Communications in **M**athematical **A**nalysis
Volume 17, Number 1, pp. 57–66 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Approximate Positive Controllability and Positive Observability for Standard Linear Systems

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(Communicated by Michal Fečkan)

Abstract

In this paper we study the approximate positive controllability and positive observability of linear systems. The characterization of the above concepts is given. The duality between the approximate positive controllability and positive observability is obtained. Necessary and sufficient conditions are formulated and proved. The abstract results are illustrated by several examples.

AMS Subject Classification: 46L57, 47B65, 46B42, 93B05, 93B07.

Keywords: Positive semigroup, Positive operator, Banach lattices, Controllability, Observability.

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1 Introduction

In the first part of this paper we are concerned with a standard linear control system of the form

$$(A,B)\quad \begin{cases}\dot{x}(t) = Ax(t) + Bu(t), & t \geq 0,\\ x(0) = x_0.\end{cases}$$

The control theory of this system has become an important mathematical discipline with applications in a variety of sciences. Especially in the past twenty years, applications resulting from technological developments gave rise to the study of infinite dimensional linear systems governed by partial differential equations. In engineering these systems are referred to as distributed parameter systems.

General questions like controllability and stabilizability of this system have been investigated in the past years intensively by means of functional analysis, especially by using theory of strongly continuous semigroups, see the books of Balakrishnan [2], Curtain and Pritchard [5]. Brammer [4], Son and Korobov [7], Saperstone and Yorke [11] and Son [14] considered controllability with the additional requirement that the controls are restricted to a certain subset of the original space, thereby making particular use of the theorems of Krein and Rutman [10] on positive operators. Since there are many physical systems whose state variables represent positive quantities (e.g. density of a population, concentration, etc.), extensive research has been conducted in the study of positivity and asymptotic behavior of the uncontrolled systems $(A,0)$. However, less attention has been paid to positive control problems. The problem we have in mind is the (approximate) controllability of positive systems in Banach lattices. More precisely, we ask what are the nonnegative states which can be reached by positive controls? To the best of our knowledge, in infinite dimension spaces, this problem is pioneered by T. Schanbacher [12] for standard linear control system (A,B) and by M. EL Azzouzi et al. [3] for boundary control systems. In this paper, we consider the system

$$(A,B)\quad \begin{cases}\dot{x}(t) = Ax(t) + Bu(t), & t \geq 0,\\ x(0) = x_0.\end{cases}$$

The state space X and the control space U are Banach lattices, the initial state $x_0 \in X^+$, $(A,D(A))$ is the generator of a positive semigroup $(T(t))_{t\geq 0}$ on X, $B \in \mathcal{L}(U,X)$ positive and $u(\cdot) \in L^1_{loc}(\mathbb{R}_+,U)]$. The mild solution of the system (A,B) is

$$x(t) = T(t)x_0 + \int_0^t T(t-s)Bu(s)ds, \quad t \geq 0.$$

In fact, the physical meaning of the solution of (A,B) requires not only that the final state of the system is positive (a condition that can be fulfilled by choice) but, also that the state $x(t)$ remains positive at all times. Clearly, this cannot be expected for any choice of the control u. We call choices of the control for which the corresponding solution of (A,B) satisfies $x(t) \in X^+$ for all $t \in \mathbb{R}_+$ admissible. In fact, any control which satisfies $u(t) \in U^+$ for all $t \in \mathbb{R}_+$ is admissible. Consequently the controllability in the usual sense is impossible. The approximate positive controllability has been studied by Tilman Schanbacher [12] in 1986 and characterized via semigroup theory. In this characterization we need to know explicitly the semigroup $(T(t))_{t\geq 0}$ which often is not possible. Here we characterize the approximate positive controllability of systems (A,B) via resolvent operators. In the second part of this work we introduce a new concept called **positive observability**. Meaning that if the output of the system is positive for all times, the initial state is positive. This definition gives the duality with the approximate positive controllability. We end this part by illustrating the above results by several examples.

2 Notations and preliminaries

To make this work more self-contained we give a brief introduction to the notions of Banach lattices, for more details see the standard references [1] and [13]. A real vector space X endowed with an order written as $\leq$ such that the ordering is compatible with the vector space structure X, that is :
$f \leq g$ implies $f + h \leq g + h$ for all f, g, h in X,
$f \geq 0$ implies $\lambda f \geq 0$ for all f in X and $\lambda \in \mathbb{R}^+$,
is called an ordered vector space. The axioms imply that the set $X^+ = \{f \in X : f \geq 0\}$ is a convex cone with vertex 0, called the positive cone of X. It follows that $f \leq g$ if and only if $g - f \in X^+$. The elements $f \in X^+$ are called positive.
An ordered vector space X is called a vector lattice if any two elements x, y in X have a supremum and an infimum denoted by $sup(x, y)$, and $inf(x, y)$ respectively. For an element x of a vector lattice we write:

a) $|x| = sup(-x, x)$ and call it the absolute value of x,

b) $x^+ = sup(x, 0)$ and call it the positive part of x,

c) $x^- = sup(-x, 0)$ and call it the negative part of x.

Then we have the following important identities:

$$x = x^+ - x^- \text{ and } |x| = x^+ + x^-.$$

A norm $\|\cdot\|$ in a vector lattice satisfying the axiom

$$|x| \leq |y| \Rightarrow \| x \| \leq \| y \|$$

is called a lattice norm. Finally, a (real) Banach lattice is a Banach space X over $\mathbb{R}$ endowed with an ordering $\leq$ such that $(X, \leq)$ is a vector lattice and the norm on X is a lattice norm.

Let E and F be two Banach lattices and $T \in \mathcal{L}(E, F)$.

i) T is called positive if $T(E^+) \subset F^+$.

ii) T is called lattice homomorphism if $|T(x)| = T(|x|)$ for all $x \in E$.

iii) A semigroup $(T(t))_{t\geq 0}$ is called positive if $T(t)$ is positive for all $t \geq 0$.

iv) A semigroup $(T(t))_{t\geq 0}$ is a called lattice semigroup if $T(t)$ is a lattice homomorphism for all $t \geq 0$.

3 Approximate positive controllability

In this section we define approximate positive controllability and give a characterization via resolvent operators.

We consider the following linear control system

$$(A,B)\begin{cases} \dot{f}(t) &= Af(t)+Bu(t), t\geq 0,\\ f(0) &= f_0. \end{cases}$$

Where the state space X and the control space U are Banach lattices, $(A,D(A))$ the generator of a positive semigroup $\mathcal{T}=(T(t))_{t\geq 0}$ on X, and $B\in\mathcal{L}(U,X)$ positive. For $t>0$ the set of reachable states from the origin in time t by means of positive controls u is

$$R_t^+ := \left\{\int_0^t T(t-s)Bu(s)ds : u(.)\in L^1(0,t,U), u(s)\in U^+, 0\leq s\leq t\right\}$$

and the set of reachable states from the origin in arbitrary time by means of positive controls u is

$$R^+ := \bigcup_{t>0} R_t^+.$$

The system (A,B) is called approximately positive controllable if

$$\overline{R^+} = X^+.$$

We recall the following propositions, which giving a useful representation of the set $\overline{R^+}$.

Proposition 3.1. *[12] Let X and U be Banach lattices, let $(A,D(A))$ be the generator of a positive semigroup $\mathcal{T}=(T(t))_{t\geq 0}$ on X, and $B\in\mathcal{L}(U,X)$ positive.*
Then $\overline{R^+} = \overline{co}\{T(s)Bu : 0\leq s, u\in U^+\}$.

Using the above proposition we obtain.

Proposition 3.2. *Let $\omega>\omega_0(A)$. Let X and U be Banach lattices, let $(A,D(A))$ be the generator of a positive semigroup $\mathcal{T}=(T(t))_{t\geq 0}$ on X, and $B\in\mathcal{L}(U,X)$ positive.*
Then

$$\overline{R^+} = \overline{co}\{R(\lambda,A)^nBu : \lambda\geq\omega, n\in\mathbb{N}^* \text{ and } u\in U^+\}.$$

Proof. By Proposition 3.1, $T(s)Bu\in\overline{R^+}$ for all $0\leq s$ and $u\in U^+$. Let $\lambda\geq\omega$ and $n\in\mathbb{N}^*$. Since $\overline{R^+}$ is a cone convex, then $\frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)Budt\in\overline{R^+}$. Since $R(\lambda,A)^nBu = \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)Budt$, (see, [6, Corollary II.1.11]), then $R(\lambda,A)^nBu\in\overline{R^+}$ for all $u\in U^+$, $\lambda\geq\omega$ and $n\in\mathbb{N}^*$. Consequently,

$$\overline{co}\{R(\lambda,A)^nBu : \lambda\geq\omega, n\in\mathbb{N}^* \text{ and } u\in U^+\}\subset\overline{R^+}.$$

For the other inclusion, let $u\in U^+$ and $0\leq s$. From the Post-Widder formula (see, [6, Corollary V.5.5]) we obtain $T(t)Bu = \lim_{n\to\infty}(\frac{n}{s}R(\frac{n}{s},A))^nBu$, and hence $T(s)Bu\in\overline{co}\{R(\lambda,A)^nBu : \lambda\geq\omega, n\in\mathbb{N}^* \text{ and } u\in U^+\}$. Therefore

$$\overline{co}\{R(\lambda,A)^nBu : \lambda\geq\omega, n\in\mathbb{N}^* \text{ and } u\in U^+\}\supseteq\overline{R^+}.$$

□

Now we state a simple result which will play a crucial role in this work.

Lemma 3.3. *[3] Let X be a Banach lattice and M a subset of X^+ such that $\lambda M \subset M$ for all $\lambda \in \mathbb{R}^+$. The following statements are equivalent.*

1) $\overline{co}M = X^+$,

2) for all $\varphi \in X'$, $\langle a, \varphi \rangle \geq 0$ *for all* $a \in M$ *implies* $\varphi \geq 0$.

We recall the following characterization of approximate positive controllability using the semigroup $(T(t))_{t\geq 0}$ from [12]. For the sake of completeness we give a short proof using the above lemma.

Proposition 3.4. *The system (A, B) is approximately positive controllable if and only if the following implication holds for all $\varphi \in X'$:*

$$\langle T(s)Bu, \varphi \rangle \geq 0 \, \textit{for all } u \in U^+, s \geq 0 \Rightarrow \varphi \geq 0.$$

Proof. By Proposition 3.1 we have

$$\overline{R^+} = \overline{co}\{T(s)Bu : 0 \leq s, u \in U^+\}.$$

We consider the set $M := \{T(s)Bu : 0 \leq s, u \in U^+\}$ which is a cone with $M \subset X^+$. By Lemma 3.3, (A, B) is approximately positive controllability if and only if

$$\text{for all, } \varphi \in X' \; \langle T(s)Bu, \varphi \rangle \geq 0 \text{ for all } u \in U^+ \text{ and } s \geq 0 \Rightarrow \varphi \geq 0.$$

□

In the above characterization we need to know explicitly the semigroup $(T(t))_{t\geq 0}$. However, using the above proposition we obtain the following simple characterization.

Theorem 3.5. *The system (A, B) is approximately positive controllable if and only if there exists $\omega \in \mathbb{R}$ such that the following implication holds for all $\varphi \in X'$:*

$$\langle R(\lambda, A)^n Bu, \varphi \rangle \geq 0 \, \textit{for all } u \in U^+, n \in \mathbb{N}^* \textit{ and } \lambda \geq \omega \Rightarrow \varphi \geq 0.$$

Proof. It is a immediate consequence of Proposition 3.2 and Lemma 3.3. □

Example 3.6. Let $X = \mathbb{R}^2$ and $U = \mathbb{R}$. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By a simple calculation we obtain $R(\lambda, A)^n = \begin{pmatrix} \frac{1}{(\lambda-1)^n} & 0 \\ \frac{n}{(\lambda-1)^{n+1}} & \frac{1}{(\lambda-1)^n} \end{pmatrix}$, then $\langle R(\lambda, A)^n Bu, \binom{x}{y} \rangle = \dfrac{nx}{\lambda - 1} + y$.

If $\langle R(\lambda, A)^n Bu, \binom{x}{y} \rangle = \dfrac{nx}{\lambda - 1} + y \geq 0$ for all $n \in \mathbb{N}$ and $\lambda \geq 2$, then $\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{R}^2)^+$. Consequently, the system (A, B) is approximately positive controllable.

By the above theorem we obtain the following corollary which giving a sufficient condition for approximate positive controllability.

Corollary 3.7. *The system (A,B) is approximately positive controllable if the following condition holds.*

There exists $\omega \in \mathbb{R}$ such that for all $\varphi \in X'$,

$$\langle R(\lambda,A)Bu,\varphi\rangle \geq 0 \textit{ for all } u \in U^+ \textit{ and } \lambda \geq \omega \Rightarrow \varphi \geq 0.$$

Remark 3.8. The property of reaching any state in X from $x_0 = 0$ by a suitable control is equivalent to the property of reaching any state in X from any other state using a suitable control (see, [6, Chapter VI, page 456]). But, the property of reaching any state in X^+ from $x_0 = 0$ by a suitable positive control is not equivalent to the property of reaching any state in X^+ from any other positive state using a suitable positive control.

Example 3.9. Let $U = X = \mathbb{R}^2$, $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B = I$.

By Corollary 3.7 ii), the system (A,B) with initial state $x_0 = 0$ is approximately positive controllable. Now, we consider the system (A,B) with initial state $x_0 = \binom{1}{2}$. The solution of (A,B) is given by

$$\begin{aligned} x(t) &= T(t)x_0 + \int_0^t T(t-s)Bu(s)ds \\ &= \begin{pmatrix} e^t + \displaystyle\int_0^t e^{t-s}u_1(s)ds \\ 2e^{2t} + \displaystyle\int_0^t 2e^{2(t-s)}u_2(s)ds \end{pmatrix}. \end{aligned}$$

Where, $u(\cdot) = \binom{u_1(\cdot)}{u_2(\cdot)} \in L^1_{loc}(\mathbb{R}^+,\mathbb{R}^2)$ is the positive control. It is easy to see that $x(t) \in \left\{\binom{x}{y} \in (\mathbb{R}^2)^+ : \binom{x}{y} \geq \binom{1}{2}\right\}$. Consequently, the system (A,B) can not be approximately positive controllable.

Example 3.10. We end this section by an example of a dynamical cells populations system

$$(CE)\begin{cases} \frac{\partial}{\partial t}n(t,s) &= -\frac{\partial}{\partial s}n(t,s) - \mu(s)n(t,s) - b(s)n(t,s) \\ & \quad + \begin{cases} 4b(2s)n(t,2s) & for \ \frac{\alpha}{2} \leq s \leq \frac{1}{2}, \\ 0 & for \ \frac{1}{2} \leq s \leq 1, \end{cases} + u(t,s) \\ n(t,\frac{\alpha}{2}) &= 0 \text{ for } t \geq 0, \\ n(0,s) &= n_0(s) \text{ for } \frac{\alpha}{2} \leq s \leq 1, \end{cases}$$

where $n(t,s)$ is the number of cells at time t having size s. Moreover, we assume that the death rate μ is a positive, continuous function on $[\frac{\alpha}{2},1]$, while the division rate b should be continuous with $b(s) > 0$ for $s \in (\alpha,1)$ and $b(s) = 0$ otherwise. The control u is an external action on the total cells population n, for more details on this equation, we refer to the monographs by Metz-Diekmann[8] and Webb[15].

Definition 3.11. On the Banach space $X := L^1[\frac{\alpha}{2}, 1]$ define the operators $A_0 f := -f' - (\mu + b)f$ with domain $D(A_0) := \left\{ f \in W^{1,1}[\frac{\alpha}{2}, 1] : f(\frac{\alpha}{2}) = 0 \right\}$,

$$Cf(s) = \begin{cases} 4b(2s)n(t,2s) & for \quad \frac{\alpha}{2} \le s \le \frac{1}{2}, \\ 0 & for \quad \frac{1}{2} \le s \le 1, \end{cases} \quad \text{for all } f \in X,$$

$A := A_0 + C$ with domain $D(A) := D(A_0)$, and control operator $B = I$.

With these definitions our partial differential equation (CE) becomes the abstract Cauchy problem

$$(A,B) \begin{cases} \dot{f}(t) &= Af(t) + Bu(t), t \ge 0, \\ f(0) &= n_0. \end{cases}$$

The authors in [6, Corollary 1.11]) have shown that the operator $A := A_0 + C$ generates a positive semigroup $(T(t))_{t\ge}$ on $X := L^1[\frac{\alpha}{2}, 1]$. The control operator satisfies $BU^+ = X^+$. Consequently by Corollary 3.7, the system (CE) is approximately positive controllable.

4 Positive observability

In this section we will introduce the definition of positive observability and we give a characterization via resolvent.

We consider the following linear system

$$(C,A) \begin{cases} \dot{x}(t) &= Ax(t), \quad t \ge 0, \\ y(t) &= Cx(t). \end{cases}$$

The state space X and the observation space Y are Banach lattices, $(A, D(A))$ is the generator of positive semigroup $\mathcal{T} = (T(t))_{t\ge 0}$ on X, and $C \in \mathcal{L}(X,Y)$ positive.

Definition 4.1. The system (C,A) is called positive observable if for all $x \in X$,

$$CT(t)x \ge 0 \text{ for all } t \ge 0 \text{ implies } x \ge 0.$$

In the above definition, we need to know explicitly the semigroup $(T(t))_{t\ge 0}$ which is not always possible. First, we have the following characterization of positive observability via resolvent operators.

Theorem 4.2. *The following conditions are equivalent*

i) The system (C,A) is positive observable

ii) There exists $\omega \in \mathbb{R}$ such that the following implication holds for all $x \in X$,

$$CR(\lambda,A)^n x \ge 0 \textit{ for all } n \in \mathbb{N} \textit{ and } \lambda \ge \omega \Rightarrow x \ge 0.$$

Here $R(\lambda,A) := (\lambda - A)^{-1}$.

Proof. **Necessity:** Let $\omega > \omega_0(A)$ and $x \in X$ such that

$$CR(\lambda, A)^n x \geq 0 \text{ for all } n \in \mathbb{N} \text{ and } \lambda \in [\omega, \infty).$$

For $t > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $\frac{n}{t} > \omega$ and hence $CR(\frac{n}{t}, A)^n x \geq 0$ for all $n \geq n_0$, then $C(\frac{n}{t}R(\frac{n}{t}, A))^n x \geq 0$ for all $n \geq n_0$. By the Post-Widder formula (see [6, Theorem II.1.10]) and by the continuity of the operator C we obtain $CT(t)x \geq 0$ for all $t > 0$ and by the strong continuity of the semigroup we obtain $CT(t)x \geq 0$ for all $t \geq 0$. Consequently $x \geq 0$.
Sufficiency: We assume that $CT(t)x \geq 0$ for all $t \geq 0$. Let $\omega > \omega_0(A)$, hence $C\frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)x\,dt \geq 0$ for all $n \in \mathbb{N}^*$.

Since $R(\lambda, A)^n x = \frac{1}{(n-1)!}\int_0^{+\infty} t^{n-1}e^{-\lambda t}T(t)x\,dt$, (see [6, Corollary V.5.5]), we obtain $CR(\lambda, A)^n x \geq 0$ for all $n \in \mathbb{N}$, $\lambda > \omega_0(A)$. Consequently, $x \geq 0$.

□

Example 4.3. Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. Let $A = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 1 \end{pmatrix}$. By a simple calculation we obtain:

$$R(\lambda, A)^n = \begin{pmatrix} \frac{1}{(\lambda-1)^n} & 0 \\ \frac{n}{(\lambda-1)^{n+1}} & \frac{1}{(\lambda-1)^n} \end{pmatrix}.$$

Then $CR(\lambda, A)^n \begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{(\lambda-1)^n}[\dfrac{nx}{\lambda-1} + y]$.

If $CR(\lambda, A)^n \begin{pmatrix} x \\ y \end{pmatrix} = \dfrac{1}{(\lambda-1)^n}[\dfrac{nx}{\lambda-1} + y] \geq 0$ for all $n \in \mathbb{N}$ and $\lambda > 1$, then $\begin{pmatrix} x \\ y \end{pmatrix} \in (\mathbb{R}^2)^+$. Consequently, the system (C, A) is positive observable.

It is obvious that if the system (C, A) is positive observable, then it is observable, however, the converse is generally not true.

Example 4.4. Let $X = Y = \mathbb{R}^2$, $A = \begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$ and $C = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$.

It is obvious that $T(t) = \begin{pmatrix} \frac{e^{-2t}+2e^t}{3} & \frac{2e^t-2e^{-2t}}{3} \\ \frac{e^t-e^{-2t}}{3} & \frac{2e^{-2t}+e^t}{3} \end{pmatrix}$.

Hence,

$$CT(t)\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ \frac{e^{-2t}+2e^t}{3} & \frac{2e^t-2e^{-2t}}{3} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{e^{-2t}+2e^t}{3}x + \frac{2e^t-2e^{-2t}}{3}y \end{pmatrix}.$$

If $CT(t)\binom{x}{y} = 0$ for all $t \geq 0$, then $\dfrac{e^{-2t}+2e^t}{3}x + \dfrac{2e^t-2e^{-2t}}{3}y = 0$ for all $t \geq 0$, hence, $x = y = 0$. Thus the system (C, A) is observable.
On the other hand, $CT(t)\binom{1}{-1} \geq 0$ for all $t \geq 0$, but $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is not positive.

The following theorem gives the converse implication.

Theorem 4.5. *We assume that the semigroup $(T(t))_{t\geq 0}$ and the observation operator are lattice homomorphisms. The system (C, A) is observable if and only if it is positive observable.*

Proof. Let $x \in X$ such that $CT(t)x \geq 0$ for all $t \geq 0$, then $(CT(t)x)^- = 0$ for all $t \geq 0$. Since the semigroup $(T(t))_{t\geq 0}$ and the observation operator C are lattice homomorphisms, so $CT(t)(x^-) = 0$ for all $t \geq 0$. Since (C,A) is approximately observable, it follows that $x^- = 0$. Consequently, $x \geq 0$. □

In the following, we assume that the Banach lattices X and U are Hilbert spaces. We know that the system (A,B) is approximately controllable if and only if the system (B^*,A^*) is observable. This duality still remains true in the positive case.

Theorem 4.6. *The system (A,B) is approximately positive controllable if and only if the system (B^*,A^*) is positive observable.*

Proof. We suppose that (A^*,B^*) is positive observable. Let $\varphi \in X'$ such that

$$< T(t)Bu, \varphi > \geq 0 \text{ for all } u \in U^+, t \geq 0.$$

Then

$$< u, B^*T^*(t)\varphi > \geq 0 \text{ for all } u \in U^+, s \geq 0.$$

So, $B^*T^*(t)\varphi \geq 0$ for all $t \geq 0$ and consequently $\varphi \geq 0$.
The converse is proved in the some way. □

Example 4.7. Let $X = \mathbb{R}^2$ and $Y = \mathbb{R}$. Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 1 & 0 \end{pmatrix}$. Then $A^* = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $C^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. By a simple calculation we obtain $R(\lambda, A^*)^n = \begin{pmatrix} \frac{1}{(\lambda-1)^n} & 0 \\ \frac{n}{(\lambda-1)^{n+1}} & \frac{1}{(\lambda-1)^n} \end{pmatrix}$ for $n \in \mathbb{N}$ and $\lambda > 1$. So, $R(\lambda,A^*)^nC^*u = \begin{pmatrix} \frac{1}{(\lambda-1)^n}u \\ \frac{n}{(\lambda-1)^{n+1}}u \end{pmatrix}$. Hence $< R(\lambda,A^*)^nC^*u, \begin{pmatrix} a \\ b \end{pmatrix} >= \frac{u}{(\lambda-1)^n}[a + \frac{n}{(\lambda-1)}b]$. Then, if $< R(\lambda,A^*)^nC^*u, \begin{pmatrix} a \\ b \end{pmatrix} > \geq 0$ for all $n \in \mathbb{N}$ and $\lambda > 1$ and $u \in \mathbb{R}^+$, thus, $\begin{pmatrix} a \\ b \end{pmatrix} \in (\mathbb{R}^2)^+$. Consequently, the system (C,A) is positive observable.

Acknowledgments

The author thanks the referees for their careful reading of the manuscript and insightful comments.

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