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Abstract

In this paper, the authors use the discrete Littlewood-Paley-Stein theory to introduce weighted multi-parameter Triebel-Lizorkin and Besov spaces associated with non-isotropic flag singular integrals under a rather weak weight condition ($w \in A_\infty$). They also obtain the boundedness of flag singular integrals on these spaces.

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1 Introduction and Statement of Main Results

The flag singular integral theory was studied extensively over the past decades. Müller, Ricci and Stein in [8] first introduced flag singular integrals when they studied the Marcinkiewicz multiplier on the Heisenberg group. Nagel, Ricci and Stein in [9] dealt with a class of product singular integrals with the flag kernel and proved the L^p boundedness of flag singular integrals. See [3] and [4] for more details.

More recently, Han and Lu in [5], [6] developed multi-parameter Hardy spaces H^p_F associated with flag singular integrals. Ruan in [11] constructed multi-parameter Hardy spaces associated with non-isotropic flag singular integrals via the discrete Littlewood-Paley-Stein

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theory and the discrete Calderón's identity, and obtained the boundedness of flag singular integrals from H_F^p to H_F^p and from H_F^p to L^p.

Ding, Lu and Ma in [2] introduced Triebel-Lizorkin and Besov spaces associated with flag singular integrals and proved the boundedness of flag singular integrals on these spaces. Wu and Liu in [13] gave characterizations of multi-parameter Triebel-Lizorkin and Besov spaces associated with flag singular integrals. The weighted multi-parameter Triebel-Lizorkin and Besov spaces in the pure product setting were first constructed by Lu and Zhu in [7]. More weighted results in multi-parameter setting can be found in [1], [10].

The main purpose of this paper is to extend the results in [11] on multi-parameter Hardy spaces to weighted multi-parameter Triebel-Lizorkin and Besov spaces. To be more precise, the authors introduce weighted multi-parameter Triebel-Lizorkin and Besov spaces related to non-isotropic flag singualr integrals. As a consequence, the boundedness of non-isotropic flag singular integrals on these spaces is presented.

Firstly, we recall the definition of product weights. For $1 < p < \infty$, we say that a non-negative locally integrable function $w \in A_p(\mathbb{R}^n \times \mathbb{R}^m)$ if there exists a constant $C > 0$ such that

$$\left(\frac{1}{|R|}\int_R w(x)\,dx\right)\left(\frac{1}{|R|}\int_R w(x)^{-1/(p-1)}\,dx\right)^{p-1} \le C \tag{1.1}$$

for all dyadic rectangles $R = I \times J$, where I and J are cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively. We say that $w \in A_1(\mathbb{R}^n \times \mathbb{R}^m)$ if there exists a constant $C > 0$ such that

$$M_s w(x) \le C w(x)$$

for almost every $x \in \mathbb{R}^{n+m}$, where M_s is the strong maximal operator defined by

$$M_s f(x) = \sup_{R \ni x} \frac{1}{|R|}\int_R |f(y)|\,dy,$$

where the supreme is taken over all dyadic rectangles $R = I \times J$ be as in (1.1). We define the class $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ by

$$A_\infty(\mathbb{R}^n \times \mathbb{R}^m) = \bigcup_{1 \le p < \infty} A_p(\mathbb{R}^n \times \mathbb{R}^m).$$

If $w \in A_q(\mathbb{R}^n \times \mathbb{R}^m)$ for some $q \ge 1$, then we use $q_w = \inf\{q : w \in A_q\}$ to denote the critical index of w. Notice that $w \in A_\infty(\mathbb{R}^n \times \mathbb{R}^m)$ implies that $q_w < \infty$.

Let $\mathcal{S}(\mathbb{R}^n)$ denote Schwartz functions on $\mathbb{R}^n$. In order to construct a test function defined on $\mathbb{R}^n \times \mathbb{R}^m$, we give the definition of the non-standard convolution $*_2$ which depends only on the second variable.

Definition 1.1.[11] We define a non-standard convolution $*_2$ by

$$\psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y) = \int_{\mathbb{R}^m} \psi^{(1)}(x, y-z)\psi^{(2)}(z)\,dz,$$

where $\psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$, $\psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$ satisfying

$$\sum_{j \in \mathbb{Z}} |\widehat{\psi^{(1)}}(2^{-j}x, 2^{-2j}y)|^2 = 1$$

for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m \setminus \{(0,0)\}$, and

$$\sum_{k\in\mathbb{Z}} |\widehat{\psi^{(2)}}(2^{-k}z)|^2 = 1$$

for all $z \in \mathbb{R}^m \setminus \{0\}$, and the cancellation conditions

$$\int_{\mathbb{R}^{n+m}} x^\alpha y^\beta \psi^{(1)}(x,y)\,dxdy = \int_{\mathbb{R}^m} z^\gamma \psi^{(2)}(z)\,dz = 0$$

for all nonnegative integers α,β and γ.

We now define the non-isotropic Littlewood-Paley-Stein square function.

Definition 1.2. [11] Let $f \in L^p, 1 < p < \infty$. The Littlewood-Paley-Stein square function f is defined by

$$g(f)(x,y) = \Big\{\sum_{j,k} |\psi_{j,k} * f(x,y)|^2\Big\}^{1/2}, \tag{1.2}$$

where

$$\psi_{j,k}(x,y) = \psi_j^{(1)} *_2 \psi_k^{(2)}(x,y),$$

$$\psi_j^{(1)}(x,y) = 2^{(n+2m)j}\psi^{(1)}(2^j x, 2^{2j} y), \quad \psi_k^{(2)}(z) = 2^{mk}\psi^{(2)}(2^k z).$$

From the Fourier transform, it is easy to see that the following continuous Calderón's identity holds on $L^2(\mathbb{R}^n \times \mathbb{R}^m)$,

$$f(x,y) = \sum_{j,k} \psi_{j,k} * \psi_{j,k} * f(x,y).$$

We formulate the definitions of product kernel and flag kernel associated with the non-isotropic dilations as follows.

Definition 1.3. [9] A distribution $K^\sharp$ on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ is said to be a product kernel on $\mathbb{R}^{n+m} \times \mathbb{R}^m$ if $K^\sharp$ is a C^∞ function away from the coordinate subspaces $\{(0,0,z) : (0,0) \in \mathbb{R}^{n+m}, z \in \mathbb{R}^m\}$ and $\{(x,y,0) : (x,y) \in \mathbb{R}^{n+m}, 0 \in \mathbb{R}^m\}$, and for all $(x,y,z) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m$ with $|x| + |y| \neq 0$ and $z \neq 0$ satisfies
(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$, $\gamma = (\gamma_1, \cdots, \gamma_m)$,

$$\left|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma K^\sharp(x,y,z)\right| \le C_{\alpha,\beta,\gamma} |(x,y)|^{-(n+2m+|\alpha|+2|\beta|)} |z|^{-m-|\gamma|}.$$

(2) (Cancellation Conditions)For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$, every normalized bump function ϕ_1 on $\mathbb{R}^m$ and every $\delta > 0$,

$$\left|\int_{\mathbb{R}^m} \partial_x^\alpha \partial_y^\beta K^\sharp(x,y,z)\phi_1(\delta z)\,dz\right| \le C_{\alpha,\beta} |(x,y)|^{-(n+2m+|\alpha|+2|\beta|)};$$

for any multi-indices $\gamma = (\gamma_1, \cdots, \gamma_m)$, every normalized bump function ϕ_2 on $\mathbb{R}^{n+m}$ and every $\delta > 0$,

$$\left|\int_{\mathbb{R}^{n+m}} \partial_z^\gamma K^\sharp(x,y,z)\phi_2(\delta x, \delta^2 y)\,dxdy\right| \le C_\gamma |z|^{-m-|\gamma|};$$

for every normalized bump function ϕ_3 on $\mathbb{R}^{n+m+m}$ and every $\delta_1, \delta_2 > 0$,

$$\left|\int_{\mathbb{R}^{n+m+m}} K^\sharp(x,y,z)\phi_3(\delta_1 x, \delta_1^2 y, \delta_2 z)\,dxdydz\right| \le C.$$

Definition 1.4. [9] A distribution K on $\mathbb{R}^{n+m}$ is said to be a flag kernel on $\mathbb{R}^n \times \mathbb{R}^m$ if K is a C^∞ function away from the coordinate subspaces $\{(0,y) : 0 \in \mathbb{R}^n, y \in \mathbb{R}^m\}$, and for all $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$ with $|x| \neq 0$ satisfies
(1) (Differential Inequalities) For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, $\beta = (\beta_1, \cdots, \beta_m)$,

$$\left|\partial_x^\alpha \partial_y^\beta K(x,y)\right| \le C_{\alpha,\beta}|x|^{-n-|\alpha|}|(x,y)|^{-2m-2|\beta|}.$$

(2) (Cancellation Conditions)For any multi-indices $\alpha = (\alpha_1, \cdots, \alpha_n)$, every normalized bump function ϕ_1 on $\mathbb{R}^m$ and every $\delta > 0$,

$$\left|\int_{\mathbb{R}^m} \partial_x^\alpha K(x,y)\phi_1(\delta y)\,dy\right| \le C_\alpha |x|^{-n-|\alpha|};$$

for any multi-indices $\beta = (\beta_1, \cdots, \beta_m)$, every normalized bump function ϕ_2 on $\mathbb{R}^n$ and every $\delta > 0$,

$$\left|\int_{\mathbb{R}^n} \partial_y^\beta K(x,y)\phi_2(\delta x)\,dx\right| \le C_\beta |y|^{-m-|\beta|};$$

every normalized bump function ϕ_3 on $\mathbb{R}^{n+m}$ and every $\delta_1, \delta_2 > 0$,

$$\left|\int_{\mathbb{R}^{n+m}} K(x,y)\phi_3(\delta_1 x, \delta_2 y)\,dxdy\right| \le C.$$

We now recall the test functions of order M, $\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, where M is a positive integer.

Definition 1.5. [11] We say $f(x,y,z) \in \mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ if f is a Schwartz test function and satisfies the following conditions:
(i) For $|\alpha|, |\beta|, |\gamma| \le M-1$,

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z)| \le C \frac{1}{(1+|(x,y)|)^{n+2m+3M+|\alpha|+2|\beta|}} \frac{1}{(1+|z|)^{m+M+|\gamma|}},$$

(ii) For $|x-x'| \le \frac{1}{2}(1+|x|)$ and $|y-y'| \le \frac{1}{2}(1+|y|)$, $|\alpha| = |\beta| = M$ and $|\gamma| \le M-1$,

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z) - \partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x',y',z)| \le C \frac{|(x-x', y-y')|}{(1+|(x,y)|)^{n+2m+6M}} \frac{1}{(1+|z|)^{m+M+|\gamma|}},$$

(iii) For $|z-z^{'}| \le \frac{1}{2}(1+|z|), |\gamma| = M$ and $|\alpha|, |\beta| \le M-1$,

$$|\partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z) - \partial_x^\alpha \partial_y^\beta \partial_z^\gamma f(x,y,z^{'})| \le C \frac{1}{(1+|(x,y)|)^{n+2m+3M+|\alpha|+2|\beta|}} \frac{|z-z^{'}|}{(1+|z|)^{m+2M}},$$

(iv) For $|x-x^{'}| \le \frac{1}{2}(1+|x|)$, $|y-y^{'}| \le \frac{1}{2}(1+|y|), |z-z^{'}| \le \frac{1}{2}(1+|z|)$, and $|\nu| = M$,

$$\begin{aligned}&|\partial_x^\nu \partial_y^\nu \nu_z^\nu f(x,y,z) - \partial_x^\nu \partial_y^\nu \partial_z^\nu f(x^{'},y^{'},z) - \partial_x^\nu \partial_y^\nu \partial_z^\nu f(x,y,z^{'}) + \partial_x^\nu \partial_y^\nu \partial_z^\nu f(x^{'},y^{'},z^{'})| \\ \le& C \frac{|(x-x^{'},y-y^{'})|}{(1+|(x,y)|)^{n+2m+6M}} \frac{|z-z^{'}|}{(1+|z|)^{m+2M}},\end{aligned}$$

(v) For $|\alpha|, |\beta|, |\gamma| \le M-1$,

$$\int_{\mathbb{R}^{n+m}} f(x,y,z) x^\alpha y^\beta \, dxdy = \int_{\mathbb{R}^m} f(x,y,z) z^\gamma \, dz = 0.$$

If $f(x,y,z) \in \mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$, the norm of f in $S_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ is defined by

$$||f||_{S_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)} = \inf\{C : (i)-(iv) \ \ hold\}.$$

The following is the test function space $\mathcal{S}_{F,M}$ on $\mathbb{R}^n \times \mathbb{R}^m$ associated with the flag structure.

Definition 1.6. [11] A function $f(x,y)$ defined on $\mathbb{R}^n \times \mathbb{R}^m$ is said to be a test function in $\mathcal{S}_{F,M}$ if there exists a function $f^\sharp \in \mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)$ such that

$$f(x,y) = \int_{\mathbb{R}^m} f^\sharp(x, y-z, z) \, dz. \tag{1.3}$$

The norm of f in $\mathcal{S}_{F,M}$ on $\mathbb{R}^n \times \mathbb{R}^m$ is defined by

$$||f||_{\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)} = \inf\{||f^\sharp||_{\mathcal{S}_M(\mathbb{R}^{n+m} \times \mathbb{R}^m)} : for\ all\ representations\ of\ f\ in\ (1.3)\}.$$

The dual space of $\mathcal{S}_{F,M}$ is denoted by $(\mathcal{S}_{F,M})^{'}$.

Since the functions $\psi_{j,k}$ constructed above belong to $\mathcal{S}_{F,M}$, the Littlewood-Paley-Stein square function $g(f)$ can be defined for all distributions in $(\mathcal{S}_{F,M})^{'}$. Thus the author in [11] defined the multi-parameter Hardy space associated with non-isotropic flag singular integral as follows.

Definition 1.7. [11] Let $0 < p < \infty$. The multi-parameter Hardy space associated with non-isotropic flag singular integrals is defined as $H_F^p(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_{F,M})^{'} : g(f) \in L^p(\mathbb{R}^n \times \mathbb{R}^m)\}$. If $f \in H_F^p(\mathbb{R}^n \times \mathbb{R}^m)$, the norm of f is defined by $||f||_{H_F^p} = ||g(f)||_p$.

Clearly, it follows that $H_F^p(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ for $1 < p < \infty$.

It is proved in [11] that the definition is independent of the choice of functions $\psi_{j,k}$ and the following boundedness result of convolution type flag singular integrals on $\mathbb{R}^n \times \mathbb{R}^m$ was established.

Theorem 1.8. *[11] Let T be the flag singular integral. Then for any* $0 < p \le 1$*, there exists a constant* $C = C(p)$ *such that*

$$\|T(f)\|_{H^p_F} \le C\|f\|_{H^p_F}.$$

In this paper, we will use the method in [11] to develop a theory of weighted multi-parameter Triebel-Lizorkin and Besov spaces with non-isotropic flag singular integrals. We first give the

Definition 1.9. Let $0 < p, q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$, $w \in A_\infty$. Let M be the integer which satisfying the inequality $\max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < \min\left\{\frac{p}{q_w}, 1, q\right\}$, then weighted Triebel-Lizorkin space $\dot{F}^{s,q}_{p,w}$ associated with non-isotropic flag singular integrals is defined by

$$\dot{F}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_{F,M})' : \|f\|_{\dot{F}^{s,q}_{p,w}} < \infty\},$$

where

$$\|f\|_{\dot{F}^{s,q}_{p,w}} = \left\|\left\{\sum_{j,k\in\mathbb{Z}} \left(2^{js_1}2^{ks_2}|\psi_{j,k} * f|\right)^q\right\}^{1/q}\right\|_{L^p(w)}.$$

And let M be the integer which satisfying the inequality $\max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < \min\left\{\frac{p}{q_w}, 1\right\}$, then weighted Besov space $\dot{B}^{s,q}_{p,w}$ associated with non-isotropic flag singular integrals is defined by

$$\dot{B}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m) = \{f \in (\mathcal{S}_{F,M})' : \|f\|_{\dot{B}^{s,q}_{p,w}} < \infty\},$$

where

$$\|f\|_{\dot{B}^{s,q}_{p,w}} = \left\{\sum_{j,k\in\mathbb{Z}} \left(2^{js_1}2^{ks_2}\|\psi_{j,k} * f\|_{L^p(w)}\right)^q\right\}^{1/q}.$$

We will prove that Definition 1.9 is independent of the choice function $\psi_{j,k}$ by Min-Max comparison principle. The main tool to prove the Min-Max comparison principle is the following discrete Calderón's identity.

Theorem 1.10. *[11] Suppose that* $\psi_{j,k}$ *are the same as in Definition 1.1. Then*

$$f(x,y) = \sum_{j,k}\sum_{I,J} |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)(\psi_{j,k} * f)(x_I,y_J), \tag{1.4}$$

where $\widetilde{\psi}_{j,k}(x,y,x_I,y_J) \in \mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$*,* $I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ *are dyadic cubes with side-length* $l(I) = 2^{-j-N}$*,* $l(J) = 2^{-k-N} + 2^{-2j-N}$ *for a fixed large integer N,* x_I, y_J *are any fixed points in* I, J*, respectively. The above series converges in the norm of* $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$ *and in the dual space* $(\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m))'$.

The above discrete Calderón's identity enables us to derive the following theorems. In what follows, we use the notation $A \approx B$ to denote that two quantities A and B are comparable independent of other substantial quantities involved in the paper. The Min-Max comparison principle on Triebel-Lizorkin spaces as follows.

Theorem 1.11. *Suppose that* $0 < p,q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$, $w \in A_\infty$ *and* $\phi^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$, $\phi^{(2)}, \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$,

$$\phi(x,y) = \phi^{(1)} *_2 \phi^{(2)}(x,y), \quad \psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y),$$

and $\phi_{j,k}, \psi_{j,k}$ *satisfy the same conditions as in Definition 1.2,* $max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < min\left\{\frac{p}{q_w}, 1, q\right\}$, *where* M *depends on* p *and* q*. Then for* $f \in (\mathcal{S}_{F,M})'$*, we have*

$$\begin{aligned} &\Big\|\Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_1}2^{ks_2} \sum_{I,J} \sup_{u\in I, v\in J} |\psi_{j,k} * f(u,v)|\Big)^q \chi_I\chi_J\Big\}^{1/q}\Big\|_{L^p(w)} \\ &\approx \Big\|\Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_1}2^{ks_2} \sum_{I,J} \inf_{u\in I, v\in J} |\phi_{j,k} * f(u,v)|\Big)^q \chi_I\chi_J\Big\}^{1/q}\Big\|_{L^p(w)}. \end{aligned}$$

$I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ *are dyadic cubes with side-length* $l(I) = 2^{-j-N}$, $l(J) = 2^{-2j-N} + 2^{-k-N}$ *for a fixed large integer* N, x_I, y_J *are any fixed points in* I, J*, respectively.*

Similarly, we have the Min-Max comparison principle on Besov spaces.

Theorem 1.12. *Suppose that* $0 < p,q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$, $w \in A_\infty$ *and* $\phi^{(1)}, \psi^{(1)} \in \mathcal{S}(\mathbb{R}^{n+m})$, $\phi^{(2)}, \psi^{(2)} \in \mathcal{S}(\mathbb{R}^m)$,

$$\phi(x,y) = \phi^{(1)} *_2 \phi^{(2)}(x,y), \quad \psi(x,y) = \psi^{(1)} *_2 \psi^{(2)}(x,y),$$

and $\phi_{j,k}, \psi_{j,k}$ *satisfy the same conditions as in Definition 1.2,* $max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < min\left\{\frac{p}{q_w}, 1\right\}$, *where* M *depends on* p *and* q*. Then for* $f \in (\mathcal{S}_{F,M})'$*, we have*

$$\begin{aligned} &\Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_1}2^{ks_2} \Big\| \sum_{I,J} \sup_{u\in I, v\in J} |\psi_{j,k} * f(u,v)| \chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q} \\ &\approx \Big\{ \sum_{j,k\in\mathbb{Z}} \Big(2^{js_1}2^{ks_2} \Big\| \sum_{I,J} \inf_{u\in I, v\in J} |\phi_{j,k} * f(u,v)| \chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q} \end{aligned}$$

$I \subset \mathbb{R}^n, J \subset \mathbb{R}^m$ *are dyadic cubes with side-length* $l(I) = 2^{-j-N}$, $l(J) = 2^{-2j-N} + 2^{-k-N}$ *for a fixed large integer* N, x_I, y_J *are any fixed points in* I, J*, respectively.*

Using discrete Calderón's identity and almost orthogonal estimates, we can prove the following theorems:

Theorem 1.13. *Let* T *be the flag singular integral. For any* $0 < p,q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$, $w \in A_\infty$, $max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < min\left\{\frac{p}{q_w}, 1, q\right\}$, *there exists a constant* $C = C(p)$ *such that*

$$\|T(f)\|_{\dot{F}^{s,q}_{p,w}} \le C\|f\|_{\dot{F}^{s,q}_{p,w}}.$$

Theorem 1.14. *Let* T *be the flag singular integral. For any* $0 < p,q < \infty$, $s = (s_1, s_2) \in \mathbb{R}^2$, $w \in A_\infty$, $max\left\{\frac{n}{n+M}, \frac{m}{m+M}\right\} < min\left\{\frac{p}{q_w}, 1\right\}$, *there exists a constant* $C = C(p)$ *such that*

$$\|T(f)\|_{\dot{B}^{s,q}_{p,w}} \le C\|f\|_{\dot{B}^{s,q}_{p,w}}.$$

2 The Min-Max Comparison Principle on Weighted Multiparameter Triebel-Lizorkin and Besov Spaces

In this section, we establish the Min-Max comparison principle on weighted multi-parameter Triebel-Lizorkin and Besov spaces associated with non-isotropic flag singular integrals.

We first recall the almost orthogonal estimates.

Lemma 2.1. *[11] For any given positive integers L and K, there exists a constant C depending only on K,L such that if* $t \vee t^{'} \leq \sqrt{s \vee s^{'}}$*, then*

$$|\psi_{t,s} * \phi_{t^{'},s^{'}}(x,y)| \leq C(\frac{t}{t^{'}} \wedge \frac{t^{'}}{t})^{L}(\frac{s}{s^{'}} \wedge \frac{s^{'}}{s})^{L}\frac{(t \vee t^{'})^{K}}{(t \vee t^{'} + |x|)^{n+K}}\frac{(s \vee s^{'})^{K}}{(s \vee s^{'} + |y|)^{m+K}},$$

and if $t \vee t^{'} > \sqrt{s \vee s^{'}}$*, then*

$$|\psi_{t,s} * \phi_{t^{'},s^{'}}(x,y)| \leq C(\frac{t}{t^{'}} \wedge \frac{t^{'}}{t})^{L}(\frac{s}{s^{'}} \wedge \frac{s^{'}}{s})^{L}\frac{(t \vee t^{'})^{K}}{(t \vee t^{'} + |x|)^{n+K}}\frac{(t \vee t^{'})^{K}}{(t \vee t^{'} + \sqrt{|y|})^{2m+K}},$$

where $\psi_{t,s}, \phi_{t^{'},s^{'}} \in \mathcal{S}_{F,M}$ *on* $\mathbb{R}^n \times \mathbb{R}^m$.

Next, we give the following lemma which is crucial in dealing with weighted multi-parameter Triebel-Lizorkin and Besov spaces.

Lemma 2.2. *[11] Given large positive integer N and* $j,k,j^{'},k^{'} \in \mathbb{Z}$*. Let* $I, I^{'}$ *and* $J, J^{'}$ *be dyadic cubes in* $\mathbb{R}^n$ *and* $\mathbb{R}^m$ *respectively, such that* $l(I) = 2^{-j-N}$, $l(J) = 2^{-2j-N} + 2^{-k-N}$, $l(I^{'}) = 2^{-j^{'}-N}$, $l(J^{'}) = 2^{-2j^{'}-N} + 2^{-k^{'}-N}$*. For any* $u, u^* \in I$, $v, v^* \in J$*, then we have when* $j \wedge j^{'} \geq \frac{k \wedge k^{'}}{2}$,

$$\sum_{I^{'},J^{'}} \frac{2^{-(j\wedge j^{'})K}2^{-(k\wedge k^{'})K}|I^{'}||J^{'}|}{(2^{-(j\wedge j^{'})} + |u - x_{I^{'}}|)^{n+K}(2^{-(k\wedge k^{'})} + |v - y_{J^{'}}|)^{m+K}}|\phi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}})|$$
$$\leq C\Big\{M_s\Big(\sum_{I^{'},J^{'}} |\phi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}})|^r \chi_{I^{'}}\chi_{J^{'}}\Big)\Big\}^{1/r}(u^*, v^*),$$

and when $j \wedge j^{'} \leq \frac{k \wedge k^{'}}{2}$,

$$\sum_{I^{'},J^{'}} \frac{2^{-2(j\wedge j^{'})K}|I^{'}||J^{'}|}{(2^{-(j\wedge j^{'})} + |u - x_{I^{'}}|)^{n+K}(2^{-(j\wedge j^{'})} + \sqrt{|v - y_{J^{'}}|})^{2m+K}}|\phi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}})|$$
$$\leq C\Big\{M\Big(\sum_{I^{'},J^{'}} |\phi_{j^{'},k^{'}} * f(x_{I^{'}}, y_{J^{'}})|^r \chi_{I^{'}}\chi_{J^{'}}\Big)\Big\}^{1/r}(u^*, v^*),$$

where M is the Hardy-Littlewood maximal function on $\mathbb{R}^n \times \mathbb{R}^m$*, and* M_s *is strong maximal function function on* $\mathbb{R}^n \times \mathbb{R}^m$*, r satisfying* $\max\Big\{\frac{n}{n+K}, \frac{m}{m+K}\Big\} < r$.

Now we are ready to give the

Proof of Theorems 1.11 and 1.12. Suppose that M satisfies the inequality $\max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < \min\Big\{\frac{p}{q_w}, 1, q\Big\}$, then we choose $p_0 > q_w$ such that $w \in A_{p_0}$ and $\max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < \min\Big\{\frac{p}{p_0}, 1, q\Big\}$.

By Theorem 1.10, we can choose N depending on M, by the discrete Calderón identity, $f \in (\mathcal{S}_{F,M})'$ can be represented by

$$f(x,y) = \sum_{j',k'} \sum_{I',J'} |I'||J'| \widetilde{\phi}_{j',k'}(x,y,x_{I'},y_{J'})(\phi_{j',k'} * f)(x_{I'},y_{J'}),$$

we write

$$(\psi_{j,k} * f)(x,y) = \sum_{j',k'} \sum_{I',J'} |I'||J'| \psi_{j,k} * \widetilde{\phi}_{j,k}(\cdot,\cdot,x_{I'},y_{J'})(x,y)(\phi_{j',k'} * f)(x_{I'},y_{J'}).$$

By Lemma 2.1 and Lemma 2.2, for any given positive integer L, we get

$$\begin{aligned}|(\psi_{j,k} * f)(x,y)| &\le \sum_{j',k'} \sum_{I',J'} |I'||J'||\psi_{j,k} * \widetilde{\phi}_{j,k}(\cdot,\cdot,x_{I'},y_{J'})||(\phi_{j',k'} * f)(x_{I'},y_{J'})| \\ &\le C \sum_{j',k'} 2^{-|j-j'|L} 2^{-|k-k'|L} \Big\{ M_s \Big(\sum_{I',J'} |(\phi_{j',k'} * f)(x_{I'},y_{J'})| \chi_{I'}\chi_{J'} \Big)^r \Big\}^{1/r} (x^*,y^*)\end{aligned}$$

for any $x,x^* \in I, x_{I'} \in I', y,y^* \in J$ and $y_{J'} \in J'$, where M_s is the strong maximal function.

Applying Hölder's inequality and summing over j,k,I,J yields

$$\begin{aligned}&\Big\{ \sum_{j,k} \Big(2^{js_1} 2^{ks_2} \sum_{I,J} \sup_{x\in I, y\in J} |(\psi_{j,k} * f)(x,y)| \chi_I \chi_J \Big)^q \Big\}^{1/q} \\ \le & C \Big\{ \sum_{j',k'} \Big(2^{j's_1} 2^{k's_2} \Big(M_s \Big(\sum_{I',J'} |(\phi_{j',k'} * f)(x_{I'},y_{J'})| \chi_{I'}\chi_{J'} \Big)^r \Big)^{1/r} \Big)^q \Big\}^{1/q},\end{aligned}$$

where $|s_1|, |s_2| < L$. Since $x_{I'}$ and $y_{J'}$ are arbitrary points in I' and J', respectively, then we have

$$\begin{aligned}&\Big\{ \sum_{j,k} \Big(2^{js_1} 2^{ks_2} \sum_{I,J} \sup_{x\in I, y\in J} |(\psi_{j,k} * f)(x,y)| \chi_I \chi_J \Big)^q \Big\}^{1/q} \\ \le & C \Big\{ \sum_{j',k'} \Big(2^{j'qs_1} 2^{k'qs_2} \Big(M_s \Big(\sum_{I',J'} \inf_{x'\in I', y'\in J'} |(\phi_{j',k'} * f)(x',y')| \chi_{I'}\chi_{J'} \Big)^r \Big)^{1/r} \Big)^q \Big\}^{1/q}.\end{aligned}$$

Since $w \in A_{p_0} \subset A_{p/r}$, then taking the L^p_w norm and applying $L^{p/r}_w(l^{q/r})$ boundedness of M_s for $\max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < r < \min\Big\{\frac{p}{p_0}, 1, q\Big\}$, then

$$\begin{aligned}&\Big\| \Big\{ \sum_{j,k} \Big(2^{js_1} 2^{ks_2} \sum_{I,J} \sup_{x\in I, y\in J} |(\psi_{j,k} * f)(x,y)| \chi_I \chi_J \Big)^q \Big\}^{1/q} \Big\|_{L^p(w)} \\ \le & C \Big\| \Big\{ \sum_{j',k'} \Big(2^{j's_1} 2^{k's_2} \sum_{I',J'} \inf_{x'\in I', y'\in J'} |(\phi_{j',k'} * f)(x',y')| \chi_{I'}\chi_{J'} \Big)^q \Big\}^{1/q} \Big\|_{L^p(w)}.\end{aligned}$$

which completes the proof of Theorem 1.11.

Now we turn to give the proof of Theorem 1.12. Assume that M satisfies the inequality $\max\left\{\frac{n}{n+M},\frac{m}{m+M}\right\}<\min\left\{\frac{p}{q_w},1\right\}$ and we choose $p_0>q_w$ such that $w\in A_{p_0}$ and $\max\left\{\frac{n}{n+M},\frac{m}{m+M}\right\}<\min\left\{\frac{p}{p_0},1\right\}$. As in the proof of Theorem 1.11, we get

$$|(\psi_{j,k}*f)(x,y)|\le\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\Big\{M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big)^r\Big\}^{1/r}(x^*,y^*).$$

Therefore, for $x^*\in I, y^*\in J$,

$$\begin{aligned}&\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I(x^*)\chi_J(y^*)\\&\le C\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\Big\{M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big)^r\Big\}^{1/r}(x^*,y^*),\end{aligned}$$

where $|s_1|,|s_2|<L$. When $1\le p<\infty$, since $w\in A_{p_0}\subset A_{p/r}$, taking the L^p_w norm and applying $L^{p/r}_w(l^{1/r})$ boundedness of M_s for $\max\left\{\frac{n}{n+M},\frac{m}{m+M}\right\}<r<\min\left\{\frac{p}{p_0},1\right\}$, we have

$$\begin{aligned}&\Big\|\sum_{I,J}\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I\chi_J\Big\|_{L^p(w)}\\&\le C\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\Big\|\Big\{M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big)^r\Big\}^{1/r}\Big\|_{L^p(w)}\\&\le C\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\Big\|\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big\|_{L^p(w)}.\end{aligned}$$

If $q\ge1$, applying Hölder's inequality and if $0<q<1$ by using usual inequality, summing over j,k, then we get

$$\begin{aligned}&\Big\{\sum_{j,k}\Big(2^{js_1}2^{ks_2}\Big\|\sum_{I,J}\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}\\&\le C\Big\{\sum_{j',k'}\Big(2^{j's_1}2^{k's_2}\Big\|\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}.\end{aligned}\tag{2.1}$$

When $0<p<1$, since $w\in A_{p_0}\subset A_{p/r}$ and taking the L^p_w norm and applying $L^{p/r}_w(l^{1/r})$ boundedness of M_s for $\max\left\{\frac{n}{n+M},\frac{m}{m+M}\right\}<r<\min\left\{\frac{p}{p_0},1\right\}$, then we have

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^m}\Big(\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I\chi_J\Big)^p w(x^*,y^*)\,dx^*dy^*\\&\le C\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\int_{\mathbb{R}^n\times\mathbb{R}^m}\Big\{M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big)^r\Big\}^{p/r}w(x^*,y^*)\,dx^*dy^*\\&\le C\sum_{j',k'}2^{-|j-j'|L}2^{-|k-k'|L}\int_{\mathbb{R}^n\times\mathbb{R}^m}\Big\{\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big\}^p w(x^*,y^*)\,dx^*dy^*\end{aligned}$$

so if $q/p \geq 1$, applying Hölder's inequality and if $0 < q/p < 1$ by using usual inequality, we get

$$\begin{aligned}&\Big\{\sum_{j,k}\Big(2^{js_1}2^{ks_2}\Big\|\sum_{I,J}\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}\\ \leq &C\Big\{\sum_{j',k'}\Big(2^{j's_1}2^{k's_2}\Big\|\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}.\end{aligned} \tag{2.2}$$

Combining (2.1) with (2.2), since $(x_I^{'}, y_J^{'})$ are arbitrary points in $I^{'} \times J^{'}$, we can get the desired result, namely

$$\begin{aligned}&\Big\{\sum_{j,k}\Big(2^{js_1}2^{ks_2}\Big\|\sum_{I,J}\sup_{(x,y)\in I\times J}|(\psi_{j,k}*f)(x,y)|\chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}\\ \leq &C\Big\{\sum_{j',k'}\Big(2^{j's_1}2^{k's_2}\Big\|\sum_{I',J'}\inf_{(x',y')\in I'\times J'}|(\phi_{j',k'}*f)(x',y')|\chi_{I'}\chi_{J'}\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q}.\end{aligned}$$

□

As a consequence of Theorem 1.11 and Theorem 1.12, we have the following characterization of $\dot{F}^{s,q}_{p,w}$ and Besov Spaces $\dot{B}^{s,q}_{p,w}$.

Corollary 2.3. *Let $0 < p,q < \infty$ and $s = (s_1, s_2) \in \mathbb{R}^2, w \in A_\infty$. Suppose that M be the integer which satisfying the inequality $max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < min\Big\{\frac{p}{q_w}, 1, q\Big\}$, then we have*

$$\|f\|_{\dot{F}^{s,q}_{p,w}} \approx \Big\|\Big\{\sum_{j,k}\Big(\sum_{I,J}2^{js_1}2^{ks_2}|(\psi_{j,k}*f)(x_I,y_J)|\chi_I\chi_J\Big)^q\Big\}^{1/q}\Big\|_{L^p(w)},$$

and let M be the integer which satisfying the inequality $max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < min\Big\{\frac{p}{q_w}, 1\Big\}$, then we have

$$\|f\|_{\dot{B}^{s,q}_{p,w}} \approx \Big\{\sum_{j,k}\Big(2^{js_1}2^{ks_2}\Big\|\sum_{I,J}|(\psi_{j,k}*f)(x_I,y_J)|\chi_I\chi_J\Big\|_{L^p(w)}\Big)^q\Big\}^{1/q},$$

where $j,k,x_I,y_J,\chi_I,\chi_J,\psi_{j,k}$ are the same in Theorem 1.11.

3 Boundedness of Flag Singular Integrals

The main purpose of this section is to obtain the boundedness of flag singular integrals on weighted multi-parameter Triebel-Lizorkin and Besov Spaces associated with non-isotropic flag singular integrals. We first give some propositions.

Proposition 3.1. *Let $0 < p,q < \infty$, $w \in A_\infty$. Then $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$ is dense in $\dot{F}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$ and $\dot{B}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$, where M satisfying the inequality $max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < min\Big\{\frac{p}{q_w}, 1, q\Big\}$ for $\dot{F}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$, and M satisfying the inequality $max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < min\Big\{\frac{p}{q_w}, 1\Big\}$ for $\dot{B}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$.*

Proof. Suppose $f \in \dot{F}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$, we get

$$f(x,y) = \sum_{j,k}\sum_{I,J} |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)(\psi_{j,k} * f)(x_I,y_J),$$

where the series converges in $\mathcal{S}_{F,M}(\mathbb{R}^n \times \mathbb{R}^m)$. It suffices to show that

$$\begin{aligned} F &= F_{M_1,M_2,s}(x,y,x_I,y_J) \\ &= \sum_{|j|\le M_1,|k|\le M_2}\sum_{I\times J\subseteq B(0,s)} |I||J|\widetilde{\psi}_{j,k}(x,y,x_I,y_J)(\psi_{j,k} * f)(x_I,y_J) \end{aligned}$$

converges to f in $\dot{F}^{s,q}_{p,w}(\mathbb{R}^n \times \mathbb{R}^m)$, as M_1, M_2 and s tend to infinity, where the $B(0,s) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m : x^2 + y^2 < s^2\}$. To do this, let W the set $\{(I,J) : I \times J \subseteq B(0,s)\}$, where the I, J are dyadic cubes in $\mathbb{R}^n$ and $\mathbb{R}^m$ with side length 2^{-j-N} and $2^{-2j-N} + 2^{-k-N}$, respectively, and let W^c be the complement of W. Let also $V = \{(j,k) : |j| \le M_1, |k| \le M_2\}$ and V^c denotes its complement.

For $(x_{I'}, y_{J'}) \in I' \times J'$, then

$$\begin{aligned} &\Big|\psi_{j',k'} * \sum_{(j,k)\in V^c}\sum_{(I,J)\in W^c} |I||J|\widetilde{\psi}_{j,k}(\cdot,\cdot,x_I,y_J)(x_{I'},y_{J'})(\psi_{j,k} * f)(x_I,y_J)\Big| \\ \le& C\sum_{(j,k)\in V^c} 2^{-|j-j'|L}2^{-|k-k'|L}\Big\{M_s\Big(\sum_{(I,J)\in W^c}|(\psi_j,k * f)(x_I,y_J)|\chi_I\chi_J\Big)^r\Big\}^{1/r}, \end{aligned}$$

where any r satisfy $\max\Big\{\frac{n}{n+M}, \frac{m}{m+M}\Big\} < r < \min\Big\{\frac{p}{p_0}, 1, q\Big\}$. Repeating the proof of Min-Max comparison principle of $\dot{F}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$, when $w \in A_\infty$, we get

$$\begin{aligned} &\Big\|\Big\{\sum_{j',k'}\sum_{I',J'} 2^{j's_1q}2^{k's_2q}|(\psi_{j',k'} * F)|^q\chi_{I'}\chi_{J'}\Big\}^{1/q}\Big\|_{L^p(w)} \\ \le& \Big\|\Big\{\sum_{(j,k)\in V^c}\sum_{(I,J)\in W^c} 2^{js_1q}2^{ks_2q}|(\psi_{j,k} * f)|^q\chi_I\chi_J\Big\}^{1/q}\Big\|_{L^p(w)}, \end{aligned}$$

where the last term tends to zero as M_1, M_2 and r tend to infinity whenever $f \in \dot{F}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$.

When $f \in \dot{B}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$, we can similarly get the desired result. □

Since $\mathcal{S}_{F,M}(\mathbb{R}^m \times \mathbb{R}^n) \subset L^2(\mathbb{R}^m \times \mathbb{R}^n)$, it is immediate to obtain that

Proposition 3.2. *$L^2(\mathbb{R}^m \times \mathbb{R}^n)$ is dense in $\dot{F}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$ and $\dot{B}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$ for $0 < p, q < \infty$.*

We now prove the boundedness of non-isotropic flag singular integrals on $\dot{F}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$ and $\dot{B}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$.

Proof of Theorem 1.13 and Theorem 1.14. For $f \in L^2(\mathbb{R}^m \times \mathbb{R}^n) \cap \dot{F}^{s,q}_{p,w}(\mathbb{R}^m \times \mathbb{R}^n)$, by discrete Calderón's identity,

$$(\psi_{j,k} * Tf)(x,y) = \sum_{j',k'}\sum_{I',J'} |I'||J'||\psi_{j,k} * K * \widetilde{\phi}_{j',k'}(\cdot - x_{I'}, \cdot - y_{J'})(x,y)|\phi_{j',k'} * f(x_{I'}, y_{J'}).$$

The author in [11] has showed

$$\begin{aligned}&|\psi_{j,k}*K*\widetilde{\phi}_{j',k'}(\cdot-x_{I'},\cdot-y_{J'})(x,y)|\\ \leq& C2^{-|j-j'|M}2^{-|k-k'|M}\\ &\times\int_{\mathbb{R}^m}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|(x-x_{I'},y-z-y_{J'})|)^{n+2m+M}}\frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')}+|z|)^{m+M}}dz.\end{aligned}$$

Similar to the proof of Lemma 3.3 in [11], there exists a constant K depending only on M such that, when $2^{-j}\vee 2^{-j'}\leq\sqrt{2^{-k}\vee 2^{-k'}}$,

$$\begin{aligned}&\int_{\mathbb{R}^m}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|(x-x_{I'},y-z-y_{J'})|)^{n+2m+M}}\frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')}+|z|)^{m+M}}dz\\ \leq& C\frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')}+|x-x_{I'}|)^{n+K}}\frac{2^{-(k\wedge k')K}}{(2^{-(k\wedge k')}+|y-y_{J'}|)^{m+K}},\end{aligned}$$

and when $2^{-j}\vee 2^{-j'}>\sqrt{2^{-k}\vee 2^{-k'}}$,

$$\begin{aligned}&\int_{\mathbb{R}^m}\frac{2^{-(j\wedge j')M}}{(2^{-(j\wedge j')}+|(x-x_{I'},y-z-y_{J'})|)^{n+2m+M}}\frac{2^{-(k\wedge k')M}}{(2^{-(k\wedge k')}+|z|)^{m+M}}dz\\ \leq& C\frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')}+|x-x_{I'}|)^{n+K}}\frac{2^{-(j\wedge j')K}}{(2^{-(j\wedge j')}+\sqrt{|y-y_{J'}|})^{2m+K}}.\end{aligned}$$

By an analogous argument to the proof of Theorem 1.11, we have

$$\psi_{j,k}*Tf(x,y)\leq\sum_{j',k'}2^{-|j-j'|M}2^{-|k-k'|M}\Big\{M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\Big)^r\Big\}^{1/r}(u^*,v^*),\tag{3.1}$$

for any $u,u^*\in I, x_{I'}\in I', v,v^*\in J$ and $y_{J'}\in J'$, where M_s is the strong maximal operator.

Applying Hölder's inequality and summing over j,k,I,J yields

$$\begin{aligned}&\Big\{\sum_{j,k}\Big(\sum_{I,J}2^{js_1}2^{ks_2}|\psi_{j,k}*Tf(x,y)(x_I,y_J)|\chi_I\chi_J\Big)^q\Big\}^{1/q}\\ \leq& C\Big\{\sum_{j',k'}\Big(2^{j's_1}2^{k's_2}\Big(M_s\Big(\sum_{I',J'}|(\phi_{j',k'}*f)(x_{I'},y_{J'})\chi_{I'}\chi_{J'}|\Big)^r\Big)^{1/r}\Big)^q\Big\}^{1/q}.\end{aligned}$$

Let $\max\{\frac{n}{n+M},\frac{m}{m+M}\}<r<\min\{\frac{p}{p_0},1,q\}$, since $w\in A_{p_0}\subset A_{p/r}$, applying $L_w^{p/r}(l^{q/r})$ boundedness of M_s, then we have

$$\begin{aligned}&\Big\|\Big\{\sum_{j,k}\Big(\sum_{I,J}2^{js_1}2^{ks_2}|\psi_{j,k}*Tf(x,y)(x_I,y_J)|\chi_I\chi_J\Big)^q\Big\}^{1/q}\Big\|_{L^p(w)}\\ \leq& C\Big\|\Big\{\sum_{j',k'}\Big(\sum_{I',J'}2^{j's_1}2^{k's_2}|(\phi_{j',k'}*f)(x_{I'},y_{J'})|\chi_{I'}\chi_{J'}\Big)^q\Big\}^{1/q}\Big\|_{L^p(w)}.\end{aligned}$$

Namely,

$$\|Tf\|_{\dot{F}^{s,q}_{p,w}} \leq C\|f\|_{\dot{F}^{s,q}_{p,w}}.$$

Since $L^2(\mathbb{R}^n\times\mathbb{R}^m)$ is dense in $\dot{F}^{s,q}_{p,w}(\mathbb{R}^n\times\mathbb{R}^m)$, then T can be extend to be a bounded operator on $\dot{F}^{s,q}_{p,w}(\mathbb{R}^n\times\mathbb{R}^m)$.

From the proof of Theorem 1.13, it is obvious that Theorem 1.14 follows from similar proof of Theorem 1.12 and Theorem 1.13. Here we omit the details. □

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