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REFINEMENTS OF HARDY-TYPE INEQUALITIES ON TIME SCALES

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Abstract

In this paper, we give many refinements and generalizations of Hardy-type inequalities on time scales for convex functions, nonnegative convex functions, monotone convex functions and nonnegative monotone convex functions. Further we give refinements for power and exponential functions. Finally we present several examples of these inequalities on different time scales.

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1 Introduction

The well-known Hardy inequality is presented in [10]. Some generalizations of this inequality are investigated in [9, 12, 13, 14]. Also in [1, 11, 15], Hardy-type inequalities are studied in refined forms. Some of Hardy-type inequalities are extended on time scales (see [17, 18, 16]). Recently, some Hardy-type inequalities with general kernels on time scales

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(via convexity) are proved in [8]. In this paper we give refinements of inequalities given in [8] in more general setting.

The theory of time scales is studied in [4, 5, 6, 7]. We start with some notions of time scales. Any nonempty closed subset of $\mathbb{R}$ is called a time scale $\mathbb{T}$. A time scale $\mathbb{T}$ may or may not be connected, keeping in mind the disconnection of time scales the forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ are defined by

$$\sigma(t) = \inf\{\tilde{t} \in \mathbb{T} : \ \tilde{t} > t\} \text{ and } \rho(t) = \sup\{\tilde{t} \in \mathbb{T} : \ \tilde{t} < t\}.$$

In general, $\sigma(t) \geq t$ and $\rho(t) \leq t$.

Let us set

$$\Upsilon_n = \{a = (a_1, a_2, \dots, a_n) : a_i \in \mathbb{T}_i, i \in \{1, \dots, n\}\}.$$

We call Υ_n an n-dimensional time scale.

Let $S \subset \Upsilon_n$ be Δ-measurable set and $f : S \to \mathbb{R}$ is Δ-measurable function, then we denote Δ-integral of f *over* S by

$$\int_S f(t_1, t_2, \dots, t_n) \Delta t_1 \Delta t_2 \cdots \Delta t_n \text{ or } \int_S f(t) \Delta t.$$

In particular, if the interval $[a, b) \subset \mathbb{T}$ contains only isolated points, then

$$\int_a^b f(t) \Delta t = \sum_{t \in [a,b)} (\sigma(t) - t) f(t).$$

In the following theorems we recall Fubini's theorem and integral Minkowski inequality on time scales (see [3]), which are used in the proof of our main results.

Theorem 1.1. *Let $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite-dimensional time scale measure spaces. If $f : X \times Y \to \mathbb{R}$ is a Δ-integrable function and if we define the functions*

$$\varphi(y) = \int_X f(x, y) d\mu_\Delta(x) \quad \text{for a.e.} \quad y \in Y$$

and

$$\psi(x) = \int_Y f(x, y) d\nu_\Delta(y) \quad \text{for a.e.} \quad x \in X,$$

then φ is Δ-integrable on Y and ψ is Δ-integrable on X and

$$\int_X d\mu_\Delta(x) \int_Y f(x, y) d\nu_\Delta(y) = \int_Y d\nu_\Delta(y) \int_X f(x, y) d\mu_\Delta(x).$$

Theorem 1.2. *Let $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite dimensional time scale measure spaces and let u, v, and f be nonnegative functions on X, Y, and $X \times Y$, respectively. If $p \geq 1$, then*

$$\left[\int_X \left(\int_Y f(x, y) v(y) d\nu_\Delta(y)\right)^p u(x) d\mu_\Delta(x)\right]^{\frac{1}{p}} \leq \int_Y \left(\int_X f^p(x, y) u(x) d\mu_\Delta(x)\right)^{\frac{1}{p}} v(y) d\nu_\Delta(y) \quad (1.1)$$

holds provided all integrals in (1.1) *exist. If* $0 < p < 1$ *and*

$$\int_X \left(\int_Y f v d\nu_\Delta \right)^p u d\mu_\Delta > 0, \quad \int_Y f v d\nu_\Delta > 0 \tag{1.2}$$

holds, then (1.1) *is reversed. For* $p < 0$*, in addition with* (1.2)*, if*

$$\int_X f^p u d\mu_\Delta > 0, \tag{1.3}$$

holds, then again (1.1) *is reversed.*

For further properties including the concept of delta integrals, we refer the readers to [4, 5].

2 Main Results

Let us consider the following hypothesis.

H1: $(X, \mathcal{M}, \mu_\Delta)$ and $(Y, \mathcal{L}, \nu_\Delta)$ be two finite dimensional time scales measure spaces.

H2: $k : X \times Y \to \mathbb{R}_+$ is such that

$$K(x) = \int_Y k(x,y)\,\Delta y < \infty, \quad x \in X.$$

H3: $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$ and $\xi : X \to \mathbb{R}_+$ is such that

$$\mathcal{T}(y) = \left(\int_X \xi(x) \left(\frac{k(x,y)}{K(x)} \right)^{\frac{q}{p}} \Delta x \right)^{\frac{p}{q}} < \infty, \quad y \in Y.$$

H4: $\Phi \in \mathrm{C}(I, \mathbb{R})$ is a nonnegative convex function, where $I \subset \mathbb{R}$ and $\varphi : I \to \mathbb{R}$ is such that $\varphi(x) \in \partial\Phi(x) = [\Phi'_+(x), \Phi'_-(x)]$ for all $x \in Int\ I$.

Throughout the paper, we consider that I is an interval in $\mathbb{R}$.

Theorem 2.1. *Assume* H1, H2, H3 *and* H4 *hold. Then*

$$\left(\int_Y \Phi(f(y)) \mathcal{T}(y) \Delta y \right)^{\frac{q}{p}} - \int_X \xi(x) \Phi^{\frac{q}{p}} (\mathcal{A}_k f(x)) \Delta x$$
$$\geq \frac{q}{p} \int_X \frac{\xi(x)}{K(x)} \Phi^{\frac{q}{p}-1} (\mathcal{A}_k f(x)) \int_Y k(x,y) \mathcal{R}_k(x,y) \Delta y \Delta x \tag{2.1}$$

holds for ν_Δ*-integrable function* f *on* Y *such that* $f(Y) \subset I$*, where*

$$\mathcal{A}_k f(x) = \frac{1}{K(x)} \int_Y k(x,y) f(y)\,\Delta y, \quad x \in X, \tag{2.2}$$

and

$$\mathcal{R}_k(x,y) = \| \Phi(f(y)) - \Phi(\mathcal{A}_k f(x)) | - |\varphi(\mathcal{A}_k f(x))| | f(y) - \mathcal{A}_k f(x) \| . \tag{2.3}$$

If Φ is a nonnegative monotone convex function on I in H4 *and $f(y) > \mathcal{A}_k f(x)$ for $y \in Y'$ ($Y' \subset Y$), then*

$$\left(\int_Y \Phi(f(y))\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^{\frac{q}{p}}(\mathcal{A}_k f(x))\Delta x \\ \geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{K(x)}\Phi^{\frac{q}{p}-1}(\mathcal{A}_k f(x))\int_Y \operatorname{sgn}(f(y)-\mathcal{A}_k f(x))k(x,y)\mathcal{S}_k(x,y)\Delta y\Delta x\right| \quad (2.4)$$

holds, where

$$\mathcal{S}_k(x,y) = \Phi(f(y)) - \Phi(\mathcal{A}_k f(x)) - |\varphi(\mathcal{A}_k f(x))|(f(y)-\mathcal{A}_k f(x)). \quad (2.5)$$

Proof. Since Φ is a convex function on I and $\varphi(x) \in \partial\Phi(x)$ for all $x \in Int\ I$, we have

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) \geq 0$$

for all $r \in Int\ I$ and $s \in I$. Now

$$\Phi(s) - \Phi(r) - \varphi(r)(s-r) = |\Phi(s) - \Phi(r) - \varphi(r)(s-r)| \\ \geq \big||\Phi(s) - \Phi(r)| - |\varphi(r)||s-r|\big|. \quad (2.6)$$

Since $\mathcal{A}_k f(x) \in I$ for all $x \in X$, let $\mathcal{A}_k f(x) \in Int\ I$, then by substituting $r = \mathcal{A}_k f(x)$ and $s = f(y)$ in (2.6) we get

$$\Phi(f(y)) - \Phi(\mathcal{A}_k f(x)) - \varphi(\mathcal{A}_k f(x))(f(y)-\mathcal{A}_k f(x)) \\ \geq \big||\Phi(f(y)) - \Phi(\mathcal{A}_k f(x))| - |\varphi(\mathcal{A}_k f(x))||f(y)-\mathcal{A}_k f(x)|\big| = \mathcal{R}_k(x,y). \quad (2.7)$$

If $\mathcal{A}_k f(x)$ is an end point of I, then (2.7) holds with value zero on both sides of the inequality for ν_Δ-a.e. $y \in Y$. Multiplying (2.7) by $\frac{k(x,y)}{K(x)} \geq 0$, then integrating it over Y with respect to the measure ν_Δ we obtain

$$\frac{1}{K(x)}\int_Y k(x,y)\Phi(f(y))\Delta y - \frac{1}{K(x)}\int_Y k(x,y)\Phi(\mathcal{A}_k f(x))\Delta y \\ - \frac{1}{K(x)}\int_Y k(x,y)\varphi(\mathcal{A}_k f(x))(f(y)-\mathcal{A}_k f(x))\Delta y \\ \geq \frac{1}{K(x)}\int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y. \quad (2.8)$$

The second integral on the left-hand side of (2.8) becomes

$$\frac{1}{K(x)}\int_Y k(x,y)\Phi(\mathcal{A}_k f(x))\Delta y = \frac{\Phi(\mathcal{A}_k f(x))}{K(x)}\int_Y k(x,y)\Delta y = \Phi(\mathcal{A}_k f(x)).$$

For the third integral we have,

$$\frac{1}{K(x)}\int_Y k(x,y)\varphi(\mathcal{A}_k f(x))(f(y)-\mathcal{A}_k f(x))\Delta y = 0.$$

Hence (2.8) takes the form

$$\Phi(\mathcal{A}_k f(x)) + \frac{1}{K(x)} \int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y \le \frac{1}{K(x)} \int_Y k(x,y)\Phi(f(y))\Delta y.$$

Since Φ is nonnegative, for $\frac{q}{p} \ge 1$, we have

$$\left(\Phi(\mathcal{A}_k f(x)) + \frac{1}{K(x)} \int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y\right)^{\frac{q}{p}} \le \left(\frac{1}{K(x)} \int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}.$$

By applying the Bernoulli's inequality on the left-hand side of the above inequality, we get

$$\begin{aligned}
&\Phi^{\frac{q}{p}}(\mathcal{A}_k f(x)) + \frac{q}{p}\frac{\Phi^{\frac{q}{p}-1}(\mathcal{A}_k f(x))}{K(x)} \int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y \qquad (2.9)\\
&\le \left(\Phi(\mathcal{A}_k f(x)) + \frac{1}{K(x)} \int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y\right)^{\frac{q}{p}}\\
&\le \left(\frac{1}{K(x)} \int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}}.
\end{aligned}$$

Multiplying (2.9) by $\xi(x)$, integrating it over X with respect to the measure μ_Δ and applying the integral Minkowski inequality on time scales, we have

$$\begin{aligned}
&\int_X \xi(x)\Phi^{\frac{q}{p}}(\mathcal{A}_k f(x))\Delta x + \frac{q}{p}\int_X \frac{\xi(x)}{K(x)}\Phi^{\frac{q}{p}-1}(\mathcal{A}_k f(x)) \int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y\Delta x\\
&\le \int_X \xi(x)\left(\frac{1}{K(x)} \int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}} \Delta x\\
&= \left(\left(\int_X \xi(x)\left(\frac{1}{K(x)} \int_Y k(x,y)\Phi(f(y))\Delta y\right)^{\frac{q}{p}} \Delta x\right)^{\frac{p}{q}}\right)^{\frac{q}{p}}\\
&\le \left(\int_Y \Phi(f(y))\left(\int_X \xi(x)\left(\frac{k(x,y)}{K(x)}\right)^{\frac{q}{p}} \Delta x\right)^{\frac{p}{q}} \Delta y\right)^{\frac{q}{p}}.
\end{aligned}$$

Let $Y' = \{y \in Y : f(y) > \mathcal{A}_k f(x)\}$ for a fixed $x \in X$, if Φ is nondecreasing on the interval I, then

$$\begin{aligned}
&\int_Y k(x,y)|\Phi(f(y)) - \Phi(\mathcal{A}_k f(x))|\Delta y \qquad (2.10)\\
&= \int_{Y'} k(x,y)[\Phi(f(y)) - \Phi(\mathcal{A}_k f(x))]\Delta y\\
&\quad + \int_{Y\setminus Y'} k(x,y)[\Phi(\mathcal{A}_k f(x)) - \Phi(f(y))]\Delta y\\
&= \int_{Y'} k(x,y)\Phi(f(y))\Delta y - \int_{Y\setminus Y'} k(x,y)\Phi(f(y))\Delta y\\
&\quad - \Phi(\mathcal{A}_k f(x))\int_{Y'} k(x,y)\Delta y + \Phi(\mathcal{A}_k f(x))\int_{Y\setminus Y'} k(x,y)\Delta y\\
&= \int_Y \operatorname{sgn}(f(y) - \mathcal{A}_k f(x))k(x,y)[\Phi(f(y)) - \Phi(\mathcal{A}_k f(x))]\Delta y.
\end{aligned}$$

Similarly, we can write

$$\int_Y k(x,y)|f(y)-\mathcal{A}_k f(x)|\Delta y$$
$$= \int_Y \operatorname{sgn}(f(y)-\mathcal{A}_k f(x))k(x,y)(f(y)-\mathcal{A}_k f(x))\Delta y. \quad (2.11)$$

From (2.1), (2.10) and (2.11), we get (2.4). The case when Φ is non increasing can be discussed in a similar way. □

Remark 2.2. (i) Let Φ be a concave function (that is $-\Phi$ is convex) in H4. Then for all $r \in Int\ I$ and $s \in I$ we have

$$\Phi(r)-\Phi(s)-\varphi(r)(r-s) \geq 0$$

and (2.6) leads

$$\Phi(r)-\Phi(s)-\varphi(r)(r-s) = |\Phi(r)-\Phi(s)-\varphi(r)(r-s)| \geq ||\Phi(s)-\Phi(r)|-|\varphi(r)||.$$

Hence, in this setting (2.1) takes the form

$$\int_X \xi(x)\Phi^{\frac{q}{p}}(\mathcal{A}_k f(x))\Delta x - \left(\int_Y \Phi(f(y))\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} \geq \frac{q}{p}\int_X \frac{\xi(x)}{K(x)}\Phi^{\frac{q}{p}-1}(\mathcal{A}_k f(x))\int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y\Delta x$$

(ii) If Φ is nonnegative monotone concave in H4, then the order of terms on the left-hand side of (2.4) is reversed.

Corollary 2.3. *Assume* H1, H2, H3 *and* H4 *hold with* $0 < p \leq q < \infty$ *and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$*, then*

$$\left(\int_Y \Phi^p(f(y))\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^q(\mathcal{A}_k f(x))\Delta x$$
$$\geq \frac{q}{p}\int_X \frac{\xi(x)}{K(x)}\Phi^{q-p}(\mathcal{A}_k f(x))\int_Y k(x,y) ||\Phi^p(f(y))-\Phi^p(\mathcal{A}_k f(x))|-|\varphi(\mathcal{A}_k f(x))||f(y)-\mathcal{A}_k f(x)||\Delta y\Delta x.$$

holds. Moreover, if Φ *is a nonnegative monotone convex function on* I *and* $f(y) > \mathcal{A}_k f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\left(\int_Y \Phi^p(f(y))\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\Phi^q(\mathcal{A}_k f(x))\Delta x$$
$$\geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{K(x)}\Phi^{q-p}(\mathcal{A}_k f(x))\int_Y \operatorname{sgn}(f(y)-\mathcal{A}_k f(x))k(x,y) \left[\Phi^p(f(y))-\Phi^p(\mathcal{A}_k f(x))-|\varphi(\mathcal{A}_k f(x))|(f(y)-\mathcal{A}_k f(x))\right]\Delta y\Delta x\right|$$

holds.

Proof. The result follows from Theorem 2.1 by replacing Φ with Φ^p. □

Corollary 2.4. *Assume* H1, H2 *and* H3 *hold. Suppose f is a nonnegative ν_Δ-integrable function (positive for $p<0$) on Y and $\mathcal{A}_k f$ is defined in* (2.2).

(*i*) *If $1<p\le q<\infty$, or $-\infty<q\le p<0$, then*

$$\left(\int_Y f^p(y)\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)(\mathcal{A}_k f(x))^q\Delta x \geq \frac{q}{p}\int_X \frac{\xi(x)}{K(x)}(\mathcal{A}_k f(x))^{q-p}\int_Y k(x,y)\mathcal{R}_{p,k}(x,y)\Delta y\Delta x \quad (2.12)$$

holds, where

$$\mathcal{R}_{p,k}(x,y) = \left|\left|f^p(y)-\mathcal{A}_k^p f(x)\right| - |p|\,|\mathcal{A}_k f(x)|^{p-1}\,|f(y)-\mathcal{A}_k f(x)|\right|. \quad (2.13)$$

If $p\in(0,1)$ and $p\le q<\infty$, the order of terms on the left-hand side of (2.12) *is reversed.*

(*ii*) *Let $f(y)>\mathcal{A}_k f(x)$ for $y\in Y'$ ($Y'\subset Y$). If $1<p\le q<\infty$, or $-\infty<q\le p<0$, then*

$$\left(\int_Y f^p(y)\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)(\mathcal{A}_k f(x))^q\Delta x \geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{K(x)}(\mathcal{A}_k f(x))^{q-p}\int_Y \operatorname{sgn}(f(y)-\mathcal{A}_k f(x))k(x,y)\mathcal{S}_{p,k}(x,y)\Delta y\Delta x\right| \quad (2.14)$$

holds, where

$$\mathcal{S}_{p,k}(x,y) = f^p(y)-\mathcal{A}_k^p f(x) - |p|(\mathcal{A}_k f(x))^{p-1}(f(y)-\mathcal{A}_k f(x)). \quad (2.15)$$

If $p\in(0,1)$ and $p\le q<\infty$, the order of terms on the left-hand side of (2.14) *is reversed.*

Proof. Use $\Phi(x)=x^p,\ x\ge 0$ in Theorem 2.1, which is nonnegative and monotone convex function for $p\in\mathbb{R}\setminus[0,1)$, concave for $p\in(0,1]$, and affine, that is, both convex and concave for $p=1$. Obviously, in this case $\varphi(x)=\Phi'(x)=px^{p-1}$. □

Corollary 2.5. *Assume* H1, H2 *and* H3 *hold with $0<p\le q<\infty$ and g is a positive ν_Δ-integrable function on Y. Then*

$$\left(\int_Y g^p(y)\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\mathcal{G}_k^q(x)\Delta x \geq \frac{q}{p}\int_X \frac{\xi(x)}{K(x)}\mathcal{G}_k^{q-p}(x)\int_Y k(x,y)\mathcal{Q}_{p,k}(x,y)\Delta y\Delta x$$

holds, where

$$\mathcal{G}_k(x) = exp\left(\frac{1}{K(x)}\int_Y k(x,y)\ln g(y)\Delta y\right) \quad (2.16)$$

and

$$\mathcal{Q}_{p,k}(x,y) = \left|\left|g^p(y) - \mathcal{G}_k^p(x)\right| - p|\mathcal{G}_k^p(x)|\left|\ln\frac{g(y)}{\mathcal{G}_k(x)}\right|\right|. \tag{2.17}$$

Moreover, if $g(y) > \mathcal{G}_k(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\left(\int_Y g^p(y)\mathcal{T}(y)\Delta y\right)^{\frac{q}{p}} - \int_X \xi(x)\mathcal{G}_k^q(x)\Delta x$$
$$\geq \frac{q}{p}\left|\int_X \frac{\xi(x)}{K(x)}\mathcal{G}_k^{q-p}(x)\int_Y \operatorname{sgn}(g(y) - \mathcal{G}_k(x))k(x,y)\mathcal{U}_{p,k}(x,y)\Delta y\Delta x\right|$$

holds, where

$$\mathcal{U}_{p,k}(x,y) = g^p(y) - \mathcal{G}_k^p(x) - p|\mathcal{G}_k^p(x)|\ln\frac{g(y)}{\mathcal{G}_k(x)}. \tag{2.18}$$

Proof. Use $\Phi(x) = e^x$, $x > 0$ and $f(x) = p\ln g(x)$ in Theorem 2.1, to obtain the required result. Note that $\mathcal{G}_k(x) = \exp(\mathcal{A}_k(\ln g(x)))$. □

For $p = q$, H3 becomes the following hypothesis:

$\hat{H}3$: $\xi : X \to \mathbb{R}_+$ is such that

$$w(y) = \int_X \frac{\xi(x)k(x,y)}{K(x)}\Delta x < \infty, \quad y \in Y.$$

Theorem 2.6. *Assume* H1, H2*,* $\hat{H}3$ *and* H4 *hold and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$*.*

(i) *If* Φ *is a convex function (need not to be nonnegative) in* H4*, then*

$$\int_Y \Phi(f(y))w(y)\Delta y - \int_X \xi(x)\Phi(\mathcal{A}_k f(x))\Delta x$$
$$\geq \int_X \frac{\xi(x)}{K(x)}\int_Y k(x,y)\mathcal{R}_k(x,y)\Delta y\Delta x \tag{2.19}$$

holds, where $\mathcal{R}_k$ *is defined in* (2.3).

If Φ *is a concave function, then the order of terms on the left-hand side of* (2.19) *is reversed.*

(ii) *If* Φ *is a monotone convex function and* $f(y) > \mathcal{A}_k f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\int_Y \Phi(f(y))w(y)\Delta y - \int_X \xi(x)\Phi(\mathcal{A}_k f(x))\Delta x$$
$$\geq \left|\int_X \frac{\xi(x)}{K(x)}\int_Y \operatorname{sgn}(f(y) - \mathcal{A}_k f(x))k(x,y)\mathcal{S}_k(x,y)\Delta y\Delta x\right|. \tag{2.20}$$

holds, where $\mathcal{S}_k$ *is defined in* (2.5).

If Φ *is monotone concave, then the order of terms on the left-hand side of* (2.20) *is reversed.*

Proof. The proof is similar to the proof of Theorem 2.1, just use $q = p$ in the proof of Theorem 2.1. □

Remark 2.7. In Theorem 2.6, since the right-hand side of (2.19) is nonnegative, we get the refinement of [8, Theorem 3.2].

Corollary 2.8. *Assume* H1, H2*,* $\hat{H}3$ *hold and* f *is a positive* ν_Δ*-integrable function on* Y*.*

(*i*) *If* $p \geq 1$ *or* $p < 0$*, then*

$$\int_Y f^p(y)w(y)\Delta y - \int_X \xi(x)\mathcal{A}_k^p f(x)\Delta x \geq \int_X \frac{\xi(x)}{K(x)} \int_Y k(x,y)\mathcal{R}_{p,k}(x,y)\Delta y\Delta x \quad (2.21)$$

holds, where $\mathcal{R}_{p,k}$ *is defined in* (2.13)*.*

If $p \in (0,1)$*, the order of terms on the left-hand side of* (2.21) *is reversed.*

(*ii*) *Let* $f(y) > \mathcal{A}_k f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*. If* $p \geq 1$ *or* $p < 0$*, then*

$$\int_Y f^p(y)w(y)\Delta y - \int_X \xi(x)\mathcal{A}_k^p f(x)\Delta x \geq \left| \int_X \frac{\xi(x)}{K(x)} \int_Y \operatorname{sgn}(f(y) - \mathcal{A}_k f(x))k(x,y)\mathcal{S}_{p,k}(x,y)\Delta y\Delta x \right| \quad (2.22)$$

holds, where $\mathcal{S}_{p,k}$ *is defined in* (2.15)*.*

If $p \in (0,1)$*, the order of terms on the left-hand side of* (2.22) *is reversed.*

Proof. Use $\Phi(x) = x^p,\ x \geq 0$ in Theorem 2.6. □

Remark 2.9. (2.21) is the refinement of inequality in [8, Corollary 3.3].

Corollary 2.10. *Assume* H1*,* H2 *and* $\hat{H}3$ *hold. If* g *is a positive* ν_Δ*-integrable function on* Y*. Then*

$$\int_Y g^p(y)w(y)\Delta y - \int_X \xi(x)\mathcal{G}_k^p(x)\Delta x \geq \int_X \frac{\xi(x)}{K(x)} \int_Y k(x,y)\mathcal{Q}_{p,k}(x,y)\Delta y\Delta x \quad (2.23)$$

holds, where $\mathcal{G}_k$ *is defined in* (2.16) *and* $\mathcal{Q}_{p,k}$ *is defined in* (2.17)*. Moreover if* $g(y) > \mathcal{G}_k(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\int_Y g^p(y)w(y)\Delta y - \int_X \xi(x)\mathcal{G}_k^p(x)\Delta x \geq \left| \int_X \frac{\xi(x)}{K(x)} \int_Y \operatorname{sgn}(g(y) - \mathcal{G}_k(x))k(x,y)\mathcal{U}_{p,k}(x,y)\Delta y\Delta x \right| \quad (2.24)$$

holds, where $\mathcal{U}_{p,k}$ *is defined in* (2.18)*.*

Proof. Use $\Phi(x) = e^x,\ x > 0$ and $f(x) = p\ln g(x)$ in Theorem 2.6. □

Remark 2.11. (2.23) is the refinement of inequality in [8, Corollary 3.4].

3 Results using Special kernels

Let us define new hypothesis for next results.

$\bar{H}1$: $X = Y$ in H1.

$\bar{H}2$: $m : Y \to \mathbb{R}_+$ is such that $\int_Y m(y)\Delta y < \infty$, for all $y \in Y$.

Theorem 3.1. *Assume $\bar{H}1$, $\bar{H}2$ and* H4 *hold. If $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$ and f is a ν_Δ-integrable function on Y such that $f(Y) \subset I$. Then*

$$\left(\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y}\right)^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(\mathcal{A}_m f(y)) \geq \frac{q}{p}\frac{1}{\int_Y m(y)\Delta y}\int_Y m(y)\mathcal{M}(y)\Delta y \tag{3.1}$$

holds, where

$$\mathcal{A}_m f(y) = \frac{1}{\int_Y m(y)\Delta y}\int_Y m(y)f(y)\Delta y \tag{3.2}$$

and

$$\mathcal{M}(y) = \big\|\Phi(f(y)) - \Phi(\mathcal{A}_m f(y))| - |\varphi(\mathcal{A}_m f(y))|\,|f(y) - \mathcal{A}_m f(y)|\big|. \tag{3.3}$$

If the function Φ is nonnegative concave, the order of terms on the left-hand side of (3.1) *is reversed.*

Moreover, if Φ is a nonnegative monotone convex function and $f(y) > \mathcal{A}_m f(y)$ for $y \in Y'$ ($Y' \subset Y$), then

$$\left(\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y}\right)^{\frac{q}{p}} - \Phi^{\frac{q}{p}}(\mathcal{A}_m f(y))$$

$$\geq \frac{q}{p}\left|\frac{1}{\int_Y m(y)\Delta y}\int_Y \operatorname{sgn}(f(y) - \mathcal{A}_m f(y))m(y)\mathcal{N}(y)\Delta y\right| \tag{3.4}$$

holds, where

$$\mathcal{N}(y) = \Phi(f(y)) - \Phi(\mathcal{A}_m f(y)) - |\varphi(\mathcal{A}_m f(y))|\,(f(y) - \mathcal{A}_m f). \tag{3.5}$$

If Φ is nonnegative monotone concave, the order of terms on the left-hand side of (3.4) *is reversed.*

Proof. The result follows from Theorem 2.1 by taking $k(x,y) = \xi(x)m(y)$ for some positive μ_Δ-integrable function ξ and positive ν_Δ-integrable function m. □

Theorem 3.2. *Assume $\bar{H}1$, $\bar{H}2$ and* H4 *hold. If f is a ν_Δ-integrable function on Y such that $f(Y) \subset I$ and Φ is a convex function in* H4*, then*

$$\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y} - \Phi(\mathcal{A}_m f(y)) \geq \frac{1}{\int_Y m(y)\Delta y}\int_Y m(y)\mathcal{M}(y)\Delta y \tag{3.6}$$

holds, where $\mathcal{A}_m f$ is defined in (3.2) *and $\mathcal{M}$ is defined in* (3.3).

If the function Φ *is concave, the order of terms on the left-hand side of* (3.6) *is reversed.*

Moreover, if Φ *is a monotone convex function on* I *and* $f(y) > \mathcal{A}_m f(y)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\frac{\int_Y m(y)\Phi(f(y))\Delta y}{\int_Y m(y)\Delta y} - \Phi(\mathcal{A}_m f(y)) \geq \left| \frac{1}{\int_Y m(y)\Delta y} \int_Y \operatorname{sgn}(f(y) - \mathcal{A}_m f(y)) m(y) \mathcal{N}(y)\Delta y \right| \quad (3.7)$$

holds, where $\mathcal{N}$ *is defined in* (3.5).

If Φ *is monotone concave, the order of terms on the left-hand side of* (3.7) *is reversed.*

Proof. The result follows from Theorem 2.6 by taking $k(x,y) = \xi(x)m(y)$ for some positive μ_Δ-integrable function ξ and positive ν_Δ-integrable function m. □

Remark 3.3. Since the right-hand side of (3.6) is nonnegative, therefore it gives the refinement of Jensen's inequality on time scales (see [2]).

Further our new hypothesis are:

$\widetilde{H}1$: $X = Y = [a,b)_{\mathbb{T}}$, where $\mathbb{T}$ is an arbitrary time scale.

$\widetilde{H}2$: Let $0 < p \leq q < \infty$ or $-\infty < q \leq p < 0$, and

$$\xi : X \to \mathbb{R}_+ \text{ is such that } \widetilde{\mathcal{T}}(y) = \left(\int_y^b \xi(x) \left(\frac{1}{\sigma(x) - a} \right)^{\frac{q}{p}} \Delta x \right)^{\frac{p}{q}} < \infty, \quad y \in Y.$$

Theorem 3.4. *Assume* $\widetilde{H}1$, $\widetilde{H}2$ *and* H4 *hold and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$*. Then*

$$\left(\int_a^b \Phi(f(y)) \widetilde{\mathcal{T}}(y) \Delta y \right)^{\frac{q}{p}} - \int_a^b \xi(x) \Phi^{\frac{q}{p}}(\mathcal{A}_1 f(x)) \Delta x \geq \frac{q}{p} \int_a^b \frac{\xi(x)}{\sigma(x) - a} \Phi^{\frac{q}{p} - 1}(\mathcal{A}_1 f(x)) \int_a^{\sigma(x)} \mathcal{R}_1(x,y) \Delta y \Delta x \quad (3.8)$$

holds, where

$$\mathcal{A}_1 f(x) = \frac{1}{\sigma(x) - a} \int_a^{\sigma(x)} f(y)\, \Delta y, \quad x \in X \quad (3.9)$$

and

$$\mathcal{R}_1(x,y) = \big| |\Phi(f(y)) - \Phi(\mathcal{A}_1 f(x))| - |\varphi(\mathcal{A}_1 f(x))| |f(y) - \mathcal{A}_1 f(x)| \big|. \quad (3.10)$$

If Φ *is a nonnegative monotone convex function and* $f(y) > \mathcal{A}_1 f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\left(\int_a^b \Phi(f(y)) \widetilde{\mathcal{T}}(y) \Delta y \right)^{\frac{q}{p}} - \int_a^b \xi(x) \Phi^{\frac{q}{p}}(\mathcal{A}_1 f(x)) \Delta x \geq \frac{q}{p} \left| \int_a^b \frac{\xi(x)}{\sigma(x) - a} \Phi^{\frac{q}{p} - 1}(\mathcal{A}_1 f(x)) \int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathcal{A}_1 f(x)) \mathcal{S}_1(x,y) \Delta y \Delta x \right| \quad (3.11)$$

holds, where

$$\mathcal{S}_1(x,y) = \Phi(f(y)) - \Phi(\mathcal{A}_1 f(x)) - |\varphi(\mathcal{A}_1 f(x))|(f(y) - \mathcal{A}_1 f(x)). \tag{3.12}$$

Proof. The statement follows from Theorem 2.1 by using

$$k(x,y) = \begin{cases} 1 & \text{if} \quad a \le y < \sigma(x) \le b, \\ 0 & \text{otherwise,} \end{cases} \tag{3.13}$$

since in this case we have

$$K(x) = \int_a^{\sigma(x)} \Delta y = (\sigma(x) - a).$$

□

For $p = q$, $\widetilde{H}2$ takes the following form

$\check{H}2$: $\xi : X \to \mathbb{R}_+$ is such that $\widetilde{w}(y) = \int_y^b \frac{\xi(x)}{\sigma(x)-a} \Delta x < \infty, \quad y \in Y.$

Theorem 3.5. *Assume* $\widetilde{H}1$, $\check{H}2$ *and* H4 *hold and* f *is a* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$.

(i) *If* Φ *is a convex function in* H4*, then*

$$\int_a^b \Phi(f(y))\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\Phi(\mathcal{A}_1 f(x))\Delta x \ge \int_a^b \frac{\xi(x)}{\sigma(x) - a} \int_a^{\sigma(x)} \mathcal{R}_1(x,y)\Delta y \Delta x \tag{3.14}$$

holds, where $\mathcal{A}_1 f$ *and* $\mathcal{R}_1$ *are defined in* (3.9) *and* (3.10) *respectively. If* Φ *is a concave function, then the order of terms on the left-hand side of* (3.14) *is reversed.*

(ii) *If* Φ *is a monotone convex function and* $f(y) > \mathcal{A}_1 f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$*, then*

$$\int_a^b \Phi(f(y))\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\Phi(\mathcal{A}_1 f(x))\Delta x \ge \left| \int_a^b \frac{\xi(x)}{\sigma(x) - a} \int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathcal{A}_1 f(x))\mathcal{S}_1(x,y)\Delta y \Delta x \right|. \tag{3.15}$$

holds, where $\mathcal{S}_1$ *is defined in* (3.12).

If Φ *is monotone concave, then the order of terms on the left-hand side of* (3.15) *is reversed.*

Proof. The statement follows from Theorem 2.6 with k defined as in (3.13). □

Corollary 3.6. *Assume* $\widetilde{H}1$ *and* $\check{H}2$ *hold and* f *is a nonnegative* ν_Δ*-integrable function on* Y *such that* $f(Y) \subset I$.

(*i*) *If* $p \geq 1$ *or* $p < 0$, *then*

$$\int_a^b f^p(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathcal{A}_1^p f(x)\Delta x \geq \int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \mathcal{R}_{p,1}(x,y)\Delta y\Delta x \quad (3.16)$$

holds, where $\mathcal{A}_1 f$ *is defined in* (3.9) *and*

$$\mathcal{R}_{p,1}(x,y) = \left|\left|f^p(y) - \mathcal{A}_1^p f(x)\right| - |p|\,|\mathcal{A}_1 f(x)|^{p-1}\,|f(y) - \mathcal{A}_1 f(x)|\right|.$$

If $p \in (0,1)$, *the order of terms on the left-hand side of* (3.16) *is reversed.*

(*ii*) *Let* $f(y) > \mathcal{A}_1 f(x)$ *for* $y \in Y'$ $(Y' \subset Y)$. *If* $p \geq 1$ *or* $p < 0$, *then*

$$\int_a^b f^p(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathcal{A}_1^p f(x)\Delta x \geq \left|\int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \operatorname{sgn}(f(y) - \mathcal{A}_1 f(x))\mathcal{S}_{p,1}(x,y)\Delta y\Delta x\right| \quad (3.17)$$

holds, where

$$\mathcal{S}_{p,1}(x,y) = f^p(y) - \mathcal{A}_1^p f(x) - |p|(\mathcal{A}_1 f(x))^{p-1}(f(y) - \mathcal{A}_1 f(x)).$$

While for $p \in (0,1)$ *the order of terms on the left-hand side of* (3.17) *is reversed.*

Proof. Use $\Phi(x) = x^p,\ x > 0$ in Theorem 3.5. □

Corollary 3.7. *Assume* $\widetilde{H}1$ *and* $\check{H}2$ *hold. If* g *is a positive* ν_Δ*-integrable function on* Y, *then*

$$\int_a^b g(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathcal{G}_1(x)\Delta x \geq \int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \mathcal{Q}_1(x,y)\Delta y\Delta x, \quad (3.18)$$

holds, where

$$\mathcal{G}_1(x) = exp\left(\frac{1}{\sigma(x)-a}\int_a^{\sigma(x)} \ln g(y)\Delta y\right)$$

and

$$\mathcal{Q}_1(x,y) = \left|\,|g(y) - \mathcal{G}_1(x)| - |\mathcal{G}_1(x)|\left|\ln\frac{g(y)}{\mathcal{G}_1(x)}\right|\,\right|.$$

If $g(y) > \mathcal{G}_1(x)$ *for* $y \in Y'$ $(Y' \subset Y)$, *then*

$$\int_a^b g(y)\widetilde{w}(y)\Delta y - \int_a^b \xi(x)\mathcal{G}_1(x)\Delta x \geq \left|\int_a^b \frac{\xi(x)}{\sigma(x)-a}\int_a^{\sigma(x)} \operatorname{sgn}(g(y) - \mathcal{G}_1(x))\mathcal{U}_1(x,y)\Delta y\Delta x\right| \quad (3.19)$$

holds, where

$$\mathcal{U}_1(x,y) = g(y) - \mathcal{G}_1(x) - |\mathcal{G}_1(x)|\ln\frac{g(y)}{\mathcal{G}_1(x)}.$$

Proof. Use $\Phi(x) = e^x,\ x > 0$ and $f(x) = \ln g(x)$ in Theorem 3.5. □

4 Examples

Example 4.1. In addition to the assumptions of Theorem 3.4, if $\mathbb{T}$ consists of only isolated points with $b = \infty$, then (3.8) takes the form

$$\left(\sum_{y\in[a,\infty)_{\mathbb{T}}} \Phi(f(y))\hat{\mathcal{T}}(y)(\sigma(y)-y)\right)^{\frac{q}{p}} - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\Phi^{\frac{q}{p}}(\hat{\mathcal{A}}_1 f(x))(\sigma(x)-x)$$
$$\geq \frac{q}{p}\sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}\Phi^{\frac{q}{p}-1}(\hat{\mathcal{A}}_1 f(x)) \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \hat{\mathcal{R}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x),$$

where

$$\hat{\mathcal{T}}(y) = \left(\sum_{x\in[y,\infty)_{\mathbb{T}}} \xi(x)\left(\frac{1}{\sigma(x)-a}\right)^{\frac{q}{p}}(\sigma(x)-x)\right)^{\frac{p}{q}}, \quad y\in Y \quad \text{and} \quad p,q\in\mathbb{R},$$

$$\hat{\mathcal{A}}_1 f(x) = \frac{1}{\sigma(x)-a}\sum_{y\in[a,\sigma(x))_{\mathbb{T}}} f(y)(\sigma(y)-y), \quad x\in X \tag{4.1}$$

and

$$\hat{\mathcal{R}}_1(x,y) = \|\Phi(f(y)) - \Phi(\hat{\mathcal{A}}_1 f(x))| - |\varphi(\hat{\mathcal{A}}_1 f(x))\|f(y) - \hat{\mathcal{A}}_1 f(x)\|;$$

(3.11) takes the form

$$\left(\sum_{y\in[a,\infty)_{\mathbb{T}}} \Phi(f(y))\hat{\mathcal{T}}(y)(\sigma(y)-y)\right)^{\frac{q}{p}} - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\Phi^{\frac{q}{p}}(\hat{\mathcal{A}}_1 f(x))(\sigma(x)-x)$$
$$\geq \frac{q}{p}\left|\sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}\Phi^{\frac{q}{p}-1}(\hat{\mathcal{A}}_1 f(x))\right.$$
$$\left.\sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \operatorname{sgn}(f(y)-\hat{\mathcal{A}}_1 f(x))\hat{\mathcal{S}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x)\right|,$$

where

$$\hat{\mathcal{S}}_1(x,y) = \Phi(f(y)) - \Phi(\hat{\mathcal{A}}_1 f(x)) - |\varphi(\hat{\mathcal{A}}_1 f(x))|(f(y) - \hat{\mathcal{A}}_1 f(x)).$$

Example 4.2. In addition to the hypothesis of Corollary 3.6 if $\mathbb{T}$ consists of only isolated points with $b = \infty$, then following statements hold.

(i) If $p \geq 1$ or $p < 0$, then

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} f^p(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathcal{A}}_1^p f(x)(\sigma(x)-x)$$
$$\geq \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}\sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \hat{\mathcal{R}}_{p,1}(x,y)(\sigma(y)-y)(\sigma(x)-x) \tag{4.2}$$

holds, where $\hat{\mathcal{A}}_1 f$ is defined in (4.1) and

$$\hat{w}(y) = \sum_{x\in[y,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a}(\sigma(x)-x), \quad y \in Y, \tag{4.3}$$

$$\hat{\mathcal{R}}_{p,1}(x,y) = \left|\left|f^p(y) - \hat{\mathcal{A}}_1^p f(x)\right| - |p|\left|\hat{\mathcal{A}}_1 f(x)\right|^{p-1}\left|f(y) - \hat{\mathcal{A}}_1 f(x)\right|\right|.$$

If $p \in (0,1)$, the order of terms on the left-hand side of (4.2) is reversed.

(ii) Let $f(y) > \hat{\mathcal{A}}_1 f(x)$ for $y \in Y'$ $(Y' \subset Y)$. If $p \geq 1$ or $p < 0$, then

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} f^p(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathcal{A}}_1^p f(x)(\sigma(x)-x) \tag{4.4}$$
$$\geq \left| \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \operatorname{sgn}(f(y)-\hat{\mathcal{A}}_1 f(x))\hat{\mathcal{S}}_{p,1}(x,y)(\sigma(y)-y)(\sigma(x)-x)\right|$$

holds, where

$$\hat{\mathcal{S}}_{p,1}(x,y) = f^p(y) - \hat{\mathcal{A}}_1^p f(x) - |p|(\hat{\mathcal{A}}_1 f(x))^{p-1}(f(y) - \hat{\mathcal{A}}_1 f(x)).$$

While for $p \in (0,1)$ the order of terms on the left-hand side of (4.4) is reversed.

Example 4.3. In addition to the hypothesis of Corollary 3.7 if $\mathbb{T}$ consists of only isolated points, then (3.18) becomes

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} g(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathcal{G}}_1(x)(\sigma(x)-x)$$
$$\geq \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \hat{\mathcal{Q}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x), \tag{4.5}$$

where $\hat{w}$ is defined in (4.3) and

$$\hat{\mathcal{G}}_1(x) = \left(\prod_{y\in[a,\sigma(x))_{\mathbb{T}}} (g(y))^{(\sigma(y)-y)}\right)^{\frac{1}{\sigma(x)-a}},$$

$$\hat{\mathcal{Q}}_1(x,y) = \left|\left|g(y) - \hat{\mathcal{G}}_1(x)\right| - |\hat{\mathcal{G}}_1(x)|\left|\ln\frac{g(y)}{\hat{\mathcal{G}}_1(x)}\right|\right|.$$

If $g(y) > \hat{\mathcal{G}}_1(x)$ for $y \in Y'$ $(Y' \subset Y)$, then (3.19) becomes

$$\sum_{y\in[a,\infty)_{\mathbb{T}}} g(y)\hat{w}(y)(\sigma(y)-y) - \sum_{x\in[a,\infty)_{\mathbb{T}}} \xi(x)\hat{\mathcal{G}}_1(x)(\sigma(x)-x) \tag{4.6}$$
$$\geq \left| \sum_{x\in[a,\infty)_{\mathbb{T}}} \frac{\xi(x)}{\sigma(x)-a} \sum_{y\in[a,\sigma(x))_{\mathbb{T}}} \operatorname{sgn}(g(y)-\hat{\mathcal{G}}_1(x))\hat{\mathcal{U}}_1(x,y)(\sigma(y)-y)(\sigma(x)-x)\right|,$$

where

$$\hat{\mathcal{U}}_1(x,y) = g(y) - \hat{\mathcal{G}}_1(x) - |\hat{\mathcal{G}}_1(x)| \ln \frac{g(y)}{\hat{\mathcal{G}}_1(x)}.$$

Example 4.4. For $\mathbb{T} = h\mathbb{N} = \{hn : n \in \mathbb{N}\}$ with $h > 0$, $a = h$, and

$$\xi(x) = \frac{1}{\sigma(x)},$$

(4.2) takes the form

$$\sum_{m=1}^{\infty} \frac{f^p(mh)}{m} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \geq \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} \hat{\mathcal{R}}_{p,1}(nh, mh),$$

where

$$\hat{\mathcal{R}}_{p,1}(nh,mh) = \left\| f^p(mh) - \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \right| \\ - |p| \left| \frac{1}{n} \sum_{m=1}^{n} f(mh) \right|^{p-1} \left| f(mh) - \frac{1}{n} \sum_{m=1}^{n} f(mh) \right\|;$$

(4.4) takes the form

$$\sum_{m=1}^{\infty} \frac{f^p(mh)}{m} - \sum_{n=1}^{\infty} \frac{1}{n+1} \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \\ \geq \left| \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \sum_{m=1}^{n} \operatorname{sgn}\left(f(mh) - \frac{1}{n} \sum_{m=1}^{n} f(mh) \right) \hat{\mathcal{S}}_{p,1}(nh,mh) \right|,$$

where

$$\hat{\mathcal{S}}_{p,1}(nh,mh) = f^p(mh) - \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^p \\ - |p| \left(\frac{1}{n} \sum_{m=1}^{n} f(mh) \right)^{p-1} \left(f(mh) - \frac{1}{n} \sum_{m=1}^{n} f(mh) \right).$$

Example 4.5. For $\mathbb{T} = \mathbb{N}^2 = \{n^2 : n \in \mathbb{N}\}$ with $a = 1$ and

$$\xi(x) = \frac{2(\sigma(x) - 1)}{(\sigma(x) - x)^2 (2\sqrt{x} + 3)},$$

(4.2) takes the form

$$\sum_{m=1}^{\infty} f^p(m^2) - \sum_{n=1}^{\infty} \frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)} \left(\sum_{m=1}^{n} (2m+1) f(m^2) \right)^p \\ \geq \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} \sum_{m=1}^{n} \hat{\mathcal{R}}_{p,1}(n^2, m^2)(2m+1),$$

where

$$\hat{\mathcal{R}}_{p,1}(n^2,m^2) = \left\| f^p(m^2) - \left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p \right| \\ -|p|\left|\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right|^{p-1}\left|f(mh) - \frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right\|;$$

(4.4) takes the form

$$\sum_{m=1}^{\infty} f^p(m^2) - \sum_{n=1}^{\infty}\frac{2\,(n(n+2))^{1-p}}{(2n+1)(2n+3)}\left(\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p \\ \geq \left|\sum_{n=1}^{\infty}\frac{2}{(2n+1)(2n+3)}\sum_{m=1}^{n}\operatorname{sgn}\left(f(m^2) - \frac{1}{n(n+2)}\sum_{m=1}^{n}f(m^2)\right)\hat{\mathcal{S}}_{p,1}(n^2,m^2)(2m+1)\right|,$$

where

$$\hat{\mathcal{S}}_{p,1}(n^2,m^2) = f^p(m^2) - \left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^p \\ -|p|\left(\frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right)^{p-1}\left(f(m^2) - \frac{1}{n(n+2)}\sum_{m=1}^{n}(2m+1)f(m^2)\right).$$

Example 4.6. For $\mathbb{T} = q^{\mathbb{N}} = \{q^n : n \in \mathbb{N}\}$ with $q > 1$, $a = q$ and

$$\xi(x) = \frac{\sigma(x) - a}{\sigma(x)(\sigma(x) - x)},$$

(4.2) takes the form

$$\sum_{m=1}^{\infty} f^p(q^m) - \sum_{n=1}^{\infty} q^{-n}(q-1)^p(q^n-1)^{1-p}\left(\sum_{m=1}^{n} q^{m-1}f(q^m)\right)^p \\ \geq \frac{q-1}{q}\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\hat{\mathcal{R}}_{p,1}(q^n,q^m)q^m,$$

where

$$\hat{\mathcal{R}}_{p,1}(q^n,q^m) = \left\| f^p(q^m) - \left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1}f(q^m)\right)^p \right| \\ -|p|\left|\left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1}f(q^m)\right)\right|^{p-1}\left|f(q^m) - \frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1}f(q^m)\right\|;$$

(4.4) takes the form

$$\sum_{m=1}^{\infty} f^p(q^m) - \sum_{n=1}^{\infty} q^{-n}(q-1)^p(q^n-1)^{1-p}\left(\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p$$
$$\geq \frac{q-1}{q}\left|\sum_{n=1}^{\infty}\frac{1}{q^n}\sum_{m=1}^{n}\operatorname{sgn}\left(f(q^m)-\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)\hat{\mathcal{S}}_{p,1}(q^n,q^m)q^m\right|,$$

where

$$\hat{\mathcal{S}}_{p,1}(q^n,q^m) = f^p(q^m) - \left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^p$$
$$-|p|\left(\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right)^{p-1}\left(f(q^m)-\frac{q-1}{q^n-1}\sum_{m=1}^{n} q^{m-1} f(q^m)\right).$$

Example 4.7. For $\mathbb{T}$, a and ξ as in Example 4.4, (4.5) takes the form

$$\sum_{m=1}^{\infty}\frac{g(mh)}{m} - \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}} \geq \sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=1}^{n}\hat{\mathcal{Q}}_1(nh,mh),$$

where

$$\hat{\mathcal{Q}}_1(nh,mh) = \left\|g(mh)-\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right|$$
$$-\left|\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right|\ln\frac{g(mh)}{\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}}\right\|;$$

(4.6) takes the form

$$\sum_{m=1}^{\infty}\frac{g(mh)}{m} - \sum_{n=1}^{\infty}\frac{1}{n+1}\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}$$
$$\geq \left|\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{m=1}^{n}\frac{1}{m}\operatorname{sgn}\left(g(mh)-\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right)\hat{\mathcal{U}}_1(nh,mh)\right|,$$

where

$$\hat{\mathcal{U}}_1(nh,mh) = g(mh) - \left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}} - \left|\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}\right|\ln\frac{f(mh)}{\left(\prod_{m=1}^{n} g(mh)\right)^{\frac{1}{n}}}.$$

Example 4.8. For $\mathbb{T}$, a and ξ as in Example 4.5, (4.5) takes the forms

$$\sum_{m=1}^{\infty} g(m^2) - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \geq \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} \sum_{m=1}^{n} \hat{Q}_1(n^2, m^2)(2m+1),$$

where

$$\hat{Q}_1(n^2,m^2) = \left\| g(m^2) - \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \right| - \left| \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \right\| \ln \frac{g(m^2)}{\left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}}} \right\|;$$

(4.6) takes the form

$$\sum_{m=1}^{\infty} g(m^2) - \sum_{n=1}^{\infty} \frac{2n(n+2)}{(2n+1)(2n+3)} \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \geq \left| \sum_{n=1}^{\infty} \frac{2}{(2n+1)(2n+3)} \sum_{m=1}^{n} \operatorname{sgn} \left(g(m^2) - \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \right) \hat{\mathcal{U}}_1(n^2,m^2)(2m+1) \right|,$$

where

$$\hat{\mathcal{U}}_1(n^2,m^2) = g(m^2) - \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} - \left| \left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}} \right| \ln \frac{g(m^2)}{\left(\prod_{m=1}^{n} (g(m^2))^{2m+1} \right)^{\frac{1}{n(n+2)}}}.$$

Example 4.9. For $\mathbb{T}$, a and ξ as in Example 4.6, (4.5) takes the form

$$\sum_{m=1}^{\infty} g(q^m) - \sum_{n=1}^{\infty} q^{-n}(q^n - 1) \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \geq \frac{q-1}{q} \sum_{n=1}^{\infty} \frac{1}{q^n} \sum_{m=1}^{n} \hat{Q}_1(q^n,q^m)q^m,$$

where

$$\hat{Q}_1(q^n,q^m) = \left\| \left| g(q^m) - \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \right| - \left| \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \right| \left| \ln \frac{g(q^m)}{\left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}}} \right| \right\|;$$

(4.6) takes the form

$$\sum_{m=1}^{\infty} g(q^m) - \sum_{n=1}^{\infty} q^{-n}(q^n-1) \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \geq \frac{q-1}{q} \left| \sum_{n=1}^{\infty} \frac{1}{q^n} \sum_{m=1}^{n} \operatorname{sgn} \left(g(q^m) - \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \right) \hat{\mathcal{U}}_1(q^n,q^m) q^m \right|,$$

where

$$\hat{\mathcal{U}}_1(q^n,q^m) = g(q^m) - \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} - \left| \left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}} \right| \ln \frac{g(q^m)}{\left(\prod_{m=1}^{n} (g(q^m))^{q^{m-1}} \right)^{\frac{q-1}{q^n-1}}}.$$

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