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An Essay Toward a Unified Theory of the L^p Version of Hardy's Uncertainty Principles and q-Analogues

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Abstract

In this paper, we give an L^p version of Hardy's uncertainty principles for a large class of integral and q-integral transforms. As an applications, we discuss an L^p version of Hardy's theorem for the generalized Fourier transform associated with the Sturm-Liouville operator and for the Jacobi-Dunkl transform.

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1 Introduction

In 1933, G. H. Hardy [14] showed that a measurable function f on $\mathbb{R}$ and its Fourier transform $\widehat{f}$ cannot, simultaneously, be very rapidly decreasing. More precisely, he proved that if

$$|f(x)| \le Ce^{-ax^2} \quad \text{and} \quad |\widehat{f}(\lambda)| \le Ce^{-b\lambda^2}, \qquad \text{for a.e.} \quad x, \lambda \in \mathbb{R},$$

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where a, b and C are positive constants, and if $ab > 1/4$, then $f \equiv 0$.

A famous generalization of Hardy's theorem, called the L^p version of Hardy's theorem, is obtained in 1983 by M. G. Cowling and J. F. Price [8]. They proved that if $p, n \in [1, +\infty]$, with at least one of them finite,

$$\|e^{ax^2} f\|_p < \infty \quad \text{and} \quad \|e^{bx^2} \widehat{f}\|_n < \infty,$$

where $\|.\|_p$ is the norm of the Lebesgue space $L^p(\mathbb{R})$, and $ab \geq \frac{1}{4}$, then $f \equiv 0$.

There has been considerable interest in a better understanding of these results and on its extensions to other settings including certain Lie groups, the motion group, the Heisenberg group, hypergroups, Dunkl theory and recently quantum calculus, see [2, 7, 9, 11–13, 15, 16, 19].

In this paper, we give a unified approach to the L^p version of Hardy's theorem for a large class of integral transform having the form

$$\mathcal{F}(f)(\lambda) = \int_\beta^\infty f(t)\varphi(\lambda, t)w(t)dt, \tag{1.1}$$

where β is either 0 or $-\infty$, the kernel φ satisfies certain analycity and growth conditions on $\mathbb{C}\times]\beta, \infty[$, and w is a positive weight function. The integral transforms considered include the classical Fourier transform, the Hankel transform, the generalized Fourier transform associated with a Sturm-Liouville operator, the one dimensional Dunkl transform and the Jacobi-Dunkl transform. We also deal with the q-analogue of this theory.

This paper is organized as follows. In Section 2, we state for $\mathcal{F}$ an L^p version of Hardy's uncertainty principle, and then derive the analogue of Hardy's theorem. Since the symmetric case has been dealt with in [11], we will focus our applications, in Section 3, on tow non-symmetric integral transforms : the generalized Fourier transform associated with a second order singular differential operator and the Jacobi-Dunkl transform. Hardy's theorem and its L^p version for these integral transforms are studied in [19] and [7] respectively, where the heat kernel, which is not explicitly given, play the role of the Gaussian kernel e^{-at^2}. We will derive the results from the approach stated in Section 2 and then show that the use of the tow kernels is equivalent. Section 4 is devoted to the q-analogues of the results presented in Section 2.

2 Hardy's theorem and its L^p version for $\mathcal{F}$

In what follows, we will assume that the following conditions are satisfied:

(H1) $\beta \in \{-\infty, 0\}$.

(H2) The kernel φ in (1.1) satisfies the following conditions:

(a) there exist C, $k > 0$ such that

$$|\varphi(x, t)| \leq Ce^{k|x||t|}, \qquad \text{for all } (x, t) \in \mathbb{C} \times (\beta, +\infty); \tag{2.1}$$

(b) for every $t \in (\beta, +\infty)$, the function $\varphi(., t) \colon \lambda \mapsto \varphi(\lambda, t)$ is entire on $\mathbb{C}$ and when $\beta = 0$, it's restriction on $\mathbb{R}$ is even.

(H3) The weight function w is $a.e$-positive on $(\beta, +\infty)$ and for all $c > 0$, the function $t \mapsto w(t)e^{-ct^2}$ is Lebesgue integrable on $(\beta, +\infty)$.

For $p \in [1, +\infty]$, we denote by $L^p_w(\beta, +\infty)$ the weighted Lebesgue spaces associated to the function w, equipped with the norm

$$\|f\|_{p,w} = \left(\int_\beta^\infty |f(t)|^p w(t) dt \right)^{1/p}, \qquad \text{if } 1 \le p < \infty$$

and

$$\|f\|_{p,w} = \|f\|_\infty = \operatorname*{ess\,sup}_{t \in (\beta, +\infty)} |f(t)|, \qquad \text{if } p = +\infty.$$

Proposition 2.1. *Let $n \in [1, +\infty[$ and let μ be the restriction on $(\beta, +\infty)$ of an even a.e-positive function for which there exist μ_1, $t_1 > 0$ such that*

$$\mu(t) \ge \mu_1, \qquad \textit{for a.e.} \quad t \ge t_1. \tag{2.2}$$

If an entire function h on $\mathbb{C}$ satisfies

$$|h(z)| \le M e^{a(\Re(z))^2}, \qquad \textit{for all } z \in \mathbb{C},$$

where a and M are positives constants, and

$$\|h\|_{n,\mu} = \left(\int_\beta^\infty |h(t)|^n \mu(t) dt \right)^{1/n} < \infty,$$

then h must vanish on $\mathbb{C}$.

Proof. The result can be obtained using (2.2) and the same technique as in [8]. □

Theorem 2.2. *Let $p \in [1, +\infty]$ and f be a measurable function on (β, ∞). Suppose that $\|e^{at^2} f\|_{p,w} < \infty$ for some constant $a > 0$. Then*

(*i*) *The function $\mathcal{F}(f)$ is entire on $\mathbb{C}$.*

(*ii*) *For every $a' \in]0, a[$, there exists $C' > 0$ such that*

$$|\mathcal{F}(f)(z)| \le C' e^{\frac{k^2 |z|^2}{4a'}}, \qquad \textit{for all } z \in \mathbb{C}, \tag{2.3}$$

where k is given in (2.1).

Proof. The proof of (i) follows from the conditions satisfied by φ, the hypothesis of the theorem, Hölder's inequality and the analyticity theorem. To prove (ii), set $n = p/(p-1)$. It follows that

$$\begin{aligned} |\mathcal{F}(f)(z)| &\le \int_\beta^\infty |\varphi(z,t)||f(t)|w(t)dt \le C \int_\beta^\infty e^{k|z||t| - at^2} e^{at^2} |f(t)| w(t) dt \\ &\le C \left(\int_\beta^\infty e^{n(k|z||t| - at^2)} w(t) dt \right)^{1/n} \|e^{at^2} f\|_{p,w}, \end{aligned}$$

for all $z \in \mathbb{C}$. Now, let $a' \in]0,a[$ and set $c = a - a'$. Then,

$$\begin{aligned}\left(\int_\beta^\infty e^{n(k|z||t|-at^2)}w(t)dt\right)^{1/n} &= \left(\int_\beta^\infty e^{n(k|z||t|-a't^2)}w(t)e^{-nct^2}dt\right)^{1/n} \\ &\le \left(\sup_{t\in(0,\infty)} e^{n(k|z|t-a't^2)}\right)^{1/n}\left(\int_\beta^\infty w(t)e^{-nct^2}dt\right)^{1/n} \\ &= \left(\int_\beta^\infty w(t)e^{-nct^2}dt\right)^{1/n} e^{\frac{k^2}{4a'}|z|^2},\end{aligned}$$

and (2.3) follows by taking $C' = C\|e^{at^2}f\|_{p,w}\|e^{-(a-a')t^2}\|_{n,w}$. □

Theorem 2.3. *Let $p \in [1,+\infty]$ and $n \in [1,+\infty[$, let μ be a function satisfying the conditions of Proposition 2.1, and let f be a measurable function on (β,∞). Suppose that*

$$\|e^{at^2}f\|_{p,w} < \infty \quad \text{and} \quad \|e^{b\lambda^2}\mathcal{F}(f)\|_{n,\mu} < \infty, \tag{2.4}$$

where a, $b > 0$. If $ab > k^2/4$, then $\mathcal{F}(f)$ must vanish on $\mathbb{C}$.

Proof. Assume that $ab > k^2/4$, let $a' \in]\frac{k^2}{4b}, a[$, and define

$$h(z) = e^{\frac{k^2}{4a'}z^2}\mathcal{F}(f)(z), \qquad z \in \mathbb{C}. \tag{2.5}$$

Since, by Theorem 2.2 (i), the function $\mathcal{F}(f)$ is entire on $\mathbb{C}$, it follows that h is entire on $\mathbb{C}$. Moreover, by Theorem 2.2 (ii), there exists $C' > 0$ such that (2.3) holds, which together with (2.5) implies that

$$|h(z)| \le C' e^{\frac{k^2}{2a'}(\Re(z))^2}, \qquad \text{for all } z \in \mathbb{C}.$$

On the other hand, by (2.4) and (2.5), we have

$$\|h\|_{n,\mu} \le \|e^{b\lambda^2}\mathcal{F}(f)\|_{n,\mu} < \infty.$$

So, by Proposition 2.1, the function h vanishes on $\mathbb{C}$, and the proof is complete. □

In the following corollary, we obtain Hardy's theorem for the transform $\mathcal{F}$.

Corollary 2.4. *Let f be a measurable function on $(\beta,+\infty)$. Suppose that*

$$|f(t)| \le Ce^{-at^2} \qquad \text{for a.e.} \quad t \in (\beta,+\infty), \tag{2.6}$$

and

$$|\mathcal{F}(f)(\lambda)| \le Ce^{-b\lambda^2}, \qquad \text{for a.e.} \quad \lambda \in (\beta,+\infty),$$

where a, b, $C \ge 0$. If $ab > k^2/4$, then $\mathcal{F}(f)$ must vanish on $\mathbb{C}$.

Proof. From (2.6) we conclude that $\|e^{at^2} f\|_\infty < \infty$. On the other hand, for an arbitrary $c \in]k^2/4a, b[$, we have

$$\begin{aligned} \|e^{c\lambda^2}\mathcal{F}(f)(\lambda)\|_1 &= \int_\beta^\infty e^{c\lambda^2}|\mathcal{F}(f)(\lambda)|d\lambda \\ &= \int_\beta^\infty e^{b\lambda^2}|\mathcal{F}(f)(\lambda)|e^{(c-b)\lambda^2}d\lambda \\ &\leq \frac{\sqrt{\pi}}{\sqrt{b-c}}\ \|e^{b\lambda^2}\mathcal{F}(f)(\lambda)\|_\infty < \infty, \end{aligned}$$

and the conclusion of the Corollary follows from Theorem 2.3. □

Remark 2.5. If $\mathcal{F}$ is injective, then under the conditions of each of the previous two results, we get $f \equiv 0$.

3 Examples

In this section, we give Hardy's theorem and its L^p version for the generalized Fourier transform associated with a Sturm-liouville operator and the Jacobi-Dunkl transform, and then compare our approach with that of [19] and [7].

3.1 Application to the generalized Fourier transform associated with a Sturm-Liouville operator

Let L_A be the Sturm-Liouville operator on $]0, +\infty[$ defined by

$$L_A(u) = -\frac{1}{A}\frac{d}{dx}\left[A\frac{d}{dx}u\right],$$

where A is a real function on $[0, +\infty[$ satisfying the following conditions :

(i) A is increasing and unbounded.

(ii) A'/A is decreasing, we then denote by 2ρ its limit at infinity

$$2\rho = \lim_{x\to+\infty}\frac{A'(x)}{A(x)}.$$

(iii) A is the form

$$A(x) = x^{2\alpha+1}B(x), \qquad \alpha > -\frac{1}{2},$$

where B is an even C^∞-function and $B(x) > 0$ for $x > 0$.

(iv) There exist δ, $x_0 > 0$ such that for all $x \geq x_0$

$$\frac{A'(x)}{A(x)} = \begin{cases} 2\rho + e^{-\delta x}D(x), & \text{if } \rho > 0; \\ \frac{2\alpha+1}{x} + e^{-\delta x}D(x), & \text{if } \rho = 0, \end{cases}$$

where D is a C^∞-function bounded together with its derivatives.

In particular, if $A(x) = x^{2\alpha+1}$ with $\alpha > -1/2$ then L_A is the Bessel operator, and if $A(x) = (2\sinh x)^{2\alpha+1}(2\cosh x)^{2\beta+1}$ with $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$, then L_A is the Jacobi operator.

We begin by summarizing some facts regarding the harmonic analysis associated with the operator L_A. For more details and proofs we refer the reader to [1], [4], [5], [10], [18], [17] and [20].

For every $\lambda \in \mathbb{C}$, the system

$$\begin{cases} L_A u = (\lambda^2 + \rho^2)u; \\ u(0) = 1, \quad u'(0) = 0, \end{cases} \tag{3.1}$$

has a unique solution on $[0, +\infty)$ extended by parity to $\mathbb{R}$ and then denoted by φ_λ. The function $(x, \lambda) \mapsto \varphi_\lambda(x)$ is C^∞ with respect to x and entire on $\mathbb{C}$ with respect to λ. Moreover, from the integral representation of Mehler type of the function φ_λ (see [10], [18] or [4]), we conclude that

$$|\varphi_\lambda(x)| \leq e^{|\Im(\lambda)||x|} \leq e^{|\lambda||x|},$$

for all $\lambda \in \mathbb{C}$ and $x \in \mathbb{R}$.

The generalized Fourier transform associated with the operator L_A is defined for $f \in L^1_A(\mathbb{R}_+)$ by

$$\mathcal{F}_A(f)(\lambda) = \int_0^{+\infty} f(t)\varphi_\lambda(t)A(t)dt, \qquad \lambda > 0.$$

Theorem 3.1 (Inversion formula)**.** *There exists a continuous function C on $(0, \infty)$ such for every $f \in L^1_A(\mathbb{R}_+)$, if $\mathcal{F}_A(f) \in L^1_\mu(\mathbb{R}_+)$, where*

$$\mu(\lambda) = \frac{1}{2\pi|C(\lambda)|^2},$$

then

$$f(x) = \int_0^{+\infty} \mathcal{F}_A(f)(\lambda)\varphi_\lambda(x)\mu(\lambda)d\lambda, \qquad \textit{for a.e} \quad \lambda > 0.$$

It was shown in [4] that there exist positive constants K_1, K_2, and λ_0 such that

$$K_1\lambda^{2\alpha+1} \leq \frac{1}{|C(\lambda)|^2} \leq K_2\lambda^{2\alpha+1}, \qquad \text{for all } \lambda \geq \lambda_0. \tag{3.2}$$

Now, we are in a situation to state the L^p version of Hardy's theorem for the generalized Fourier transform $\mathcal{F}_A$.

Theorem 3.2. *Let $p \in [1, +\infty]$, let $n \in [1, +\infty[$, and let f be a measurable function on $(0, \infty)$. Suppose that $\|e^{at^2} f\|_{p,A} < \infty$ and $\|e^{bt^2}\mathcal{F}_A(f)\|_{n,\mu} < \infty$ for some constants a, $b > 0$. If $ab > 1/4$, then $f \equiv 0$.*

Proof. We need only show μ satisfies the condition of Proposition 2.1 and A satisfies (H3). The first assertion follows immediately from (3.2). For the second one, since A is continuous on $[0, +\infty)$ and $\lim_{x\to+\infty} A'(x)/A(x) = 2\rho$, there exists a positive constant M, such that

$$0 < A(x) \leq Me^{(2\rho+1)x}, \qquad \text{for all } x \in (0, +\infty),$$

which implies that for all $c > 0$, the function $x \mapsto e^{-cx^2} A(x)$ is Lebesgue integrable on $[0, +\infty)$, and (H3) is satisfied. □

Corollary 3.3. *Let f be a measurable function on $(0, \infty)$. Suppose that*

$$|f(x)| \leq Ce^{-ax^2} \quad \textit{and} \quad |F(f)(\lambda)| \leq Ce^{-b\lambda^2}, \qquad \textit{for a.e.} \quad x, \lambda \in (0, \infty),$$

where a, b and C are positives constantes. If $ab > 1/4$, then $f \equiv 0$.

3.2 Application to the Jacobi-Dunkl transform

In this subsection, we assume, as in [6], that α and β are real parameters satisfying $\alpha \geq \beta \geq -1/2$ and $\alpha \neq -1/2$. The Jacobi-Dunkl operator $\Lambda_{\alpha,\beta}$ is the differential-difference operator defined for smooth function f on $\mathbb{R}$ by

$$\Lambda_{\alpha,\beta} f(x) = \frac{df}{dx}(x) + \frac{A'_{\alpha,\beta}(x)}{A_{\alpha,\beta}(x)} \frac{f(x) - f(-x)}{2},$$

where

$$A_{\alpha,\beta}(x) = 2^{2\rho} (\sinh |x|)^{2\alpha+1} (\cosh |x|)^{2\beta+1} \quad \text{and} \quad \rho = \alpha + \beta + 1. \tag{3.3}$$

From [6] we know that for every $\lambda \in \mathbb{C}$, the system

$$\begin{cases} \Lambda_{\alpha,\beta} f(x) = i\lambda f(x); \\ f(0) = 0, \end{cases}$$

admits a unique solution $\psi_\lambda^{(\alpha,\beta)}$ called the Jacobi-Dunkl Kernel and given by

$$\psi_\lambda^{(\alpha,\beta)}(x) = \begin{cases} \varphi_\mu^{(\alpha,\beta)}(x) - \frac{i}{\lambda} \frac{d}{dx} \varphi_\mu^{(\alpha,\beta)}(x), & \text{if } \lambda \neq 0; \\ 1, & \text{if } \lambda = 0, \end{cases}$$

which can be written also as [3]

$$\psi_\lambda^{(\alpha,\beta)}(x) = \varphi_\mu^{(\alpha,\beta)}(x) + i \frac{\lambda}{4(\alpha+1)} \sinh 2x \, \varphi_\mu^{(\alpha+1,\beta+1)}(x), \tag{3.4}$$

where $\varphi_\mu^{(\alpha,\beta)}$ is the Jacobi function given by the use of the Gauss function as

$$\varphi_\mu^{(\alpha,\beta)}(x) = {}_2F_1\left(\frac{\rho + i\mu}{2}, \frac{\rho - i\mu}{2}; \alpha + 1; -(\sinh x)^2\right),$$

with μ such that $\lambda^2 = \mu^2 + \rho^2$. We recall that the Jacobi function $\varphi_\mu^{(\alpha,\beta)}$ is the solution of (3.1) when L_A is the Jacobi operator. From this and (3.4) it follows that the function $(x, \lambda) \mapsto \psi_\lambda(x)$ is even, C^∞ with respect to x and entire on $\mathbb{C}$ with respect to λ. Furthermore, the Jacobi-Dunkl kernel has the integral representation of Laplace type

$$\psi_\lambda^{(\alpha,\beta)}(x) = \int_{-|x|}^{|x|} K(x,t) e^{i\lambda t} dt, \qquad x \in \mathbb{R} \setminus \{0\}, \lambda \in \mathbb{C},$$

where, for $x \in \mathbb{R} \setminus \{0\}$, the function $K(x,\cdot)\colon t \mapsto K(x,t)$ is nonnegative on $\mathbb{R}$, continuous on $]-|x|,|x|[$, supported in $[-|x|,|x|]$ and

$$\int_{-|x|}^{|x|} K(x,t)dt = 1,$$

which implies that

$$|\psi_\lambda(x)| \le e^{|x||\lambda|}, \qquad \text{for all } (\lambda,x) \in \mathbb{C}\times\mathbb{R}.$$

The Jacobi-Dunkl transform $\mathcal{F}_{\alpha,\beta}$ is defined for $f \in L^1_{A_{\alpha,\beta}}(\mathbb{R})$ by

$$\mathcal{F}_{\alpha,\beta}(f)(\lambda) = \int_{-\infty}^{\infty} f(t)\psi_\lambda^{\alpha,\beta}(t)A_{\alpha,\beta}(t)dt, \qquad \lambda \in \mathbb{R}.$$

In order to give inversion formula for $\mathcal{F}_{\alpha,\beta}$, we shall introduce the function σ as follows:

$$\sigma(\lambda) = \frac{|\lambda|}{8\pi\sqrt{\lambda^2-\rho^2}|c_{\alpha,\beta}(\sqrt{\lambda^2-\rho^2})|}\mathbf{1}_{\mathbb{R}\setminus]-\rho,\rho[}(\lambda), \qquad \lambda \in \mathbb{R}, \tag{3.5}$$

with $\mathbf{1}_{\mathbb{R}\setminus]-\rho,\rho[}$ is the characteristic function of $\mathbb{R}\setminus]-\rho,\rho[$ and

$$c_{\alpha,\beta}(y) = \frac{2^{\rho-iy}\Gamma(\alpha+1)\Gamma(iy)}{\Gamma(\frac{1}{2}(\rho+iy))\Gamma(\frac{1}{2}(\alpha-\beta+1+iy))}, \qquad y \in \mathbb{C}\setminus(i\mathbb{N}).$$

Theorem 3.4 (Inversion formula)**.**
If $f \in L^1_A(\mathbb{R}_+)$ and $\mathcal{F}_{\alpha,\beta}(f) \in L^1_\sigma(\mathbb{R})$, then

$$f(x) = \int_{-\infty}^{\infty} \mathcal{F}_{\alpha,\beta}(f)(\lambda)\psi_{-\lambda}^{\alpha,\beta}(x)\sigma(\lambda)d\lambda, \qquad \textit{for a.e.} \quad x \in \mathbb{R}. \tag{3.6}$$

Let us now state an L^p version of Hardy's theorem for the Jacobi-Dunkl transform.

Theorem 3.5. *Let $p \in [1,+\infty]$, let $n \in [1,+\infty[$, and let f be a measurable function on $\mathbb{R}$. Suppose that*

$$\|e^{at^2}f\|_{p,A_{\alpha,\beta}} < \infty \quad \textit{and} \quad \|e^{b\lambda^2}\mathcal{F}_{\alpha,\beta}(f)\|_{n,\sigma} < \infty,$$

where a, $b > 0$, and σ is the weight function given by (3.5)*. If $ab > \frac{1}{4}$, then $f \equiv 0$.*

Proof. It follows immediately from (3.3) that the weight function $A_{\alpha,\beta}$ satisfies (H3). On the other hand, it was shown in [18, p. 157] that there exists $C > 0$ such that $|c_{\alpha,\beta}(x)|^{-2} \ge C|x|^{2\alpha+1}$ at infinity. This together with (3.5) implies that the weight σ satisfies the conditions of Proposition 2.1. Therefore, the conclusion of the theorem follows from Theorem 2.3 and (3.6). □

The following result gives a Hardy's uncertainty principle for the Jacobi-Dunkl transform.

Corollary 3.6. *Let f be a measurable function on $\mathbb{R}$. Suppose that there exist positive constants a, b and C such that*

$$|f(x)| \leq Ce^{-ax^2} \quad \textit{and} \quad |\mathcal{F}_{\alpha,\beta}(f)(\lambda)| \leq Ce^{-b\lambda^2}, \qquad \textit{for a.e.} \quad x, \lambda \in \mathbb{R}.$$

If $ab > \frac{1}{4}$ then $f \equiv 0$.

3.3 Remarks

In [7], the authors proved an L^p version of Hardy's theorem for the Jacobi-Dunkl transform via a transmutation approach. They used the heat kernel associated with the Jacobi-Dunkl operator, defined for $t > 0$ by

$$E_t(x) = \mathcal{F}^{-1}_{\alpha,\beta}(e^{-t\lambda^2})(x), \qquad x \in \mathbb{R},$$

to show that for a measurable function f on $\mathbb{R}$, if

$$E^{-1}_{1/4a} f \in L^p_{A_{\alpha,\beta}}(\mathbb{R}) \quad \text{and} \quad e^{b\lambda^2}\mathcal{F}_{\alpha,\beta}(f) \in L^q_{\sigma}(\mathbb{R}),$$

where $1 \leq p, q \leq +\infty$ with at least one of them is finite, and a, b are positives constants such that $ab \geq 1/4$, then $f \equiv 0$.

For this end, we point out that from the fact (see [10], page 251) that there exist $C_1(t) > 0$ and $C_2(t) > 0$, such that for all $x \in \mathbb{R}$,

$$C_1(t)\frac{e^{-x^2/(4t)}}{\sqrt{B_{\alpha,\beta}(x)}} \leq E_t(x) \leq C_2(t)\frac{e^{-x^2/(4t)}}{\sqrt{B_{\alpha,\beta}(x)}},$$

where $B_{\alpha,\beta}(x) = 2^{2\rho}\left(\frac{\sinh x}{x}\right)^{2\alpha+1}(\cosh x)^{2\beta+1}$, one can see that every measurable function f satisfying (3.3) satisfies also (3.5). Furthermore, the heat kernel E_t is not explicitly given, which proves that our approach is simpler and more constructive.
Furthermore, the heat kernel E_t is not explicitly given, which proves that our approach is simpler and more constructive.

The same remark can be stated comparing our approach by that developed in [19] for the generalized Fourier transform associated with the Sturm-Liouville operator.

4 An L^p-version of Hardy's Theorem in Quantum Calculus

In this section, we give the q-analogues of the results of Section 2. Let $q \in]0, 1[$ and write

$$\mathbb{R}^+_q = \{q^n \mid n \in \mathbb{Z}\} \quad \text{and} \quad \{\mathbb{R}_q = \pm q^n \mid n \in \mathbb{Z}\}.$$

The Jackson's q-integrals from 0 to $+\infty$ and from $-\infty$ to $+\infty$ are defined as follows. For a function f defined on $\mathbb{R}_q^+$,

$$\int_0^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n,$$

provided the series converges absolutely, and when the function f is defined on $\mathbb{R}_q$,

$$\int_{-\infty}^\infty f(x)d_qx = (1-q)\sum_{n=-\infty}^{\infty} f(q^n)q^n + (1-q)\sum_{n=-\infty}^{\infty} f(-q^n)q^n,$$

provided the tow series above converge absolutely.

Let $\beta \in \{0, +\infty\}$ and define

$$(\beta,+\infty)_q = \begin{cases} \mathbb{R}_q, & \text{if } \beta = -\infty; \\ \mathbb{R}_q^+, & \text{if } \beta = 0. \end{cases}$$

Let w be a positive function on $(\beta,+\infty)_q$ and $p \in [1,+\infty]$. For a function f defined on $(\beta,+\infty)_q$, define

$$\|f\|_{p,w,q} = \left(\int_\beta^{+\infty} |f(t)|w(t)d_qt\right)^{\frac{1}{p}}, \qquad \text{if } 1 \le p < +\infty$$

and

$$\|f\|_{p,w,q} = \|f\|_\infty = \sup_{t\in(\beta,+\infty)_q} |f(t)|, \qquad \text{if } p = +\infty.$$

We consider the q-integral transform

$$\mathcal{F}_q(f)(\lambda) = \int_\beta^\infty f(t)\varphi(\lambda,t)w(t)d_qt,$$

with the following properties :

(H′1) The kernel φ is entire with respect to λ, and when $\beta = -\infty$ it is even with respect to t. Moreover, there exist $C, k > 0$ such that

$$|\varphi(x,t)| \le Ce^{k|x||t|}, \qquad \text{for all } (x,t) \in (\beta,+\infty)_q \times \mathbb{C},$$

(H′2) The weight function w is positive on $(\beta,+\infty)_q$ and there exists $c > 0$ such that the function $t \mapsto w(t)e^{-ct^2}$ is q-integrable on $(\beta,+\infty)_q$.

The operator $\mathcal{F}_q$ can be seen as a q-analogue of the operator $\mathcal{F}$. The proofs of the following results can follow the same steps of the corresponding classical ones presented in Section 2, by replacing the Lebesgue integral by the Jackson's q-integral.

Theorem 4.1. *Let $p \in [1,+\infty]$, and let f be a function on $(\beta,\infty)_q$. Suppose that there exists $a > 0$ such that $\|e^{at^2}f\|_{p,q,w} < \infty$. Then*

(*i*) *The function $\mathcal{F}_q(f)$ is entire on $\mathbb{C}$.*

(*ii*) *For every* $a' \in]0,a[$, *there exists* $C' > 0$ *such that*

$$|\mathcal{F}_q(f)(z)| \leq C' e^{\frac{k^2|z|^2}{4a'}}, \qquad \textit{for all } z \in \mathbb{C}.$$

Theorem 4.2. *Let* $p \in [1,+\infty]$, $n \in [1,+\infty[$ *and* μ *be a function satisfying the conditions of Proposition 2.1. Let* f *be a function defined on* $(\beta,\infty)_q$. *Suppose that there exist a, b and* $C \geq 0$ *such that*

$$||e^{at^2} f||_{p,q,w} < \infty \quad \textit{and} \quad ||e^{b\lambda^2} \mathcal{F}_q(f)||_{n,\mu} < \infty,$$

If $ab > k^2/4$, *then* $\mathcal{F}_q(f) \equiv 0$. *Moreover, if* $\mathcal{F}_q$ *is injective then* $f \equiv 0$.

Corollary 4.3. *Let* f *be a function defined on* $(\beta,\infty)_q$. *Suppose that there exist a, b and* $C \geq 0$ *such that*

$$|f(t)| \leq Ce^{-at^2}, \quad \textit{for all } t \in (\beta,+\infty)_q$$

and

$$|\mathcal{F}_q(f)(\lambda)| \leq Ce^{-b\lambda^2}, \quad \textit{for a.e.} \quad \lambda \in (\beta,+\infty).$$

If $ab > \dfrac{k^2}{4}$, *then* $\mathcal{F}_q(f) = 0$. *Moreover, if* $\mathcal{F}_q$ *is injective then* $f \equiv 0$.

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