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Existence of Positive Solutions to Nonlinear Elliptic Problem in Nta-Conical Domains

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Abstract

We study the existence and the asymptotic behavior of positive solutions of the nonlinear equation

$$\Delta u + \varphi(.,u) + \psi(.,u) = 0,$$

in NTA- cones in $\mathbb{R}^n$($n \geq 3$).

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1 Inroduction

We work in the Euclidean space $\mathbb{R}^n$, where $n \geq 3$. By G_Ω, we denote the Green function for the Laplacian in a domain Ω. We write $\delta_\Omega(x)$ for the distance from $x \in \Omega$ to the Euclidean boundary $\partial\Omega$ of Ω. By the symbol A, we denote an absolute positive constant whose value is unimportant and may change from line to line. If necessary, we use $A_0, A_1, \ldots$ to specify them. $B(x,r)$ denote the open ball and the sphere of center x and radius r. We write $B(r) = B(0,r)$ for simplicity. We say that a bounded domain Ω is uniform if there exists a constant $A_0 > 1$ such that each pair of points x and y in Ω can be connected by a rectifiable curve γ in $\overline{\Omega}$ for which

$$\begin{cases} \ell(\gamma) \leq A_0 \ |x-y|, \\ \min\{\ell(\gamma(x,z)),\ \ell(\gamma(z,y))\} \leq \ A_0\ \delta_\Omega(z), \quad \text{for all } z \in \gamma, \end{cases}$$

where $\ell(\gamma(x,z))$ denotes the length of the subarc $\gamma(x,z)$ of γ from x to z. A non-tangentially accessible (abbreviated **NTA**) domain, as introduced by Jerison and Kenig in [11], is a

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uniform domain satisfying the *exterior corkscrew condition* : there exists a constant $r_0 > 0$ such that for any $z \in \partial\Omega$ and $0 < r < r_0$, we find a point $x \in \mathbb{R}^n \setminus \overline{\Omega}$ such that $|x - z| = r$ and $\delta_\Omega(x) \geq \frac{r}{A_0}$. By a ray from 0 we mean an 'open' half-line starting from 0 (thus excluding 0). A cone of vertex 0 is a domain (non-empty connected open set) C consisting of rays from 0 and such that its exterior $\mathbb{R}^n \setminus C$ is not empty. From [16], an NTA-cone of vertex 0 is a cone C of vertex 0 such that $C \cap B(0,1)$ is an NTA-domain. By Γ, we denote an NTA-cone of vertex 0.

Let z_0 be a fixed point in Γ (the reference point), $\zeta \in \partial\Gamma \cup \{\infty\}$, and $\left\{y_j\right\}$ a sequence in Γ converging to ζ. Then some subsequence of $\left\{G_\Gamma(.,y_j)/G_\Gamma(z_0,y_j)\right\}_j$ converges to a positive harmonic function in Γ. All limit functions obtained in this way are called **Martin kernels at** ζ. Note from [[1], Theorem 3] and the Kelvin transformation, that for all $\zeta \in \partial\Gamma \cup \{\infty\}$, there exists a unique (minimal) Martin kernel $K_\Gamma(.,\zeta)$ at ζ. Moreover, from [[16],p 472], there exist a nonnegative constant α and a positive bounded continuous function ω on $\Gamma \cap S(0,1)$ such that

$$K_\Gamma(x,0) = |x|^{2-n-\alpha}\,\omega\left(\frac{x}{|x|}\right) \quad \text{and} \quad K_\Gamma(x,\infty) = |x|^{\alpha}\,\omega\left(\frac{x}{|x|}\right), \tag{1.1}$$

where $S(0,1)$ is the unit sphere. Our motivation in this paper comes from [19], where Shi and Yao investigated the existence of nonnegative solutions for the elliptic problem

$$\begin{cases} \Delta u + K(x)u^{-\gamma} + \lambda u^{\alpha} = 0, & \text{in } \Omega, \\ u(x) > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain with smooth compact boundary, γ and α in $(0,1)$ are two constants, λ real parameter and K is in $C^{0,\beta}(\overline{\Omega})$. Using this result. Sun and Li [20] gave a similar result in $\mathbb{R}^n$ ($n \geq 2$). In fact they proved an existence result for the problem

$$\begin{cases} \Delta u + p(x)u^{-\gamma} + q(x)u^{\alpha} = 0 & \text{in } \mathbb{R}^n \\ u(x) > 0 & \text{in } \mathbb{R}^n \\ u(x) \longrightarrow 0, \quad \text{as } |x| \longrightarrow \infty, \end{cases}$$

where γ and α in $(0,1)$ are two constant and p,q are two nonnegative functions in $C^{\beta}_{loc}(\mathbb{R}^n)$ such that $p + q \neq 0$. The pure singular elliptic problem

$$\Delta u(x) + p(x)\;(u(x))^{-\gamma} = 0, \quad \gamma > 0,\; x \in D \subset \mathbb{R}^n$$

has been extensively studied for both bounded and unbounded domains with smooth compact boundary (see for example [3] [4], [5], [8], [6], [7], [9] and [10]). In [2] Breizis and Kamin study the sublinear elliptic equation

$$\begin{aligned} &\Delta u + \rho(x)\,(u(x))^{\gamma} = 0 \quad \text{in } \mathbb{R}^n, \\ &\liminf_{|x| \longrightarrow \infty} u(x) = 0, \end{aligned}$$

with $0 < \gamma < 1$ and ρ is a nonnegative measurable function satisfying some appropriate conditions. They proved the existence and the uniqueness of positive solution. In this paper we combine a singular term and a sublinear term in the nonlinearity. Indeed, we consider the boundary value problem

$$\begin{cases} \Delta u + \varphi(.,u) + \psi(.,u) = 0 & \text{in } \Gamma, \\ u > 0 & \text{in } \Gamma, \\ u = 0 & \text{on } \partial\Gamma, \\ \lim_{x \longrightarrow \infty} u(x) = 0, \end{cases} \tag{1.2}$$

where φ and ψ are required to satisfy some appropriate hypotheses related to a functional class $K(\Gamma)$, introduced and studied by K. Hirata in [17].

Definition 1.1. (Kato class)(see[17]) We say that a measure ν on Γ belongs to the extended Kato class $K(\Gamma)$ if ν satisfies the following conditions :

$$\lim_{r \longrightarrow 0} \left(\sup_{x \in \Gamma} \int_{\Gamma \cap B(x,r)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)} G_\Gamma(x,y)\, d\nu(y) \right) = 0, \tag{1.3}$$

$$\lim_{R \longrightarrow +\infty} \left(\sup_{x \in \Gamma} \int_{\Gamma \setminus B(0,R)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)} G_\Gamma(x,y)\, d\nu(y) \right) = 0. \tag{1.4}$$

We also say that a Borel measurable function ϕ in Γ belongs to the extended Kato class $K(\Gamma)$ if the measure $d\nu = |\phi|\ dy$ belongs to $K(\Gamma)$.

Example 1.1. (see[17]) Suppose that $0 \le \alpha < 1$ and $\omega(z) \simeq \delta_\Gamma(z)$, and let

$$W(y) = (1 + |y|)^{\alpha p - q}\ |y|^{p(1-\alpha)}\ \delta_\Gamma(y)^{-p}.$$

Then $W \in K(\Gamma)$ if and only if $p < 2 < q$.

In [3] M. Zribi studied (1.2) and prove an existence result in a bounded domain Ω with smooth compact boundary, when φ and ψ satisfies some appropriate hypothesis related to the Kato class $K(\Omega)$. His discussion was based on the explicit estimates for the Green function. For the reader's convenience, we recall the definition of $K(\Omega)$.

Definition 1.2. (see [13] and [14]) A Borel measurable function ϕ in Ω belongs to the Kato class $K(\Omega)$ if ϕ satisfies

$$\lim_{r \longrightarrow 0} \left(\sup_{x \in \Omega} \int_{\Omega \cap B(x,r)} \frac{\delta_\Omega(y)}{\delta_\Omega(x)} H_\Omega(x,y)\ |\phi(y)|\ dy \right) = 0,$$

where

$$H_\Omega(x,y) = \begin{cases} \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^{n-2}\left(|x-y|^2 + \delta_\Omega(x)\delta_\Omega(y)\right)} & \text{if } n \ge 3 \\ \ln\left(1 + \frac{\delta_\Omega(x)\delta_\Omega(y)}{|x-y|^2}\right) & \text{if } n = 2. \end{cases}$$

Our aim in this paper is to prove an existence result and asymptotic behavior for positive solutions of the problem (1.2) by applying the sharp estimates for the Green function established by K. Hirata in [17]. The following notations will be adopted :

i) $C(\Gamma)$ will denote the set of continuous functions in Γ.

ii) $C_0(\overline{\Gamma}\cup\{\infty\}) = \left\{v \in C(\overline{\Gamma}\cup\{\infty\}) \ : \ \lim_{x\longrightarrow\partial\Gamma} v(x) = \lim_{x\longrightarrow\infty} v(x) = 0\right\}$. We recall that this space endowed with the uniform norm is Banach $\|v\|_\infty = \sup_{x\in\Gamma}|v(x)|$.

iii) For two positive functions f_1 and f_2, we write $f_1 \simeq f_2$ if there exists a constant $A \geq 1$ such that $A^{-1}f_1 \leq f_2 \leq Af_1$. The constant A will be called the constant of comparison.

Let $B(\Gamma)$ be the set of Borel measurable functions in Γ and $B^+(\Gamma)$ the set of non negative one. We define the potential kernel V on $B^+(\Gamma)$ by

$$V\phi(x) = \int_\Gamma G_\Gamma(x,y)\,\phi(y)\,dy.$$

We note that, for any $\phi \in B^+(\Gamma)$ such that $\phi \in L^1_{loc}(\Gamma)$ and $V\phi \in L^1_{loc}(\Gamma)$, we have in the distributional sense (see [18], p.49)

$$\Delta(V\phi) = -\phi \qquad \text{in } \Gamma. \tag{1.5}$$

We point out that for any $\phi \in B^+(\Gamma)$ such that $V\phi \not\equiv \infty$, we have $V\phi \in L^1_{loc}(\Gamma)$, (see [18], p.51). The following hypothesis on φ and ψ are adopted :

(H$_1$) φ is a nonnegative Borel measurable function on $\Gamma\times]0,+\infty[$, continuous and non-increasing with respect to the second variable.

(H$_2$) For all $c > 0$, $x \longmapsto \varphi\left(x, c\,\dfrac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right)$, belongs to $K(\Gamma)$.

(H$_3$) ψ is a nonnegative Borel measurable function on $\Gamma\times]0,+\infty[$, continuous with respect to the second variable such that there exist a nontrivial nonnegative function p and a non negative function $q \in K(\Gamma)$ satisfying for $x \in \Gamma$ and $t > 0$,

$$p(x)\,h(t) \leq \psi(x,t) \leq q(x)\,f(t),$$

where h is a measurable nondecreasing function on $[0,+\infty[$ satisfying

$$\lim_{t\longrightarrow 0^+} \frac{h(t)}{t} = +\infty,$$

and f is a nonnegative measurable function locally bounded on $[0,+\infty[$ satisfying

$$\limsup_{t\longrightarrow+\infty} \frac{f(t)}{t} < \frac{1}{\|Vq\|_\infty}.$$

Note that in [12] Mâagli and Masmoudi studied the case $\varphi = 0$, under similar conditions to those in (**H$_3$**). Indeed the authors gave an existence result for

$$\Delta u + \psi(.,u) = 0, \quad \text{in } D,$$

with some boundary conditions, where D is an unbounded domain in $\mathbb{R}^n$ ($n \geq 2$) with a compact smooth nonempty boundary. Typical examples of nonlinearities satisfying $(\mathbf{H_1})-(\mathbf{H_3})$ are

$$\begin{aligned}\varphi(x,t) &= p(x)\left(\frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right)^{\gamma} t^{-\gamma}, \quad \text{for } \gamma \geq 0,\\ \psi(x,t) &= p(x)\, t^{\alpha}\, \ln(1+t^{\beta}), \quad \text{for } \alpha,\beta \geq 0 \text{ such that } \alpha+\beta < 1,\end{aligned}$$

where p is a non negative function in $K(\Gamma)$. Our main result is the following

Theorem 1.1. *Assume* $(\mathbf{H_1})-(\mathbf{H_3})$. *Then the problem* (1.2) *has a positive solution* $u \in C_0(\overline{\Gamma}\cup\{\infty\})$ *satisfying for each* $x \in \Gamma$

$$\theta\,\frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}} \leq u(x) \leq V\left(\varphi\left(.,\theta\,\frac{K_\Gamma(.,\infty)}{(1+|.|)^{n-2+2\alpha}}\right)\right)(x) + bV\,q(x),$$

where θ, b *are positive constants.*

This paper consists of 4 section devoted to the following topics : In section 2, we recall some helpful results established by Hirata in [17]. In section 3 we establish some preliminaries results that will be necessary to prove Theorem 1.1 in section 4.

2 recall of some helpful results

Lemma 2.1. *There exists a constant* $A_1 > 0$ *depending only on* Γ *such that for all* $x,y \in \Gamma$, *we have*

$$G_\Gamma(x,y) \geq A_1\,\frac{K_\Gamma(x,\infty)\,K_\Gamma(y,\infty)}{\Big((1+|x|)\,(1+|y|)\Big)^{n-2+2\alpha}} \tag{2.1}$$

Theorem 2.1. (3-G Inequalities). *There exists a constant* A *depending only on* Γ *such that for* $x,y,z \in \Gamma$,

$$\frac{G_\Gamma(x,y)\,G_\Gamma(y,z)}{G_\Gamma(x,z)} \leq A\left(\frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\,G_\Gamma(x,y) + \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\,G_\Gamma(y,z)\right). \tag{2.2}$$

Lemma 2.2. *Let* $r > 0$ *and* $R > 0$. *Then there exists a constant* A *depending only on* r, R *and* Γ *such that for* $x,y \in \Gamma\cap B(R)$ *with* $|x-y| \geq r$,

$$G_\Gamma(x,y) \leq A\,K_\Gamma(x,\infty)\,K(y,\infty). \tag{2.3}$$

Proposition 2.1. *If* ϕ *is a Borel measurable function in* Γ *such that* $\phi \in K(\Gamma)$, *then*

$$\|\phi\|_H = \sup_{x\in\Gamma}\int_\Gamma \frac{K_\Gamma(y,\infty)}{K_\Gamma(x,\infty)}\,G_\Gamma(x,y)\,|\phi(y)|\,dy < +\infty,$$

moreover for each $R > 0$, $\displaystyle\int_{\Gamma\cap B(R)} K_\Gamma^2(y,\infty)\,|\phi(y)|\,dy < +\infty$.

Corollary 2.1. *For $x, y \in \Gamma$ with $2|y| \leq |x|$,*

$$G_\Gamma(x,y) \simeq |x|^{2-n-2\alpha}\ K_\Gamma(x,\infty)\ K_\Gamma(y,\infty) = K_\Gamma(x,0)\ K_\Gamma(y,\infty),$$

where the constant of comparaison depends only on Γ.

Corollary 2.2. *Let ϕ be a Borel measurable function in Γ such that $\phi \in K(\Gamma)$. Then, for each $R > 0$,*

$$\int_{\Gamma\cap B(R)} K_\Gamma(y,\infty)\ |\phi(y)|\ dy < +\infty. \tag{2.4}$$

Lemma 2.3. *Let ϕ be a Borel measurable function in Γ such that $\varphi \in K(\Gamma)$. Then, for each $x_0 \in \overline{\Gamma}$,*

$$\lim_{r\longrightarrow 0} \int_{\Gamma\cap B(x_0,r)} K_\Gamma(y,\infty)^2\ |\phi(y)|\ dy = 0.$$

3 preliminaries results

Proposition 3.1. *Let ϕ be a Borel measurable function in Γ such that $\phi \in K(\Gamma)$ and h be a positive superharmonic function in Γ. Then*

a)

$$\lim_{r\longrightarrow 0} \sup_{x\in\Gamma} \frac{1}{h(x)} \int_{B(x_0,r)\cap\Gamma} G_\Gamma(x,y)\, h(y)\ |\phi(y)|\ dy = 0, \quad \forall x_0 \in \overline{\Gamma}\,. \tag{3.1}$$

$$\lim_{M\longrightarrow +\infty} \sup_{x\in\Gamma} \frac{1}{h(x)} \int_{\Gamma\setminus B(M)} G_\Gamma(x,y)\, h(y)\ |\phi(y)|\ dy = 0\,. \tag{3.2}$$

b) *For all $x \in \Gamma$ and A as in Theorem 2.1,*

$$\int_\Gamma G_\Gamma(x,y)\, h(y)\ |\phi(y)|\ dy \leq 2\, A\ \|\phi\|_H\ h(x)\,. \tag{3.3}$$

Proof. Let h be a positive superharmonic function in Γ. Then by [[7],Theorem 2.1, p.164], there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of positive measurable functions in Γ such that

$$h(y) = \sup_{n\in\mathbb{N}} \int_\Gamma G_\Gamma(y,z)\, f_n(z)\, dz.$$

Hence we need to verify (3.1), (3.2) and (3.3) only for $h(y) = G_\Gamma(y,z)$, uniformly for $z \in \Gamma$.

a) Let $r > 0$. By using Theorem 2.1, we get

$$\frac{1}{G_\Gamma(x,z)} \int_{B(x_0,r)\cap\Gamma} G_\Gamma(x,y)\, G_\Gamma(y,z)\ |\phi(y)|\ dy \leq 2\, A \sup_{z\in\Gamma} \int_{B(x_0,r)\cap\Gamma} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)\ |\phi(y)|\ dy.$$

Let $\varepsilon > 0$. Since $\phi \in K(\Gamma)$, there exist positive numbers r_1 and R_1 such that

$$\sup_{z\in\Gamma} \int_{\Gamma\cap B(z,r_1)} \frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\, G_\Gamma(z,y)|\phi(y)|\ dy \leq \varepsilon$$

and

$$\sup_{z\in\Gamma}\int_{\Gamma\setminus B(R_1)}\frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\,G_\Gamma(z,y)\;|\phi(y)|\;dy\le\varepsilon\,.$$

Let $r>0$ and $z\in\Gamma$. Then, we have by Lemma 2.2

$$\begin{aligned}\int_{\Gamma\cap B(x_0,r)}\frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\,G_\Gamma(z,y)\,|\phi(y)|\;dy &\le 2\varepsilon+\int_{\Gamma\cap B(x_0,r)\cap B(R_1)\setminus B(z,r_1)}\frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\,G_\Gamma(z,y)\,|\phi(y)|\;dy\\ &\le 2\varepsilon+A\int_{\Gamma\cap B(x_0,r)}K_\Gamma(y,\infty)^2\;|\phi(y)|\;dy.\end{aligned}$$

Hence, (3.1) follows from Lemma 2.3. On the other hand, we have

$$\frac{1}{G_\Gamma(x,z)}\int_{(|y|\ge M)\cap\Gamma}G_\Gamma(x,y)\,G_\Gamma(y,z)\;|\phi(y)|\;dy\le 2\,A\sup_{z\in\Gamma}\int_{(|y|\ge M)\cap\Gamma}\frac{K_\Gamma(y,\infty)}{K_\Gamma(z,\infty)}\,G_\Gamma(z,y)\;|\phi(y)|\;dy,$$

which converges to zero as $M\longrightarrow+\infty$. This gives (3.2).

b) By using Theorem 2.1, we obtain

$$\frac{1}{G_\Gamma(x,z)}\int_\Gamma G_\Gamma(x,y)\,G_\Gamma(y,z)\;|\phi(y)|\;dy\le 2\,A\;\|\phi\|_H\,.$$

Corollary 3.1. *Let ϕ be a Borel measurable function in Γ such that $\phi\in K(\Gamma)$. Then, we have*

$$\sup_{x\in\Gamma}\int_\Gamma G_\Gamma(x,y)\;|\varphi(y)|\;dy<\infty,\tag{3.4}$$

$$\int_\Gamma\frac{K_\Gamma(y,\infty)}{(1+|y|)^{n+2\alpha-2}}\;|\phi(y)|\;dy<+\infty.\tag{3.5}$$

Proof. Inequality (3.4) is a consequence of (3.3) with $h=1$ in Γ and Proposition 2.1. Let $x_0\in\Gamma$. Then by Lemma 2.1 and (3.4) we get

$$\int_\Gamma\frac{K_\Gamma(y,\infty)}{(|y|+1)^{n+2\alpha-2}}\;|\phi(y)|\;dy\le A_1\,\frac{(|x_0|+1)^{n+2\alpha-2}}{K_\Gamma(x_0,\infty)}\,\sup_{x\in\Gamma}\int_\Gamma G_\Gamma(x,y)\;|\phi(y)|\;dy<\infty.$$

By using (2.1) and (3.5), we get

Proposition 3.2. *Let ϕ be a Borel measurable function in Γ such that $\phi\in K(\Gamma)$. Then, there exists a constant $A_2>0$ such that for all $x\in\Gamma$*

$$V\phi(x)\ge A_2\,\frac{K_\Gamma(x,\infty)}{(1+|x|)^{n+2\alpha-2}}.\tag{3.6}$$

Proposition 3.3. *Let ϕ be a Borel measurable function in Γ such that $\phi\in K(\Gamma)$. Then, the function $V\phi$ is in $C_0(\overline{\Gamma}\cup\{\infty\})$.*

Proof. Let $x_0 \in \overline{\Gamma}$ and $\delta > 0$. Let $x, x' \in \Gamma \cap B(x_0, \frac{\delta}{2})$.

$$\begin{aligned}\left|V\phi(x) - V\phi(x')\right| \leq \quad & 2 \sup_{x\in\Gamma} \int_{\Gamma\setminus B(\delta^{-1})} G_\Gamma(x,y)\ |\phi(y)|\ dy + 2 \sup_{x\in\Gamma} \int_{\Gamma\cap B(x_0,\delta)} G_\Gamma(x,y)\ |\phi(y)|\ dy\ + \\ & + \int_{\Gamma\cap B(\delta^{-1})\setminus B(x_0,\delta)} \left|G_\Gamma(x,y) - G_\Gamma(x',y)\right|\ |\phi(y)|\ dy\ .\end{aligned}$$

By (3.1) and (3.2), the first two quantities of the right hand side are bounded by ε whenever δ is sufficiently small. For δ sufficiently small, $G_\Gamma(.,y)$ can be extended continuously to $B(x_0, \frac{\delta}{2}) \cap \overline{\Gamma}$ whenever $y \in \Gamma \setminus B(x_0,\delta)$. Moreover, by (2.3) and (1.1), there exists $A > 0$ such that

$$G_\Gamma(x,y) \leq A\ K_\Gamma(y,\infty), \quad \forall (x,y) \in \Big(B(x_0, \frac{\delta}{2}) \cap (\Gamma \cup \partial_r\Gamma)\Big) \times \Big(\Gamma \cap B(\delta^{-1}) \setminus B(x_0,\delta)\Big).$$

Then by (2.4) and Lebesgue's theorem, we have

$$\int_{(\Gamma\cap B(\delta^{-1}))\setminus B(x_0,\delta)} \left|G_\Gamma(x,y) - G_\Gamma(x',y)\right|\ |\phi(y)|\ dy \underset{|x-x'|\longrightarrow 0}{\longrightarrow} 0\ .$$

Hence, $V\phi$ is continuous in $\overline{\Gamma}$.
Now we will show that $\lim\limits_{x\longrightarrow\partial\Gamma} V\phi(x) = \lim\limits_{|x|\longrightarrow+\infty} V\phi(x) = 0$.
Let $x_0 \in \partial\Gamma$, $\delta \in]0,1[$ and $x \in B(x_0, \frac{\delta}{2}) \cap \Gamma$. Then

$$\begin{aligned}|V\phi(x)| \quad &\leq \quad \int_\Gamma G_\Gamma(x,y)\ |\phi(y)|\ dy \\ &\leq \quad \sup_{z\in\Gamma} \int_{B(x_0,\delta)\cap\Gamma} G_\Gamma(z,y)\ |\phi(y)|\ dy + \sup_{z\in\Gamma} \int_{\Gamma\setminus B(\delta^{-1})} G_\Gamma(z,y)\ |\phi(y)|\ dy \\ &\quad + \int_{\Gamma\cap B(\delta^{-1})\setminus B(x_0,\delta)} G_\Gamma(x,y)\ |\phi(y)|\ dy\ .\end{aligned}$$

By Lemma 2.2, we get

$$\int_{\Gamma\cap B(\delta^{-1})\setminus B(x_0,\delta)} G_\Gamma(x,y)\ |\phi(y)|\ dy \leq\ A\ K_\Gamma(x,\infty) \int_{\Gamma\cap B(\delta^{-1})} K_\Gamma(y,\infty)\ |\phi(y)|\ dy.$$

Then, we obtain, by (2.4), (3.1) and (3.2) with $h = 1$, that $\lim\limits_{x\longrightarrow\partial\Gamma} V\phi(x) = 0$.
We next consider $x_0 = \infty$. Let $M > 0$. Then

$$|V\phi(x)| \leq \int_{\Gamma\cap B(M)} G_\Gamma(x,y)\ |\phi(y)|\ dy + \int_{\Gamma\setminus B(M)} G_\Gamma(x,y)\ |\phi(y)|\ dy$$

By (3.2), the second term of the right hand side is bounded by ε uniformly for x, whenever M is sufficiently large. Using (1.1) and Corollary 2.1 , we get

$$G_\Gamma(x,y) \leq A\ \frac{K_\Gamma(y,\infty)}{|x|^{n-2+\alpha}}, \quad \text{for } x \in \Gamma \setminus B(2M) \text{ and } y \in \Gamma \cap B(M).$$

It follows from Corollary 2.2 that $\lim\limits_{|x|\longrightarrow+\infty} |V\phi(x)| = 0$.

For a fixed nonnegative function q in $K(\Gamma)$, we put

$$\mathcal{M}_q = \{\Psi \in B(\Gamma),\ |\Psi| \leq q\}.$$

by the same way in the proof of Proposition 3.3, we prove

Proposition 3.4. *Let q be a nonnegative function in $K(\Gamma)$, then the family of functions*

$$V(\mathcal{M}_q) = \left\{V\Psi\ :\ \Psi \in \mathcal{M}_q\right\}$$

is relatively compact in $C_0(\overline{\Gamma} \cup \{\infty\})$.

4 Proof of Theorem 1.1

The proof is based on Schauder fixed point argument. In the sequel, we suppose that Γ is a NTA-cone in $\mathbb{R}^n$ ($n \geq 3$). Let $\mathcal{K}$ be a compact of Γ such that using (**H$_3$**), we have

$$0 < a\ := \int_{\mathcal{K}} \frac{K_\Gamma(y,\infty)}{(1+|y|)^{n+2\alpha-2}}\ p(y)\ dy < +\infty.$$

Let $\beta = \min\limits_{x \in \mathcal{K}} \dfrac{K_\Gamma(x,\infty)}{(1+|x|)^{n+2\alpha-2}} \neq 0.$

Since $\lim\limits_{t \longrightarrow 0^+} \dfrac{h(t)}{t} = +\infty$, then there exists $\theta > 0$, such that

$$a\, A_1\ h(\theta\beta) \geq \theta. \tag{4.1}$$

Let $\mu > 0$ such that $\limsup\limits_{t \longrightarrow +\infty} \dfrac{f(t)}{t} < \mu < \dfrac{1}{\|Vq\|_\infty}$. Then there exist $r > 0$, such that for all $t \geq r$

$$f(t) \leq \mu t.$$

Hence for all $t \geq 0$, we obtain

$$0 \leq f(t) \leq \mu t + \left(\sup_{t \in [0,r]} f(t)\right) = \sigma. \tag{4.2}$$

From (**H$_2$**) and Proposition 3.4, we observe that

$$\left\| V\varphi\left(.,\theta\ \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n-2+2\alpha}}\right)\right\|_\infty < \infty, \quad V\varphi\left(.,\theta\ \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n-2+2\alpha}}\right) \text{ and } Vq\ \in C_0(\overline{\Gamma} \cup \{\infty\}).$$

Let

$$b = \left(\frac{\theta}{A_2}\ \vee\ \frac{\mu\ \left\|V\varphi(.,\theta\ \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n-2+2\alpha}})\right\|_\infty + \sigma}{1 - \mu\|Vq\|_\infty}\right)$$

and consider the closed convex set

$$F = \left\{u \in C_0(\overline{\Gamma} \cup \{\infty\})\ :\ \theta\ \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}} \leq u(x) \leq V\varphi\left(.,\theta\ \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right) + \ b\ Vq(x), \quad \forall x \in \Gamma\right\}.$$

It is clear from (3.6) that $F \neq 0$. For $u \in F$, we define

$$Tu(x) = \int_\Gamma G_\Gamma(x,y)\ (\varphi(y,u(y))\ +\ \psi(y,u(y)))\ dy, \quad x \in \Gamma.$$

Lemma 4.1. *$T(F) \subset F$. Moreover $T(F)$ is relatively compact in $C_0(\overline{\Gamma} \cup \{\infty\})$.*

Proof. Let $u \in F$ and $x \in \Gamma$, then by (4.2) we have

$$\begin{aligned} Tu(x) &\leq V\varphi\left(.,\theta \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right)(x) + \int_\Gamma G_\Gamma(x,y)\, q(y)\, f(u(y))\, dy \\ &\leq V\varphi\left(.,\theta \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right)(x) + \int_\Gamma G_\Gamma(x,y)\, q(y)\, (\mu\, u(y) + \sigma)\, dy \\ &\leq V\varphi\left(.,\theta \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right)(x) + b\, Vq(x). \end{aligned}$$

Moreover, since h is nondecreasing, then it follows from (2.1) and (4.1) that

$$\begin{aligned} Tu(x) &\geq \int_\Gamma G_\Gamma(x,y)\, \psi(y,u(y))\, dy \\ &\geq A_1 \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\, h(\theta\beta) \int_{\mathcal{K}} \frac{K_\Gamma(y,\infty)}{(1+|y|)^{n-2+2\alpha}}\, p(y)\, dy \\ &\geq A_1\, a\, h(\theta\beta) \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}} \\ &\geq \theta \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}, \ \forall x \in \Gamma. \end{aligned}$$

On the other hand, for all $u \in F$, we have

$$\varphi(.,u) \leq \varphi\left(.,\theta \frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\right), \tag{4.3}$$

and

$$\psi(.,u) \leq \mu\left(\left\| V\varphi(.,\theta \frac{K_\Gamma(x,\infty)}{(|x|+1)^{n-2+2\alpha}})\right\|_\infty + b\, \|Vq\|_\infty + \sigma\right) q. \tag{4.4}$$

Thus we deduce by Proposition 3.4, that $T(F)$ is relatively compact in $C_0(\overline{\Gamma} \cup \{\infty\})$. Hence $T(F) \subset F$.

Lemma 4.2. *T is continuous in F.*

Proof. Let $(u_k)_{k\in\mathbb{N}}$ be a sequence in F which converges uniformly to $u \in F$. Since φ and ψ are continuous with respect to the second variable, then it follows by the dominated convergence theorem that for all $x \in \overline{\Gamma} \cup \{\infty\}$ $\lim_{k\longrightarrow+\infty} Tu_k(x) = Tu(x)$. Hence, Tu_k converges pointwisely to Tu in Γ as $k \longrightarrow +\infty$. Since $T(F)$ is relatively compact in $C_0(\overline{\Gamma} \cup \{\infty\})$, the pointwise convergence implies the uniform convergence. Thus $\lim_{k\longrightarrow+\infty} \|Tu_k - Tu\|_\infty = 0$. Hence T is continuous on F.

Proof. of Theorem 1.1 Let us recall that F is a nonempty closed convex set in $C_0(\overline{\Gamma} \cup \{\infty\})$. Since T is a compact mapping from F to it self, it follows from Schauder's fixed point theorem, that there exists $u \in F$ such that $T(u) = u$, that is

$$u(x) = \int_\Gamma G_\Gamma(x,y)\, (\varphi(y,u(y)) + \psi(y,u(y)))\, dy, \quad \forall x \in \Gamma. \tag{4.5}$$

Since q and $\varphi\Big(.,\theta\,\frac{K_\Gamma(x,\infty)}{(1+|x|)^{n-2+2\alpha}}\Big)$ are in $K(\Gamma)$, it follows by (4.3), (4.4) that the function $y \longmapsto \varphi(y,u(y))+\psi(y,u(y))$ belongs to $L^1_{loc}(\Gamma)$. On the other hand, since $u \in C_0(\overline{\Gamma}\cup\{\infty\})$, we deduce from (4.5) that

$$V(\varphi(.,u)\ +\ \psi(.,u)) \in C_0(\overline{\Gamma}\cup\{\infty\}) \subset L^1_{loc}(\Gamma).$$

Hence

$$\Delta u+\varphi(.,u)+\psi(.,u)=0, \quad \text{in } \Gamma \text{ in the sense of distribution},$$

and so u is a solution of (1.2).

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