wavenumber downshift of solitons, caused by the pseudo-SRS, may be compensated by an upshift provided by the increasing SPM and SOD coefficients. An analytical solution for solitons is obtained in an approximate form. Analytical and numerical results agree well.

AMS Subject Classification: 35Q51 - Solitons

Keywords: Extended Nonlinear Schrödinger Equation, Soliton Solutions, Stimulated Scattering, Damped Low-Frequency Waves, Inhomogeneity, Second-Order Dispersion, Self-Phase Modulation, Analytical Solutions

1 Introduction

The great interest to the dynamics of solitons is motivated by their ability to travel long distances keeping the particular localized shape, thus transferring the energy and information with little loss. It is commonly known that soliton solutions play a profound role in nonlinear models which deal with the propagation of intensive wave fields in dispersive media: optical pulses and beams in fibers and spatial waveguides, electromagnetic waves in plasma, surface waves on deep water, etc. [1]-[7]. Recently, solitons have also drawn a great deal of interest in plasmonics [8]-[10]. Dynamics of long high-frequency (HF) wave packets is described by the second-order nonlinear dispersive wave theory. The fundamental equation of the theory is the nonlinear Schrödinger equation (NLSE) [11],[12], which includes the second-order dispersion (SOD) and self-phase modulation (SPM). Soliton solutions in this case arise as a result of the balance between the dispersive stretch and nonlinear compression of wave packets. Modeling the dynamics of narrow HF wave packets requires the use of the mode sophisticated third-order nonlinear dispersive wave theory [1], which takes into account higher-order effects, such as the nonlinear dispersion (self-steeping) [14], stimulated Raman scattering (SRS) [14]-[16] and third-order dispersion (TOD). The basic equation of the third-order theory is the accordingly extended version of the NLSE [16]-[20].. Soliton solutions in the framework of the extended NLSE including the TOD and nonlinear dispersion were found in Refs. [21]-[28]. In addition to the solitons, stationary kink waves (shocks) were found in Refs. [29] and [30] as solutions to the extended NLSE with the SRS and nonlinear dispersion terms. This solution exists as the equilibrium between the nonlinear dispersion and SRS. For localized nonlinear wave packets (solitons), the SRS gives rise to the downshift of the soliton spectrum [14]-[16] and eventually leads to destabilization of the solitons. The use of the balance between the SRS and the slope of the gain for the stabilization of solitons in long fiber-optic links was proposed in Ref. [31]. The compensation of the SRS by emission of quasi-linear radiation waves from the soliton's core was considered in Ref. [32]. In addition, the compensation of the SRS in inhomogeneous media was considered in several situations, viz., periodic SOD [33]-[34], shifting zero-dispersion point of the SOD [35], and dispersion-decreasing fibers [36].

Intense short pulses of HF electromagnetic or Langmuir waves in plasmas, as well as HF surface waves in deep stratified water, suffer effective induced damping due to interaction with low-frequency (LF) waves, which, in turn, are subject to the action of viscosity. Such naturally present LF modes may be ion-sound waves in the plasmas, and internal waves in the stratified fluid. The first model for the effective damping of the HF envelop waves induced by the interaction with the dissipative LF modes was proposed in Refs. [23]-[39]. This model gives rise to an extended NLSE with the spatial-domain counterpart of the SRS term, that was named a pseudo-SRS one. The equation containing this term was derived from the model of the HF-LF wave interaction based on a system of equations of the Zakharov's type [40],[41] for the coupled Langmuir and ion-acoustic waves in plasmas. The pseudo-SRS leads to the self-wavenumber downshift of the soliton, similar to what is well known in the temporal domain [1],[13]-[16] and, eventually (also similarly to the situation in nonlinear optical fibers), to destabilization of the solitons. The model elaborated in Refs. [23]-[39] also included smooth spatial variation of the SOD, accounted for by a spatially decreasing SOD coefficient, which may also be a natural ingredient of the corresponding physical settings. The latter property leads to an increase of the soliton's wavenumber, making it possible to compensate the destructive effect of the pseudo-SRS. In this work, we aim to study the dynamics of HF wave packets in media described by the extended NLSE with the pseudo-SRS term, in the combination with SOD and SPM terms whose coefficients are spatially inhomogeneous. The pseudo-SRS term is derived from the underlying Zakharov's system, as outlined above. The balance between this term and the inhomogeneous dispersion parameters, namely, increasing SPM and SOD coefficients, leads to stabilization of the soliton's wavenumber spectrum. An analytical soliton solution for the soliton is found in an approximate form, and corroborated by numerical results.

2 The basic equation and integrals relations

We consider the evolution of slowly varying envelope $U(\xi,t)$ of the intense HF wave field, $U(\xi,t)\exp(i\omega t - ik\xi)$, in the medium with inhomogeneous SOD, inhomogeneous group velocity of the LF waves, $V(\xi)$, which are coupled to the LF ones by the Zakharov's system, which includes the viscosity acting on the LF modes. The respective system of evolution equations for HF amplitude, $U(\xi,t)$, and the LF field, $n(\xi,t)$ (such as the local perturbation of the refractive index in optics), is a modification of the Zakharov's system derived in Refs. [40], [41]:

$$2i\frac{\partial U}{\partial t} + \frac{\partial}{\partial \xi}\left(q(\xi)\frac{\partial U}{\partial \xi}\right) - nU = 0, \tag{2.1}$$

$$\frac{\partial n}{\partial t} + V(\xi)\frac{\partial n}{\partial \xi} - \nu\frac{\partial^2 n}{\partial \xi^2} = -\frac{\partial\left(|U|^2\right)}{\partial \xi} \tag{2.2}$$

where $q(\xi)$ is the SOD coefficient, and ν is the viscosity coefficient. In particular, this system describes the dynamics of intense electromagnetic or Langmuir waves ($U(\xi,t)$) in isotropic plasmas with the striction nonlinearity, taking into account the viscous losses of the ion-sound waves, $n(\xi,t)$. Assuming that the heterogeneity scale of the LF group-wave coefficient, $V(\xi)$, is much larger than the size of the envelope-wave packet, in the third-order approximation of the theory (for the HF wave packets with $\nu/\Delta << V$, where Δ is extension of the wave packet), the nonlinear response of the LF wave to the action of the HF packet includes a weakly nonlocal correction: $n \approx -|U|^2/V(\xi) - \nu\left(\partial |U|^2/\partial\xi\right)/V(0)$. The latter term causes effective losses of the HF waves, induced by the viscosity acting on the LF modes. Thus, the system of Eqs. (2.1), (2.2) is reduced to the following version of the extended NLSE:

$$2i\frac{\partial U}{\partial t}+\frac{\partial}{\partial \xi}\left(q(\xi)\frac{\partial U}{\partial \xi}\right)+2\alpha(\xi)U\,|\,U\,|^2+\mu U\frac{\partial\left(|\,U\,|^2\right)}{\partial \xi}=0, \tag{2.3}$$

where $\alpha(\xi)=1/(2V(\xi))$ is the SPM coefficient, and $\mu=\nu/V(0)$. The last term in Eq. (2.3) represents the spatial counterpart of SRS term, which is well known in the temporal domain. Equation (2.3) with zero boundary conditions at infinity, $U\big|_{\xi\to\pm\infty}\to 0$, gives rise to the following integral relations for field moments, which will be used below:

$$\frac{dN}{dt}\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}|\,U\,|^2 d\xi=0, \tag{2.4}$$

$$2\frac{d}{dt}\int_{-\infty}^{+\infty}K\,|\,U\,|^2 d\xi=-\mu L-\int_{-\infty}^{+\infty}\frac{dq}{d\xi}\left|\frac{\partial U}{\partial \xi}\right|^2+\int_{-\infty}^{+\infty}\frac{d\alpha}{d\xi}\,|\,U\,|^4 d\xi, \tag{2.5}$$

$$\begin{aligned}\frac{dZ}{dt}\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}\left|\frac{\partial U}{\partial \xi}\right|^2 d\xi=-\mu\int_{-\infty}^{+\infty}K\left(\frac{\partial\,|\,U\,|^2}{\partial \xi}\right)^2 d\xi+2\int_{-\infty}^{+\infty}\frac{d\alpha}{d\xi}K\,|\,U\,|^4 d\xi\\+\int_{-\infty}^{+\infty}\alpha K\frac{\partial\left(|\,U\,|^4\right)}{\partial \xi}d\xi+\frac{i}{2}\int_{-\infty}^{+\infty}\frac{dq}{d\xi}\left(\frac{\partial^2 U}{\partial \xi^2}\frac{\partial U^*}{\partial \xi}-\frac{\partial^2 U^*}{\partial \xi^2}\frac{\partial U}{\partial \xi}\right)d\xi,\end{aligned} \tag{2.6}$$

$$\frac{dL}{dt}\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}\left(\frac{\partial\left(|\,U\,|^2\right)}{\partial \xi}\right)^2 d\xi=2\int_{-\infty}^{+\infty}\frac{\partial^2\left(|\,U\,|^2\right)}{\partial \xi^2}\frac{\partial\left(qK\,|\,U\,|^2\right)}{\partial \xi}d\xi, \tag{2.7}$$

$$\frac{dM}{dt}\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}|\,U\,|^4 d\xi=\int_{-\infty}^{+\infty}qK\frac{\partial\left(|\,U\,|^4\right)}{\partial \xi}d\xi, \tag{2.8}$$

$$N\frac{d\bar{\xi}}{dt}\equiv\frac{d}{dt}\int_{-\infty}^{+\infty}qK\,|\,U\,|^2 d\xi, \tag{2.9}$$

where the complex field is represented as $U\equiv\exp(i\phi)$, $U^*\equiv|\,U\,|\exp(-i\phi)$, and $K\equiv\partial\phi/\partial\xi$ is the local wavenumber.

3 Analytical results

3.1 The approximation for the weakly inhomogeneous medium

For the analytical consideration of the wave-packet dynamics, we assume that the scales of the inhomogeneity of both the SOD and SPM coefficients, as well as that of the local wavenumber, K, are much larger than the spatial width of the envelope wave-packet:

$D_{q,\alpha,K} >> D_{|U|}$. Accordingly, we approximate the spatial variation of the wavenumber by the lowest-order expansion, $K(\xi,t) \approx K(\bar{\xi},t) + (\partial K/\partial \xi)_{\bar{\xi}}(\xi - \bar{\xi})$, where the center-of-mass coordinate of the soliton is $\bar{\xi} \equiv N^{-1} \int_{-\infty}^{+\infty} \xi \mid U \mid^2 d\xi$. Then it follows from the imaginary part of Eq. (2.3), under condition $(\partial \mid U \mid /\partial \xi)_{\bar{\xi}} = 0$ (which means that the peak of the soliton's shape is located at its center):

$$\left(\frac{\partial K}{\partial \xi}\right)_{\bar{\xi}} = -\left(\frac{2}{q \mid U \mid}\frac{\partial \mid U \mid}{\partial t} + \frac{1}{q}\frac{dq}{d\xi}K\right)_{\bar{\xi}}. \tag{3.1}$$

For soliton-like packets, taking into account Eqs. (2.4) and (3.1), we obtain

$$K(\xi,t) = k(t)\left(1 - \frac{\alpha'(\bar{\xi})}{\alpha(\bar{\xi})}\left(\xi - \bar{\xi}\right)\right), \tag{3.2}$$

where $k(t) \equiv K(\bar{\xi})$, $\alpha'(\bar{\xi}) = (d\alpha/d\xi)_{\bar{\xi}}$. Then, the system of equations (2.4)-(2.7) can be cast in the form of evolution equations for the following parameters of the wave packets:

$$2N\frac{dk}{dt} = -\mu L - q'(\bar{\xi})Z + \alpha'(\bar{\xi})M, \tag{3.3}$$

$$\frac{dZ}{dt} = \left(-\mu L - 3q'(\bar{\xi})Z + 2k^2 q'(\bar{\xi})N + 2\alpha'(\bar{\xi})\right)k, \tag{3.4}$$

$$\frac{dL}{dt} = 3\left(\frac{q(\bar{\xi})}{\alpha(\bar{\xi})}\alpha'(\bar{\xi}) - q'(\bar{\xi})\right)kL, \tag{3.5}$$

$$\frac{dM}{dt} = \left(\frac{q(\bar{\xi})}{\alpha(\bar{\xi})}\alpha'(\bar{\xi}) - q'(\bar{\xi})\right)kM, \tag{3.6}$$

$$\frac{d\bar{\xi}}{dt} = kq(\bar{\xi}), \tag{3.7}$$

where $q'(\bar{\xi}) = (dq/d\xi)_{\bar{\xi}}$. The fixed point (equilibrium state) of the system of Eqs. (3.3)-(3.7) is achieved under conditions

$$k = 0, \mu L_0 = \alpha'(\bar{\xi}_0)M_0 - q'(\bar{\xi}_0)Z_0, \tag{3.8}$$

where $L_0 = L(0)$, $M_0 = M(0)$, $Z_0 = Z(0)$ are initial values of the respective variables. Thus, in the equilibrium regime, the wave packet propagates with the integral moments keeping their initial values,N, L_0, Z_0, M_0, and the central wavenumber remaining zero. To analyze the dynamics of the wave packet with non-equilibrium parameters in an explicit form, we assume that both the SOD and SPM coefficients are exponential spatial functions,

$$q(\xi) \equiv q_0 \exp(\xi/D_q), \quad \alpha(\xi) \equiv \alpha_0 \exp(\xi/D_\alpha). \tag{3.9}$$

In particular, the realization of fibers with exponential profiles of the SOD and SPM coefficients was demonstrated experimentally in Ref. [42]. By means of substitutions, $\rho \equiv \bar{\xi}/D_q$, $\delta \equiv D_q/D_\alpha$, $y \equiv k\sqrt{N/Z_0}$, $\tau \equiv tq_0\sqrt{N}/\left(D_q\sqrt{Z_0}\right)$, $q_0 = q(0)$,$\alpha_0 = \alpha(0)$, $y_0 = y(0)$, and taking into regard integral relations, $Z/Z_0 = y^2 - y_0^2 exp(-2\rho) + \exp(2\delta\rho - 2\rho)$, $M = M_0 \exp(\delta\rho - \rho)$, $L = L_0 exp(3\delta\rho - 3\rho)$, Eqs. (3.3)-(3.7) reduce to

$$2\frac{dy}{d\tau} = -\lambda \exp(3\delta\rho - 3\rho) + (2\delta - 1)\exp(2\delta\rho - \rho) - y^2 + y_0^2 \exp(-\rho), \tag{3.10}$$

$$\frac{d\rho}{d\tau} = y \exp(\rho), \tag{3.11}$$

where $\lambda \equiv \mu L_0 D_q/(q_0 Z_0)$. The equilibrium state of Eqs. (3.10), (3.11) is $y = 0$, $\lambda = 2\delta - 1$. For $\delta < 1/2$ and $\delta > 2$, it is a fixed point of the center type, while for $1/2 \leq \delta \leq 2$ it is a saddle. The first integral of Eqs. (3.10), (3.11) for initial soliton-like packets is

$$\exp(\rho)y^2 - y_0^2 + \frac{\lambda}{3(\delta-1)}[\exp(3\delta\rho - 3\rho) - 1] + y_0^2[\exp(-\rho) - 1] + 1 - \exp(2\delta\rho - \rho) = 0 \tag{3.12}$$

In Fig. 1, this relation between variables y and ρ is plotted for initial condition $y_0 = 0$ with different δ and λ .

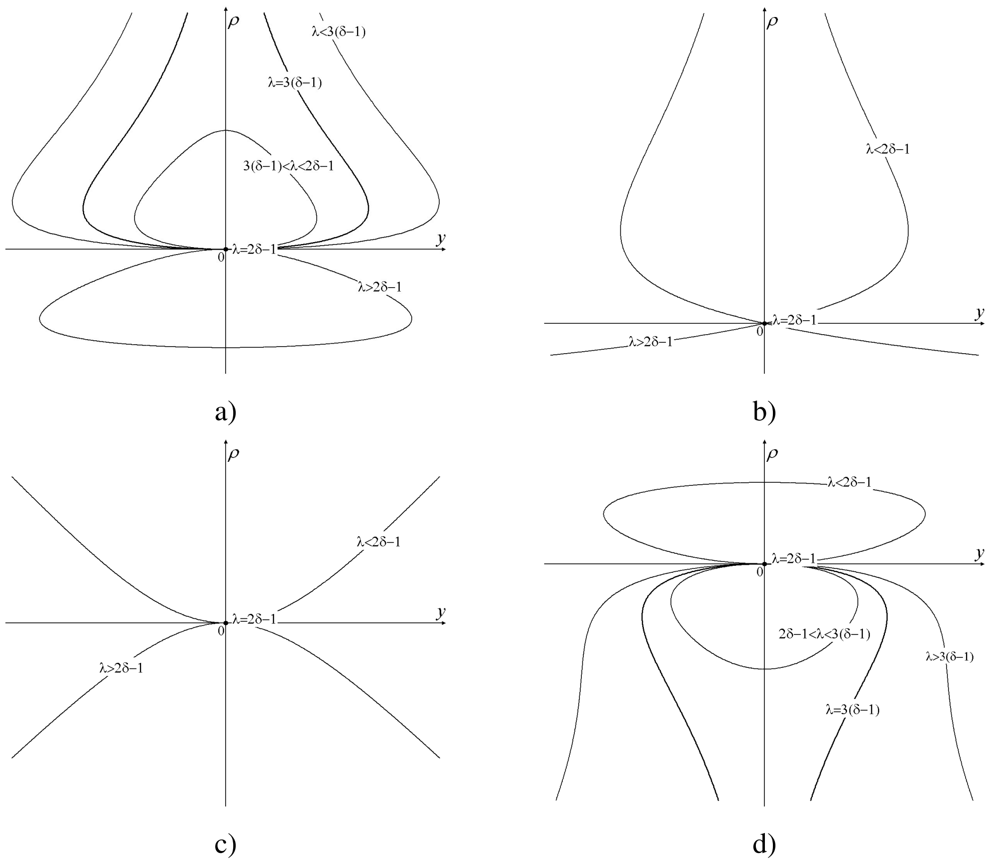


Figure 1. The first integral (3.12) of Eqs. (3.10), (3.11) in the plane (y,ρ) for initial condition $y_0 = 0$ with different values of δ $[(a) : \delta < 1/2, (b) : 1/2 \leq \delta < 1, (c) : 1 \leq \delta \leq 2, (d) : \delta > 2]$, and different values of constant.

3.2 The soliton solution

We now consider solutions of Eq. (2.3) in the form of $U(\xi,t) = \psi(\xi)\exp(i\Omega t)$ for exponential profiles of the SOD and SPM modulations, $q(\xi) \equiv q_0 exp(\xi/D_q)$ and $\alpha(\xi) \equiv \alpha_0 \exp(\xi/D_\alpha)$:

$$q_0 \exp\left(\frac{\xi}{D_q}\right)\frac{d^2\psi}{d\xi^2} + \frac{q_0}{D_q}\exp\left(\frac{\xi}{D_q}\right)\frac{d\psi}{d\xi} + 2\alpha_0 \exp\left(\frac{\xi}{D_\alpha}\right)\psi^3 - 2\Omega\psi + \mu U\frac{d\psi^2}{d\xi} = 0. \tag{3.13}$$

In the spirit of the above analysis, we assume that the corresponding modulation scales are much larger than the width of the envelope-packet, $D_{q,\alpha} >> D_\psi \equiv D_{|U|}$ and use the respective expansion, $\exp(\xi/D_{\alpha,q}) \approx 1 + \xi/D_{\alpha,q}$. Then, a solution to Eq. (3.13) is looked for as $\psi = \psi_0 + \psi_1$, where $\psi_1 \sim \epsilon\psi_0$ is a small correction to ψ_0 , with $\epsilon \sim \xi/D_{\alpha,q} \sim D_\psi/D_{\alpha,q} \sim \mu << \alpha_0, q_0$. Keeping terms of order ϵ, we obtain from (3.13)

$$q_0\frac{d^2\psi_0}{d\xi^2} + 2\alpha_0\psi_0^3 - 2\Omega\psi_0 = 0, \tag{3.14}$$

$$q_0\frac{d^2\psi_1}{d\xi^2} + 2(3\alpha_0\psi_0^2 - \Omega)\psi_1 = -2\frac{\alpha_0}{D_\alpha}\psi_0^3\xi - \frac{q_0}{D_q}\frac{d^2\psi_0}{d\xi^2}\xi - \frac{2}{3}\mu\frac{d(\psi_0^3)}{d\xi} - \frac{q_0}{D_q}\frac{d\psi_0}{d\xi}. \tag{3.15}$$

Equation (3.14) has the standard soliton solution, $\psi_0 = A_0 \sec(\xi/\Delta)$, where $\Delta \equiv \sqrt{q_0}/\left(A_0\sqrt{\alpha_0}\right)$, $\Omega \equiv \alpha_0 A_0^2/2$. Further, Eq. (3.15), with the substitution of $\eta \equiv \xi/\Delta$ and $\Psi \equiv \psi_1/(A_0^2 q'_\eta)$. takes the form

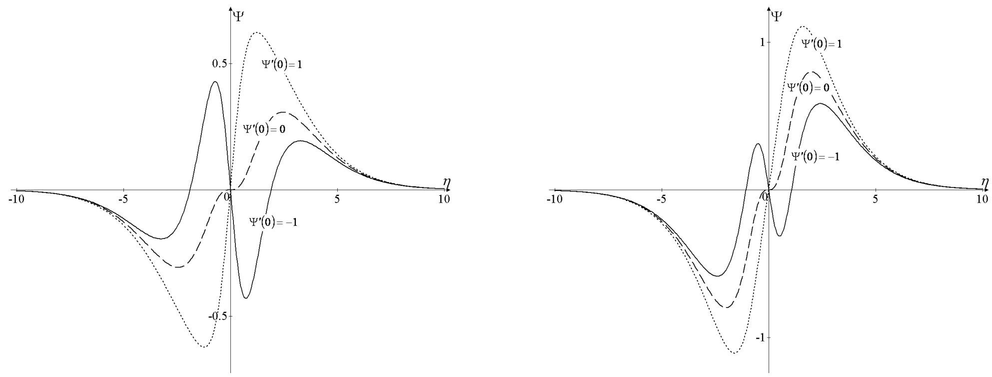
$$\frac{d^2\Psi}{d\eta^2} + \left(\frac{6}{\cosh^2\eta - 1}\right)\Psi = -\frac{\eta}{\cosh\eta} + \frac{2\eta(1-\alpha)}{\cosh^3\eta} + \frac{5}{4}\frac{\mu(2\alpha-1)}{\mu_*}\frac{\sinh\eta}{\cosh^2\eta}, \tag{3.16}$$

where $\mu_* = 5(2\alpha-1)\sqrt{\alpha_0}/(8D_q A_0\sqrt{q_0})$ is the equilibrium value of the strength of the pseudo-SRS term, and $\alpha = \alpha_0 D_q/(q_0 D_\alpha) \equiv \delta\alpha_0/q_0$. With boundary condition $\Psi(0) = 0$, Eq. (3.16) has an *exact solution*,

$$\begin{aligned}\Psi(\eta) \quad = \quad & \frac{1}{2}\left(2\Psi'(0)\tanh\eta + \eta^2\tanh\eta - \eta + \tanh\eta + \frac{\mu(2\alpha-1)}{2\mu_*}(\tanh\eta)\ln(\cosh\eta)\right)\sec\eta + \\ & \frac{1}{12}(1-2\alpha)\left(1 - \frac{\mu}{\mu_*}\right)(\sinh\eta)\tanh^2\eta, \end{aligned} \tag{3.17}$$

cf. a similar solution reported in Ref. [43]. For $\mu = \mu_*$, solution (3.17) satisfies localization conditions $\Psi(\eta \to \pm\infty) \to 0$. This solution exists due to the balance between the pseudo-SRS term and the inhomogeneous SOD and SPM. In Fig. 2, distributions $\Psi(\eta)$ for $\mu = \mu_*$ and different values of $\Psi'(0)$ are shown.

Solution $\Psi(\eta)$ has an asymmetric shape. Solitons with asymmetric tails also arise in other settings e.g., in the well-known system of linearly coupled NLSEs which describes arrays of tunnel-coupled nonlinear optical fibers [33].



a) b)

Figure 2. Distributions $\Psi(\eta)$ for $\mu = \mu_*$, different values of a $[(a) : a = 0, (b) : a = 3]$, and different values of $\Psi'(0)$.

4 Numerical results

We now aim to numerically solve the initial-value problem for the dynamics of the wave packet, $U(\xi, t = 0) = \sec\xi$, in the framework of Eq. (2.3), for $q(\xi) = \exp(\xi/30)$, $\alpha(\xi) = \exp(\xi/10)$ and different values of μ . The analytically predicted equilibrium value of the strength of the pseudo-SRS term for the initial pulse is $\mu_* = 5/48$. In direct simulations, the initial pulse for $\mu = 5/48$ is transformed into a stationary localized distribution (the solid curve in Fig. 3) with zero wavenumber.

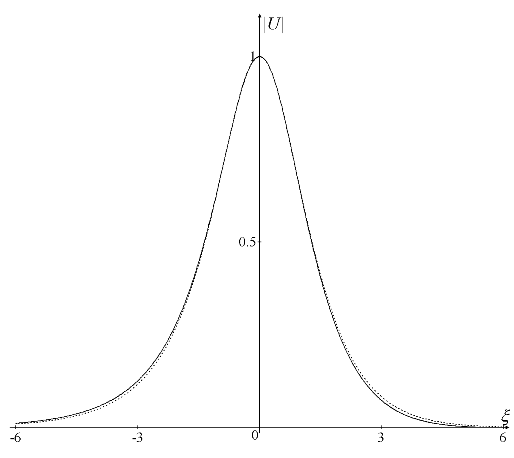
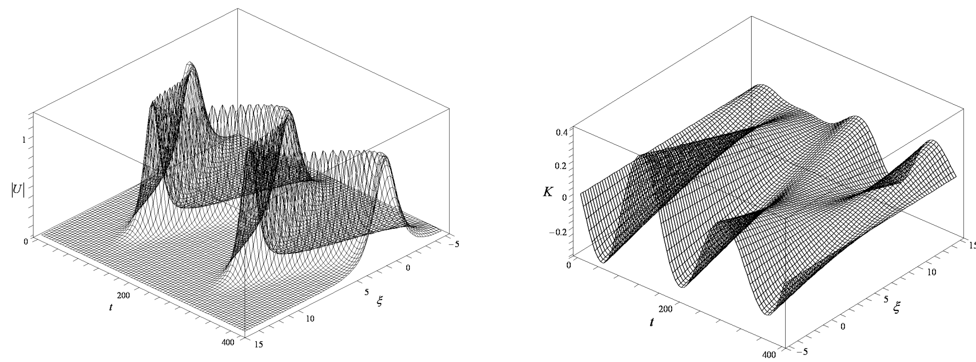


Figure 3. The numerical result (the solid curve) for the soliton's envelope, $|\,U(\xi)\,|$, within the time interval $5 < t \leq 400$ for $\mu = 5/48 \equiv \mu_*$ (evidently, the soliton keeps the established shape in the course of this interval of time). The dotted curve shows the profile of the analytical solution (4.1) (see the text).

This numerically found distribution is very close to the analytical solution of the system of Eqs. (3.14), (3.15) for $q(\xi) = \exp(\xi/30)$, $\alpha(\xi) = \exp(\xi/10)$ and $\mu = 5/48 \equiv \mu_*$:

$$| U |= \left(1+\frac{1}{60}\left(\xi^2 \tanh\xi - 1 + \tanh\xi + \frac{5}{2}\tanh\xi \ln(\cosh\xi)\right)\right)\operatorname{sec}\xi. \qquad (4.1)$$

Variation of parameter μ leads to a variation oe the soliton‘s parameters (wavenumber and amplitude). The corresponding spatial distributions of $| U |$ and local wavenumber K at different moments of time for $\mu = 3/48 \equiv (3/5)\mu_*$ are shown on Fig. 4.



a) b)

Figure 4. Numerical results for space-time distributions of $| U(\xi,t) |$ (a) and $K(\xi,t)$ (b) for $\mu = 3/48 \equiv (3/5)\mu_*$.

In Fig. 5 numerical (solid curves) and analytical (ddotted curves) results for the local wavenumber at the point of the maximum of the wave-packet's envelope are displayed, as functions of t , for $q(\xi) = \exp(\xi/30)$, $\alpha(\xi) = \exp(\xi/10)$ and different values of μ .
For $\mu = 5/48 \equiv \mu_*$, the local wavenumber at the soliton's center does not vary. It corresponds to the exact equilibrium between the pseudo-SRS term and the inhomogeneous SOD and SPM. For $\mu \neq 5/48$, the analytical and numerical results are seen to agree well. A similar picture is observed at other values of the parameters.

5 Conclusion

In the work the soliton dynamics is studied in the framework of the extended inhomogeneous NLSE, which includes the pseudo-SRS term (induced by the interaction of the HF waves with damped LF modes) and the exponentially modulated SOD an SPM terms. The results were obtained by means of numerical and analytical methods. The solitons exist due to the balance between the self-wavenumber downshift, caused by the pseudo-SRS term, and the upshift induced by the inhomogeneous SOD and SPM. The analytical soliton solution, found in the approximate form, is very close to its numerical counterpart.
In this work the soliton dynamics was considered in the model neglecting the nonlinear dispersion and third-order linear dispersion. The compensation of the pseudo-SRS term in inhomogeneous media which include these higher-order effects will be considered elsewhere.

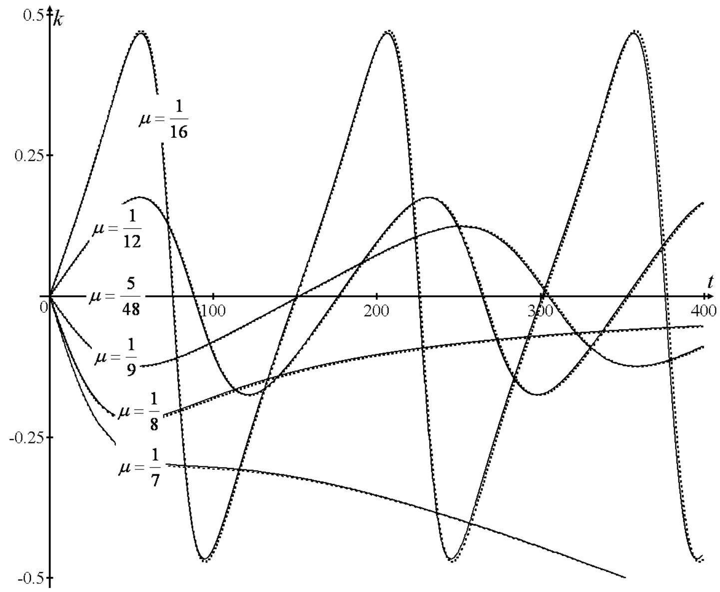


Figure 5. Numerically (solid curves) and analytically (dotted curves) found local wavenumbers at the point of the maximum of the wave-packet envelope vs. t for $q(\xi) = \exp(\xi/30)$, $\alpha(\xi) = \exp(\xi/10)$ and different μ .

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