Communications in Mathematical Analysis
Volume 16, Number 2, pp. 48–56 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

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(Communicated by Palle Jorgensen)

Abstract

Recent work of Bui, Duong and Yan in [1] defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth. This note aims to extend Theorem 5.12 in [1], the decomposition of Besov spaces associated with Schrödinger operators, to more general α, p, q.

AMS Subject Classification: 42B35, 46E35, 47F05

Keywords: Besov space, Decomposition, Schrödinger operator

1 Introduction

The theory of Besov spaces has been an active area of research in the last few decades because of its important role in the study of approximation of functions and regularity of solutions to partial differential equations.

Classical theory of Besov spaces, for example, can be found in [2, 3, 6, 10, 9, 13, 14]. Some of more recent results on Besov spaces are [12, 15, 7, 5].

Recent work of Bui, Duong and Yan in [1] defined Besov spaces associated with a certain operator L under the weak assumption that L generates an analytic semigroup e^{-tL} with Poisson kernel bounds on $L^2(\mathcal{X})$ where $\mathcal{X}$ is a (possibly non-doubling) quasi-metric space of polynomial upper bound on volume growth. When L is the Laplace operator $-\Delta$ or its square root $\sqrt{-\Delta}$ acting on the Euclidean space $\mathbb{R}^n$, this class of Besov spaces associated with the operator L are equivalent to the classical Besov spaces. Depending on the choice of L, the Besov spaces are natural settings for generic estimates for certain singular integral operators such as the fractional powers L^α.

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In Theorem 5.12 of [1], the decomposition of Besov spaces associated with Schrödinger operators is given, but only for the case $\alpha = 0$, $p = q = 1$. This note aims to extend that result to more general α, p, q.

The paper is organized as follows. In Section 2, we give some preliminaries on Schrödinger operators. In Section 3, we define the notion of molecules, then our main result.

2 Schrödinger operators

Suppose that V is a fixed non-negative function on $\mathbb{R}^n$, $n \geq 3$, satisfying a *reverse Hölder inequality* $RH_S(\mathbb{R}^n)$ for some $s > \frac{n}{2}$; that is, there is a $C = C(s,V) > 0$ with the property that

$$\Big(\frac{1}{|B|}\int_B V(x)^s\,dx\Big)^{1/s} \leq \frac{C}{|B|}\int_B V(x)\,dx \tag{2.1}$$

for all balls $B \subset \mathbb{R}^n$. Let us consider the time independent Schrödinger operator with the potential V on $L^2(\mathbb{R}^n)$:

$$L = -\Delta + V(x). \tag{2.2}$$

We note that the operator L is non-negative self-adjoint on $L^2(\mathbb{R}^n)$ and it generates a semigroup

$$e^{-tL}f(x) = \int_{\mathbb{R}^n} p_t(x,y)f(y)\,dy, \quad f \in L^2(\mathbb{R}^n), \quad t > 0,$$

where the kernel $p_t(x,y)$ is dominated by the heat kernel of the Laplacian on $\mathbb{R}^n$, thus $p_t(x,y)$ has a Gaussian upper bound.

Let us recall some estimates for the heat kernel of e^{-tL}. In the same way as in [11], we shall define a function $\rho(x;V) = \rho(x)$ by

$$\rho(x) = \sup\Big\{r > 0 : \frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1\Big\}.$$

In this paper we make the assumption that $V \not\equiv 0$, hence $0 < \rho(x) < \infty$. Using a result in [11], there exist $k_0 \geq 1$ and $c > 0$ such that for every $x, y \in \mathbb{R}^n$,

$$c^{-1}\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{-k_0} \leq \rho(y) \leq c\rho(x)\Big(1 + \frac{|x-y|}{\rho(x)}\Big)^{\frac{k_0}{k_0+1}}. \tag{2.3}$$

In particular, we have $\rho(x) \sim \rho(y)$ when $r \leq \rho(x)$ and $y \in B(x,r)$. Furthermore, when $r = \rho(x)$, we have

$$\frac{1}{r^{n-2}}\int_{B(x,r)} V(y)\,dy \leq 1.$$

Lemma 2.1. *Suppose that $V \in RH_S(\mathbb{R}^n)$, $s > \frac{n}{2}$. Then for every N there exists a constant C_N such that the kernel $p_t(x,y)$ of the semigroup e^{-tL} satisfies*

$$0 \leq p_t(x,y) \leq C_N t^{-\frac{n}{2}} \exp\Big(-\frac{|x-y|^2}{5t}\Big)\Big(1 + \frac{\sqrt{t}}{\rho(x)} + \frac{\sqrt{t}}{\rho(y)}\Big)^{-N}. \tag{2.4}$$

Proof. For a proof, we refer the reader to p. 332, Proposition 2 in [4]. □

We will require estimates for the kernel of the operator $t^2Le^{-t^2L}$,

$$q_t(x,y) = t^2 \frac{\partial p_s(x,y)}{\partial s}\Big|_{s=t^2}, \tag{2.5}$$

as follows.

Proposition 2.2. *There are constants $c,\sigma > 0$ such that for every N there exists a constant $C_N > 0$ so that*

(i) $|q_t(x,y)| \le C_N t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-N}$;

(ii) $|q_t(x+h,y) - q_t(x,y)|$
$\le C_N\Big(\frac{|h|}{t}\Big)^{\sigma} t^{-n} \exp\Big(-\frac{|x-y|^2}{ct^2}\Big)\Big(1+\frac{t}{\rho(x)}+\frac{t}{\rho(y)}\Big)^{-N}$ *for all* $|h| \le t$;

(iii) $\Big|\int_{\mathbb{R}^n} q_t(x,y)\,dy\Big| \le C_N\Big(\frac{t}{\rho(x)}\Big)^{\sigma}\Big(1+\frac{t}{\rho(x)}\Big)^{-N}$.

Proof. For a proof, we refer the reader to p. 332, Proposition 4 in [4]. □

3 Molecular decomposition of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$

Let us define the notion of molecules. In the following, the definition of a molecule associated with a cube $Q = \{x \in \mathbb{R}^n : a_i \le x_i \le b_i, i = 1,2,\dots,n\}$ involves the "lower left corner of Q", $x_Q = a = (a_1,a_2,\dots,a_n)$, and $\ell(Q)$, the side length of Q.

Definition 3.1. Let $\epsilon \in (0,1]$, $\alpha \in (-1,1)$ and $p \ge 1$. A function m_Q is called an (ϵ,α,p)-molecule for L associated to the cube Q if $m_Q = Lg_Q$ for some g_Q, and the following conditions hold:

$$|m_Q(x)| + \ell(Q)^{-2}|g_Q(x)| \le \ell(Q)^{\alpha-n/p}\Big\{1+\frac{|x-x_Q|}{\ell(Q)}\Big\}^{-n-\epsilon} \quad \text{for } x \in \mathbb{R}^n; \tag{3.1}$$

$$\int_{|y|\le\ell(Q)} \|m_Q(x+y)-m_Q(x)\|_{L^p(dx)} \frac{dy}{|y|^{n+\alpha}} \le 1. \tag{3.2}$$

The following result is a molecular characterization of $\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$. It is an extension of Theorem 5.12 in [1], where only the case $\alpha = 0$, $p = q = 1$ is considered, to more general α, p, q. In the following, given $j \in \mathbb{Z}$, we use $\mathbb{D}_j$ to denote the set of all dyadic cubes of sidelength 2^{-j}.

Theorem 3.2. *Suppose that $L = -\Delta + V$, where $V \not\equiv 0$ is a non-negative potential in $RH_s(\mathbb{R}^n)$ for some $s > \frac{n}{2}$. Assume that $f \in L^1(\mathbb{R}^n)$ and let σ be the constant from Proposition 2.2. Let $-1 < \alpha < \min\{1,\sigma\}$ and $1 \le p \le q < \infty$. Then in the following we have (a) ⇒ (b) and (b) ⇒ (c):*

(a) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$.

(b) *For any $0 < \epsilon \le 1$, there exist a sequence of coefficients $\{s_Q\}$, $0 \le s_Q < \infty$, where Q ranges over the dyadic cubes, and a sequence $\{m_Q\}$ of (ϵ,α,p)-molecules for L, such that*

$$f = \sum_Q s_Q m_Q \qquad in \quad \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n), \tag{3.3}$$

and

$$\Big(\sum_{j\in\mathbb{Z}}\Big(\sum_{Q\in\mathbb{D}_j} |s_Q|^p\Big)^{q/p}\Big)^{1/q} \approx \|f\|_{\dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)}. \tag{3.4}$$

(c) $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)$.

Proof of Theorem 3.2. The proof is a modification of that in Theorem 5.12 in [1].

We shall show that (b) $\Rightarrow$ (c).

Let m_Q be an (ϵ,α,p)-molecule for L associated to a cube Q. We will prove that $\|m_Q\|_{\dot{B}^{\alpha,L}_{p,q}} \le C$.

We first split

$$\|m_Q\|_{\dot{B}^{\alpha,L}_{p,q}} = \Big\{\Big(\int_0^{\ell(Q)} + \int_{\ell(Q)}^{\infty}\Big)\big\|t^2Le^{-t^2L}m_Q\big\|^q_{L^p}\frac{dt}{t^{1+\alpha q}}\Big\}^{1/q} \le I + II$$

where

$$I = \Big\{\int_0^{\ell(Q)}\big\|t^2Le^{-t^2L}m_Q\big\|^q_{L^p}\frac{dt}{t^{1+\alpha q}}\Big\}^{1/q},$$
$$II = \Big\{\int_{\ell(Q)}^{\infty}\big\|t^2Le^{-t^2L}m_Q\big\|^q_{L^p}\frac{dt}{t^{1+\alpha q}}\Big\}^{1/q}.$$

Let us estimate the second term. Firstly the bounds for g_Q in (3.1) allow us to obtain

$$\|g_Q\|_{L^1} \lesssim \ell(Q)^{\alpha+2+n(1-1/p)}.$$

Next using that $m_Q = Lg_Q$ for some g_Q, the kernel bounds in Proposition 2.2 (i), and Minkowski's inequality, we have

$$\begin{aligned}
II &= \Big\{\int_{\ell(Q)}^{\infty}\Big(\int_{\mathbb{R}^n}\big|(t^2L)^2e^{-t^2L}g_Q(x)\big|^p dx\Big)^{q/p}\frac{dt}{t^{1+q(\alpha+2)}}\Big\}^{1/q}\\
&\lesssim \Big\{\int_{\ell(Q)}^{\infty}\Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} e^{-p|x-y|^2/ct^2}dx\Big)^{1/p}|g_Q(y)|\,dy\Big)^q\frac{dt}{t^{1+q(\alpha+2+n)}}\Big\}^{1/q}\\
&\lesssim \|g_Q\|_{L^1}\Big\{\int_{\ell(Q)}^{\infty}\frac{dt}{t^{1+q(\alpha+2+n(1-1/p))}}\Big\}^{1/q} \lesssim C.
\end{aligned}$$

To estimate the first term we write

$$\begin{aligned}
I &= \Big\{\int_0^{\ell(Q)}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} q_t(x,y)\,m_Q(y)\,dy\Big|^p dx\Big)^{q/p}\frac{dt}{t^{1+\alpha q}}\Big\}^{1/q}\\
&= \Big\{\int_0^{\ell(Q)}\Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} q_t(x,y)\,[m_Q(y)+m_Q(x)-m_Q(x)]\,dy\Big|^p dx\Big)^{q/p}\frac{dt}{t^{1+\alpha q}}\Big\}^{1/q}\\
&\le I_1 + I_2
\end{aligned}$$

where

$$I_1 = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} q_t(x,y)\, m_Q(x)\, dy\Big|^p dx\Big)^{q/p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q},$$
$$I_2 = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} q_t(x,y)\,[m_Q(y)-m_Q(x)]\, dy\Big|^p dx\Big)^{q/p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q}.$$

Let us estimate I_1. By using (iii) of Proposition 2.2, Minkowski's inequality and the assumption that $p \le q$ we obtain

$$\begin{aligned}
I_1 &\le \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big|\int_{\mathbb{R}^n} q_t(x,y)\, dy\Big|^p |m_Q(x)|^p\, dx\Big)^{q/p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q} \\
&\le C_N\Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big|\frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N}\Big|^p |m_Q(x)|^p\, dx\Big)^{q/p} \frac{dt}{t^{1+\alpha q}}\Big\}^{1/q} \\
&= C_N\Bigg(\Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big|\frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N}\Big|^p |m_Q(x)|^p\, dx\Big)^{q/p} \frac{dt}{t^{1+\alpha q}}\Big\}^{p/q}\Bigg)^{1/p} \\
&\le C_N\Bigg(\int_{\mathbb{R}^n}\Big\{\int_0^{\ell(Q)} \Big|\frac{(t/\rho(x))^\sigma}{(1+t/\rho(x))^N}\Big|^q \frac{dt}{t^{1+\alpha q}}\Big\}^{p/q} |m_Q(x)|\, dx\Bigg)^{1/p} \\
&\le C_{N,\sigma}.
\end{aligned}$$

We estimate the second term by splitting the region of integration in the y variable into two regions: $|x-y| \ge \ell(Q)$ and $|x-y| < \ell(Q)$. That is,

$$\begin{aligned}
I_2 &\le \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big(\int_{\mathbb{R}^n} e^{-|x-y|^2/ct^2}|m_Q(y)-m_Q(x)|\, dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\
&\le I_{2.1} + I_{2.2}
\end{aligned}$$

where

$$I_{2.1} = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|\ge\ell(Q)} e^{-|x-y|^2/ct^2}|m_Q(y)-m_Q(x)|\, dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q},$$
$$I_{2.2} = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|<\ell(Q)} e^{-|x-y|^2/ct^2}|m_Q(y)-m_Q(x)|\, dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q}$$

For the first case we integrate

$$I_{2.1} \le I_{2.1.1} + I_{2.1.2}$$

where

$$I_{2.1.1} = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|\ge\ell(Q)} e^{-|x-y|^2/ct^2}|m_Q(y)|\, dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q},$$
$$I_{2.1.2} = \Big\{\int_0^{\ell(Q)} \Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|\ge\ell(Q)} e^{-|x-y|^2/ct^2}|m_Q(x)|\, dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q}$$

Then for any $\delta > \frac{n}{2}(1-1/p)+\alpha$, Minkowski's inequality gives

$$\begin{aligned} I_{2.1.1} &\leq \Big\{\int_0^{\ell(Q)}\Big(\int_{\mathbb{R}^n}\Big(\int_{|x-y|\geq \ell(Q)} e^{-p|x-y|^2/ct^2}dx\Big)^{1/p}|m_Q(y)|\,dy\Big)^q \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\lesssim \ell(Q)^{-2\delta/p}\,\|m_Q\|_{L^1}\Big\{\int_0^{\ell(Q)} \frac{dt}{t^{1+q(\alpha+n(1-1/p)-2\delta/p)}}\Big\}^{1/q} \lesssim C. \end{aligned}$$

In the last step we used the estimate

$$\|m_Q\|_{L^1} \lesssim \ell(Q)^{n+\alpha-n/p}$$

which holds via the bounds in (3.1).

Next, for any $\delta > 0$, $x \in \mathbb{R}^n$ and cube Q we have

$$\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2}dy \lesssim t^{n+2\delta}\ell(Q)^{-2\delta}.$$

Applying this with some $\delta > \alpha/2$ gives

$$\begin{aligned} I_{2.1.2} &= \Big\{\int_0^{\ell(Q)}\Big(\int_{\mathbb{R}^n}|m_Q(x)|^p\Big(\int_{|x-y|\geq \ell(Q)} e^{-|x-y|^2/ct^2}dy\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\lesssim \ell(Q)^{2\delta}\,\|m_Q\|_{L^p}\Big\{\int_0^{\ell(Q)} \frac{dt}{t^{1+q(\alpha-2\delta)}}\Big\}^{1/q} \lesssim C. \end{aligned}$$

In the last step we used the estimate

$$\|m_Q\|_{L^p} \lesssim \ell(Q)^{\alpha}$$

which holds via again the bounds in (3.1).

Next, with a change of variable $y = x+w$, and applying Minkowski's inequality twice, we obtain

$$\begin{aligned} I_{2.2} &= \Big\{\int_0^{\ell(Q)}\Big(\int_{\mathbb{R}^n}\Big(\int_{|w|\leq \ell(Q)} e^{-|w|^2/ct^2}|m_Q(x+w)-m_Q(x)|\,dw\Big)^p dx\Big)^{q/p} \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\leq \Big\{\int_0^{\ell(Q)}\Big(\int_{|w|\leq \ell(Q)} e^{-|w|^2/ct^2}\|m_Q(\cdot+w)-m_Q(\cdot)\|_{L^p}\,dw\Big)^q \frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q} \\ &\leq \int_{|w|\leq \ell(Q)} \|m_Q(\cdot+w)-m_Q(\cdot)\|_{L^p}\Big\{\int_0^{\ell(Q)} e^{-q|w|^2/ct^2}\frac{dt}{t^{1+q(n+\alpha)}}\Big\}^{1/q}dw \\ &\lesssim \int_{|w|\leq \ell(Q)} \|m_Q(\cdot+w)-m_Q(\cdot)\|_{L^p}\frac{dw}{|w|^{n+\alpha}} \lesssim C \end{aligned}$$

In the last step we applied (3.2).

We show (a) $\Rightarrow$ (b).

Let $f \in \dot{B}^{\alpha,L}_{p,q}(\mathbb{R}^n)\cap L^2(\mathbb{R}^n)$. Applying the Calderón reproducing formula I to f we obtain

$$f = \frac{1}{8}\int_0^\infty (t^2L)^2e^{-2t^2L}f\frac{dt}{t} = \frac{1}{8}\int_0^\infty\int_{\mathbb{R}^n} q_t(x,y)(t^2Le^{-t^2L}f)(y)\frac{dy\,dt}{t}$$

We then "discretize" the right hand side by splitting $\mathbb{R}^n$ into dyadic cubes. Let Q be a dyadic cube. We define

$$\mathcal{T}(Q) := \{(x,t) \in \mathbb{R}^{n+1}_+ : x \in Q,\ \ell(Q)/2 < t \leq \ell(Q)\}$$

to be the "half-cube" in $\mathbb{R}^{n+1}_+$ over Q.

We then have

$$\begin{aligned} f(x) &= \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathbb{D}_j} \frac{1}{8} \iint_{\mathcal{T}(Q)} q_t(x,y)(t^2Le^{-t^2L}f)(y)\frac{dy\,dt}{t} \\ &= \sum_{j\in\mathbb{Z}} \sum_{Q\in\mathbb{D}_j} s_Q\, m_Q(x) \end{aligned}$$

where

$$\begin{aligned} s_Q &= \frac{1}{\ell(Q)^{\alpha+n(1-1/p)}} \iint_{\mathcal{T}(Q)} \left|t^2Le^{-t^2L}f(y)\right|\frac{dy\,dt}{t} \\ m_Q(x) &= \frac{1}{8s_Q} \iint_{\mathcal{T}(Q)} q_t(x,y)(t^2Le^{-t^2L}f)(y)\frac{dy\,dt}{t}. \end{aligned}$$

We now show that m_Q satisfies (3.1) and (3.2).

We first check (3.2). By using estimate (ii) from Proposition 2.2, we have

$$\begin{aligned} &\int_{|z|\leq\ell(Q)} \|m_Q(\cdot+z)-m_Q(\cdot)\|_{L^p}\frac{dz}{|z|^{n+\alpha}} \\ &\leq \frac{1}{8s_Q} \iint_{\mathcal{T}(Q)} \left|t^2Le^{-t^2L}f(y)\right|\Big(\int_{|z|\leq\ell(Q)} \|q_t(\cdot+z,y)-q_y(\cdot,y)\|_{L^p}\frac{dz}{|z|^{n+\alpha}}\Big)\frac{dy\,dt}{t} \\ &\lesssim \frac{1}{s_Q} \iint_{\mathcal{T}(Q)} \left|t^2Le^{-t^2L}f(y)\right|\frac{dy\,dt}{t^{1+\sigma+n(1-1/p)}} \int_{|z|\leq\ell(Q)} \frac{dz}{|z|^{n+\alpha-\sigma}} \\ &\lesssim \frac{1}{\ell(Q)^{\alpha+n(1-1/p)}\ s_Q} \iint_{\mathcal{T}(Q)} \left|t^2Le^{-t^2L}f(y)\right|\frac{dy\,dt}{t} \ \lesssim\ C \end{aligned}$$

In the next to last step we used the condition that $\sigma > \alpha$ in the second integral. We also used that $(y,t) \in \mathcal{T}(Q)$ implies $t \approx \ell(Q)$.

We now check (3.1). For each $x \in \mathbb{R}^n$, and any $\epsilon > 0$

$$\begin{aligned} |m_Q(x)| &\lesssim \frac{1}{s_Q} \iint_{\mathcal{T}(Q)} |q_t(x,y)|\left|t^2Le^{-t^2L}f(y)\right|\frac{dy\,dt}{t} \\ &\lesssim \ell(Q)^{\alpha+n(1-1/p)} \sup_{(y,t)\in\mathcal{T}(Q)} |q_t(x,y)| \\ &\lesssim \ell(Q)^{\alpha-n/p} \sup_{(y,t)\in\mathcal{T}(Q)} e^{-|x-y|^2/ct^2} \\ &\lesssim \ell(Q)^{\alpha-n/p}\Big(1+\frac{|x-x_Q|}{\ell(Q)}\Big)^{-n-\epsilon}. \end{aligned}$$

By a similar argument using (2.4) it follows that (3.1) is true for g_Q. □

Acknowledgments

The author would like to thank his PhD advisor X.T. Duong for proposing this work and for the useful discussions and advice on the topic of the paper. The author is also grateful to Fu Ken Ly for helpful comments and discussions.

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