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Essential Ascent of Closed Operator and Some Decomposition Theorems

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Abstract

The aim of this work is to study the essential ascent and the related essential ascent spectrum of closed unbounded operators on a Banach space. Our approach is based on the concept of paracomplete subspaces of Banach spaces. We prove an unbounded spectral mapping theorem for the ascent spectrum and the essential ascent spectrum. A characterization of closed unbounded operators with finite essential ascent as direct sum of a suitable operators is proved. The new notion of a-essential index for closed unbounded operators with finite essential ascent is introduced. We also give some perturbations results for such operators. This paper extends some results proved in [1] to closed unbounded operators.

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1 Introduction and terminology

Let X be an infinite-dimensional complex Banach space. We denote by $\varphi(\mathrm{X})$ the class of all closed linear operators with domain $\mathcal{D}(T) \subseteq \mathrm{X}$ and range $\mathrm{Im}(T) \subseteq \mathrm{X}$. Let $\mathcal{B}(\mathrm{X})$ be the Banach algebra of bounded linear operators on X. We denote by $\ker(T)$ the kernel of an operator $T \in \varphi(\mathrm{X})$ and by $\alpha(T) = \dim\ker(T)$ and $\beta(T) = \dim \mathrm{X}/\mathrm{Im}(T)$ its nullity and defect, respectively. The resolvent set of $T \in \varphi(\mathrm{X})$ is defined by

$$\varrho(T) = \{z \in \mathbb{C} : \ (zI - T)^{-1} \in \mathcal{B}(\mathrm{X})\},$$

i.e., $z \in \varrho(T)$ if and only if $\mathrm{Im}(zI - T) = \mathrm{X}$ and $(zI - T)$ is injective and has continuous inverse $(zI - T)^{-1} \in \mathcal{B}(\mathrm{X})$. The spectrum of $T \in \varphi(\mathrm{X})$ is defined as $\sigma(T) = \mathbb{C}\backslash\varrho(T)$.

Recall that $T \in \varphi(\mathrm{X})$ is said to be *upper semi-Fredholm* if T has closed range and $\alpha(T) < +\infty$; and T is said to be *lower semi-Fredholm* if $\beta(T) < +\infty$. We say that T is *semi-Fredholm* if it is upper or lower semi-Fredholm, and we denote by $\Phi_{\pm}(\mathrm{X})$ the class of all semi-Fredholm operators. For an operator $T \in \Phi_{\pm}(\mathrm{X})$ we define the *index* of T by

$$\mathrm{ind}(T) = \alpha(T) - \beta(T).$$

An operator is *Fredholm* if it is semi-Fredholm with finite index. We denote by $\Phi(\mathrm{X})$ (resp. $\Phi_{+}(\mathrm{X})$, $\Phi_{-}(\mathrm{X})$) the class of all Fredholm (resp. upper semi-Fredholm, lower semi-Fredholm) operators. The Fredholm spectrum (known in literature also as essential spectrum) is defined by

$$\sigma_e(T) = \{\lambda \in \mathbb{C} : \lambda I - T \notin \Phi(\mathrm{X})\}.$$

We define the generalized kernel of $T \in \varphi(\mathrm{X})$ by $\ker^{\infty}(T) = \bigcup\limits_{n\in\mathbb{N}} \ker(T^n)$ and the generalized range of T by $\mathrm{Im}^{\infty}(T) = \bigcap\limits_{n\in\mathbb{N}} \mathrm{Im}(T^n)$.

Also from [13] we recall that for $T \in \varphi(\mathrm{X})$, the *ascent*, $\boldsymbol{a}(T)$, and the *descent*, $\boldsymbol{d}(T)$, are defined by $\boldsymbol{a}(T) = \inf\{n \geq 0 \ : \ \ker(T^n) = \ker(T^{n+1})\}$ and $\boldsymbol{d}(T) = \inf\{n \geq 0 \ : \ \mathrm{Im}(T^n) = \mathrm{Im}(T^{n+1})\}$, respectively; the infimum over the empty set is taken to be ∞.

An operator $T \in \varphi(\mathrm{X})$ is called s-regular (semi-regular) if T has closed range and $\ker^{\infty}(T) \subseteq \mathrm{Im}^{\infty}(T)$.

For $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ an unbounded operator and n, $k \in \mathbb{N}$, we define the following three quantities :

$$\alpha_n^k(T) = \dim\ker(T^{n+k})/\ker(T^n),$$

$$\beta_n^k(T) = \dim \mathrm{Im}(T^n)/\mathrm{Im}(T^{n+k}),$$

$$S_n^k(T) = \dim[\mathrm{Im}(T^n) \cap \ker(T^k)]/[\mathrm{Im}(T^{n+1}) \cap \ker(T^k)].$$

Let us recall the following useful relations :

$$\alpha_n^k(T) = \dim\ker(T^k) \cap \mathrm{Im}(T^n),$$

$$\begin{aligned} \beta_n^k(T) &= \dim \mathcal{D}(T^n)/[\mathrm{Im}(T^k) + \ker(T^n)] \cap \mathcal{D}(T^n), \\ &\leq \dim \mathrm{X}/[\mathrm{Im}(T^k) + \ker(T^n)]. \end{aligned}$$

It is clear that, for every $j \in \mathbb{N}$, $(\alpha_n^j(T))_{n\geq 0}$ and $(\beta_n^j(T))_{n\geq 0}$ are both decreasing sequences, whereas $(\alpha_j^k(T))_{k\geq 0}$ and $(\beta_j^k(T))_{k\geq 0}$ are both increasing sequences.

For $T \in \varphi(X)$, the *essential ascent*, $\boldsymbol{a_e}(T)$, is defined by

$$\boldsymbol{a_e}(T) = \inf\{n \in \mathbb{N} : \alpha_n^1(T) < +\infty\},$$

where as usual the infimum over the empty set is taken to be ∞. The *essential descent*, $\boldsymbol{d_e}(T)$, is defined by

$$\boldsymbol{d_e}(T) = \inf\{n \in \mathbb{N} : \beta_n^1(T) < +\infty\},$$

if no such n exists, then by definition $\boldsymbol{d_e}(T) = +\infty$.

The paper is organized as follows. In the next section, we prove some algebraic results needed in this paper. Section 3 focuses on some properties of essential ascent of closed unbounded operators in Banach spaces. Some results that deal with the connection between, on the one hand, the essential ascent resolvent set, and on the other, the s-regular set and the upper semi-Fredholm resolvent set of closed unbounded operators are given. In Section 4, we prove an unbounded spectral mapping theorems for the essential ascent spectrum and ascent spectrum. The notion of an a-essential index of closed unbounded operator with finite essential ascent is introduced in Section 5. We prove a decomposition theorem for closed unbounded operators with finite essential ascent and such that $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is topological complemented in X as direct sum of a suitable operators having some specific properties. Finally, in Section 6, we show some perturbation results for closed unbounded operators with finite essential ascent.

In this paper, some results from [1] related to essential ascent for bounded operators are extended to closed unbounded operators. However, the techniques used in this work are different from those used in [1]. Our approach here is based in the concept of paracomplete subspaces of Banach spaces (see, [8, Chapter II]).

2 Algebraic preliminaries

Throughout this paper the symbol $\dot{+}$ denotes the standard algebraic sum, while $\oplus$ denotes the direct sum of closed subspaces, i.e., $X_0 = X_1 \oplus X_2$ if the linear space $X_0 = X_1 + X_2$ is closed and $X_1 \cap X_2 = \{0\}$. We shall say that X_1 is topological complemented in X_0 if there is a closed subspace $X_2 \subseteq X_0$ such that $X_0 = X_1 \oplus X_2$. In this case the subspace X_2 is said to be a topological complement of X_1. Also X_1 and X_2 are said to be topological complementary subspaces.

For $T : \mathcal{D}(T) \subseteq X \longrightarrow X$ and $k \in \mathbb{N}$, the k-degree of stable iteration, $p_k(T)$, is defined by

$$\begin{aligned} p_k(T) &= \inf\{n \in \mathbb{N} : \ker(T^k) \cap \mathrm{Im}(T^n) = \ker(T^k) \cap \mathrm{Im}(T^m), \forall\, m \geq n\}, \\ &= \inf\{n \in \mathbb{N} : S_m^k(T) = 0, \forall\, m \geq n\}, \end{aligned}$$

where the infimum over the empty set is taken to be ∞.

We note that if $\boldsymbol{a_e}(T) < +\infty$, then

$$p_k(T) = \inf\{n \in \mathbb{N} : \alpha_n^k(T) = \alpha_m^k(T),\ \forall\, m \geq n\}.$$

Define

$$\mathcal{A}(X) = \{T \in \varphi(X) : \mathcal{D}(T^i) + \mathrm{Im}(T^j) = X,\ \forall\, i, j \in \mathbb{N}\}.$$

Clearly, $\mathcal{A}(\mathrm{X}) \neq \emptyset$, because $T \in \mathcal{A}(\mathrm{X})$, when T is a closed surjective operator.

For $T \in \mathcal{A}(\mathrm{X})$, we can see the following

$$\begin{array}{rcl} \beta_n^k(T) & = & \dim \mathrm{Im}(T^n)/\mathrm{Im}(T^{n+k}), \\ & = & \dim \mathcal{D}(T^n)/[\mathrm{Im}(T^k)+\ker(T^n)]\cap \mathcal{D}(T^n), \\ & = & \dim[\mathcal{D}(T^n)+\mathrm{Im}(T^k)]/[\mathrm{Im}(T^k)+\ker(T^n)], \\ & = & \dim \mathrm{X}/[\mathrm{Im}(T^k)+\ker(T^n)]. \end{array} \tag{2.1}$$

Throughout this paper, we use the following notation

$$\begin{array}{rcccc} \widetilde{T}_k & : & \mathcal{D}(\widetilde{T}_k) \subseteq \mathrm{X}/\ker(T^k) & \longrightarrow & \mathrm{X}/\ker(T^k) \\ & & \overline{x} & \longmapsto & \overline{Tx}, \end{array}$$

where $T \in \varphi(\mathrm{X})$ and $k \in \mathbb{N}$.

We start our study with the following algebraic results for later use.

Lemma 2.1. *Let $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$, $n \in \mathbb{N}$ and $k, m \in \mathbb{N}\backslash\{0\}$. Then*

$$\alpha_{nm}^1(T) \leq \alpha_n^k(T^m) \leq mk\,\alpha_{nm}^1(T), \tag{1}$$

$$\beta_{nm}^1(T) \leq \beta_n^k(T^m) \leq mk\beta_{nm}^1(T). \tag{2}$$

Proof. Let us first observe that

$$\alpha_n^k(T^m) = \dim\ker[(T^m)^{n+k}]/\ker[(T^m)^n] = \dim\ker(T^{mn+mk})/\ker(T^{mn}).$$

It follows that

$$\alpha_n^k(T^m) = \sum_{i=0}^{mk-1} \dim\ker(T^{mn+i+1})/\ker(T^{mn+i}) = \sum_{i=0}^{mk-1} \alpha_{mn+i}^1(T).$$

Therefore

$$\alpha_{nm}^1(T) \leq \alpha_n^k(T^m) \leq mk\,\alpha_{nm}^1(T).$$

This prove (1). To prove (2), note that

$$\beta_n^k(T^m) = \dim \mathrm{Im}[(T^m)^n]/\mathrm{Im}[(T^m)^{n+k}] = \dim \mathrm{Im}(T^{mn})/\mathrm{Im}(T^{mn+mk}).$$

In particular, this allows us to see

$$\beta_n^k(T^m) = \sum_{i=0}^{mk-1} \dim \mathrm{Im}(T^{mn+i})/\mathrm{Im}(T^{mn+i+1}) = \sum_{i=0}^{mk-1} \beta_{mn+i}^1(T).$$

Hence

$$\beta_{nm}^1(T) \leq \beta_n^k(T^m) \leq mk\beta_{nm}^1(T).$$

The proof is complete. □

Lemma 2.2. *Let $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ and $n, k \in \mathbb{N}$, then*

$$\alpha_n^k(T) = S_n^k(T) + \alpha_{n+1}^k(T).$$

Proof. Let $\mathrm{M} \subseteq \ker(T^k) \cap \mathrm{Im}(T^n)$ such that

$$\ker(T^k) \cap \mathrm{Im}(T^n) = \ker(T^k) \cap \mathrm{Im}(T^{n+1}) \dot{+} \mathrm{M}.$$

Then

$$\dim \mathrm{M} = \dim[\ker(T^k) \cap \mathrm{Im}(T^n)]/[\ker(T^k) \cap \mathrm{Im}(T^{n+1})] = S_n^k(T).$$

Consequently,

$$\alpha_n^k(T) = S_n^k(T) + \alpha_{n+1}^k(T),$$

and this completes the proof. □

Lemma 2.3. *Let* $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ *and let* $i, k \in \mathbb{N}$, *one has*

$$\begin{aligned} S_i^k(T) &= \dim[\ker(T) \cap \mathrm{Im}(T^i)]/[\ker(T) \cap \mathrm{Im}(T^{i+k})], \\ &= \dim[\ker(T^{i+1}) + \mathrm{Im}(T^k)]/[\ker(T^i) + \mathrm{Im}(T^k)]. \end{aligned}$$

Proof. Let T_i denote the restriction of T to the invariant subspace $\mathrm{Im}(T^i)$. Since

$$\begin{aligned} S_i^k(T) &= \dim[\ker(T^k) \cap \mathrm{Im}(T^i)]/[\mathrm{Im}(T^i) \cap \ker(T^k) \cap \mathrm{Im}(T^{i+1})] \\ &= \dim \ker[(T_i)^k]/[\ker[(T_i)^k] \cap \mathrm{Im}(T_i)], \end{aligned}$$

from [6, Lemma 3.5], we deduce that

$$\begin{aligned} S_i^k(T) &= \dim \ker(T_i)/[\ker(T_i) \cap \mathrm{Im}[(T_i)^k]] \\ &= \dim[\ker(T) \cap \mathrm{Im}(T^i)]/[\ker(T) \cap \mathrm{Im}(T^{i+k})]. \end{aligned}$$

This prove the first equality. Let us show the second equality. First, denote by $\overline{y}$ the class of $y \in [\ker(T) \cap \mathrm{Im}(T^i)]$ modulo $[\ker(T) \cap \mathrm{Im}(T^{i+k})]$. Define ψ by setting $\psi(x) = \overline{T^i x}$, for each $x \in \ker(T^{i+1})$. It is clear that ψ is a linear operator from $\ker(T^{i+1})$ onto $[\ker(T) \cap \mathrm{Im}(T^i)]/[\ker(T) \cap \mathrm{Im}(T^{i+k})]$ and $\ker(\psi) = [\mathrm{Im}(T^k) + \ker(T^i)] \cap \ker(T^{i+1})$. Consequently,

$$\begin{aligned} [\ker(T) \cap \mathrm{Im}(T^i)]/[\ker(T) \cap \mathrm{Im}(T^{i+k})] &\approx \ker(T^{i+1})/[\mathrm{Im}(T^k) + \ker(T^i)] \cap \ker(T^{i+1}) \\ &\approx [\ker(T^{i+1}) + \mathrm{Im}(T^k)]/[\ker(T^i) + \mathrm{Im}(T^k)]. \end{aligned}$$

This completes the proof of the lemma. □

Remark 2.4. Let $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ and $k \in \mathbb{N}\backslash\{0\}$, then $p_1(T) = p_k(T)$. Indeed, let $d = p_k(T)$, from Lemma 2.3, we have immediately

$$\ker(T) \cap \mathrm{Im}(T^d) = \ker(T) \cap \mathrm{Im}(T^{i+k}), \quad \forall\, i \geq d.$$

Hence, for every $i \in \mathbb{N}$,

$$\ker(T) \cap \mathrm{Im}(T^{d+i}) \subseteq \ker(T) \cap \mathrm{Im}(T^d) = \ker(T) \cap \mathrm{Im}(T^{d+i+k}) \subseteq \ker(T) \cap \mathrm{Im}(T^{d+i}),$$

which implies that $p_1(T) \leq p_k(T)$. On the other hand, put $l = p_1(T)$, then

$$\ker(T) \cap \mathrm{Im}(T^l) = \ker(T) \cap \mathrm{Im}(T^{i+k}), \quad \forall\, i \geq l.$$

Now, from Lemma 2.3, for all $i \geq l$, we have $S_i^k(T) = 0$, and consequently, $p_k(T) \leq p_1(T)$.

Lemma 2.5. *Let $T \in \mathcal{A}(\mathrm{X})$ and let $n, k \in \mathbb{N}$. Then*

$$\beta_n^k(T) = S_n^k(T) + \beta_{n+1}^k(T).$$

Proof. First, from (2.1), it follows that

$$\beta_n^k(T) = \dim \mathrm{X}/[\mathrm{Im}(T^k) + \ker(T^n)]$$

and

$$\beta_{n+1}^k(T) = \dim \mathrm{X}/[\mathrm{Im}(T^k) + \ker(T^{n+1})].$$

On the other hand, since

$$\mathrm{Im}(T^k) + \ker(T^n) \subseteq \mathrm{Im}(T^k) + \ker(T^{n+1}) \subseteq \mathrm{X},$$

we see that

$$\beta_n^k(T) = \dim[\mathrm{Im}(T^k) + \ker(T^{n+1})]/[\mathrm{Im}(T^k) + \ker(T^n)] + \beta_{n+1}^k(T).$$

Hence, by Lemma 2.3,

$$\beta_n^k(T) = S_n^k(T) + \beta_{n+1}^k(T).$$

This complete the proof. □

3 Ascent spectrum and essential ascent spectrum of closed operator

The goal of this section is to extend some results in [1] to closed unbounded operators of Banach spaces.

For the rest of this article, we denote by

$$\varphi_\infty(\mathrm{X}) = \{T \in \varphi(\mathrm{X}) : \ T^n \in \varphi(\mathrm{X}), \forall\, n \in \mathbb{N}\}.$$

We note that if $\varrho_e^+(T) = \{\lambda \in \mathbb{C} : \lambda I - T \in \Phi_+(\mathrm{X})\} \neq \emptyset$, then $P(T) \in \varphi(\mathrm{X})$, for every complex polynomial P.

Let us recall the following definition [8, Definition 2.1.1].

Definition 3.1. A subspace M of X is said to be paracomplete in X, if M is a Banach space and the canonical injection of M in X is continuous.

The following lemma follows immediately from [8, Proposition 2.1.3] and [8, Proposition 2.1.4].

Lemma 3.2. *Let $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ be a paracomplete operator and let $k \in \mathbb{N}$. Then $\mathcal{D}(T^k)$, $\mathrm{Im}(T^k)$ and $\ker(T^k)$ are paracomplete subspaces in* X.

We have the following lemma, which will be needed in the sequel.

Lemma 3.3. *Let $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$ be a paracomplete operator such that $\boldsymbol{a_e}(T) < +\infty$ and let $k \in \mathbb{N}$. The following statements are equivalent :*

1) $\mathrm{Im}(T^k)+\ker(T^n)$ *is closed for some* $n \geq \boldsymbol{a_e}(T)$;

2) $\mathrm{Im}(T^k)+\ker(T^n)$ *is closed for all* $n \geq \boldsymbol{a_e}(T)$.

Proof. Only the implication "1) $\Longrightarrow$ 2)" requires a proof. Let $n_0 \geq \boldsymbol{a_e}(T)$ such that $\mathrm{Im}(T^k)+\ker(T^{n_0})$ is closed. First, we prove that $\mathrm{Im}(T^k)+\ker(T^{n_0+1})$ is closed. Since $\alpha^1_{n_0}(T) < +\infty$, then there is a finite dimensional subspace $\mathrm{M} \subseteq \ker(T^{n_0+1})$ such that $\ker(T^{n_0+1}) = \ker(T^{n_0}) \dot{+} \mathrm{M}$. Hence,

$$\mathrm{Im}(T^k)+\ker(T^{n_0+1}) = \mathrm{Im}(T^k)+\ker(T^{n_0})+\mathrm{M} \text{ is closed.}$$

Suppose that $n_0 > \boldsymbol{a_e}(T)$. The lemma is proved if we prove that $\mathrm{Im}(T^k)+\ker(T^{n_0-1})$ is closed. Since $\alpha^1_{n_0-1}(T) < +\infty$, there exists a finite dimensional subspace $\mathrm{N} \subseteq \ker(T^{n_0})$ such that $\ker(T^{n_0}) = \ker(T^{n_0-1}) \dot{+} \mathrm{N}$. Therefore $[\mathrm{Im}(T^k)+\ker(T^{n_0-1})]+\mathrm{N}$ and $[\mathrm{Im}(T^k)+\ker(T^{n_0-1})] \cap \mathrm{N}$ are both closed. Consequently, by applying Lemma 3.2, [8, Proposition 2.1.1] and [8, Proposition 2.2], we deduce that $\mathrm{Im}(T^k)+\ker(T^{n_0-1})$ is closed. This completes the proof. □

Lemma 3.4. *Let $T \in \varphi_\infty(\mathrm{X})$ such that $\boldsymbol{a_e}(T)$ is finite. Let $k \in \mathbb{N}\backslash\{0\}$ and $j \geq p_k(T)$. If* $\mathrm{Im}(T^k)+\ker(T^{\boldsymbol{a_e}(T)})$ *is closed, then*

$$\begin{array}{rccc} \widetilde{T}\ : & \mathcal{D}(\widetilde{T}) \subseteq \mathrm{X}/\ker(T^j) & \longrightarrow & \mathrm{X}/\ker(T^j) \\ & \overline{x} & \longmapsto & \overline{T^k x} \end{array}$$

is both s-regular and upper semi-Fredholm operator.

Proof. First, recall that from Lemma 3.3, we have $\mathrm{Im}(T^k)+\ker(T^{\boldsymbol{a_e}(T)+n})$ is closed for all $n \in \mathbb{N}$. Define the following map :

$$\begin{array}{rccc} \pi\ : & \mathrm{X}\times\mathrm{X} & \longrightarrow & (\mathrm{X}/\ker(T^j))\times(\mathrm{X}/\ker(T^j)) \\ & (x,y) & \longmapsto & (\overline{x},\overline{y}). \end{array}$$

Let $\mathrm{G}(\widetilde{T})$ denote the graph of $\widetilde{T}$, evidently $\mathrm{G}(\widetilde{T}) = \pi(\mathrm{G}(T^k))$. Recall that by [8, Proposition 2.1.4], $\mathrm{G}(\widetilde{T})$ is paracomplete. On the other hand, it is clear that $\alpha(\widetilde{T}) = \dim\ker(T^{j+k})/\ker(T^j)$ is finite and $\mathrm{Im}(\widetilde{T}) = [\mathrm{Im}(T^k)+\ker(T^j)]/\ker(T^j)$ is closed. Hence, from [8, Proposition 2.2.3], $\widetilde{T}$ is closed, and consequently, $\widetilde{T}$ is upper semi-Fredholm. Now let $x \in \ker(T^{j+k})$, then

$$T^j x \in \ker(T^k)\cap\mathrm{Im}(T^j) = \ker(T^k)\cap\mathrm{Im}(T^{j+nk}), \quad \forall\, n \in \mathbb{N},$$

which implies that, for every $n \in \mathbb{N}$, there is $x_n \in \mathrm{X}$ such that $T^j x = T^{j+nk}x_n$. Hence, $x = (x - T^{kn}x_n) + T^{kn}x_n \in \mathrm{Im}(T^{kn})+\ker(T^j)$ and consequently, $\ker(\widetilde{T}) \subseteq \mathrm{Im}^\infty(\widetilde{T})$. This finishes the proof of Lemma 3.4. □

We should note that the techniques given in Lemma 3.4 were influenced by a Lemma 2.1 [1].

To simplify notation, for the remainder of the paper we write simply T_λ in place of $\lambda I - T$, for all $T \in \varphi(\mathrm{X})$ and $\lambda \in \mathbb{C}$.

The *ascent resolvent* set of an operator $T \in \varphi(\mathrm{X})$ is defined by

$$\varrho_{asc}(T) = \{\lambda \in \mathbb{C} : T_\lambda \in \varphi_\infty(\mathrm{X}), \boldsymbol{a}(T_\lambda) < +\infty, \mathrm{Im}(T_\lambda)+\ker[(T_\lambda)^{\boldsymbol{a}(T_\lambda)}] \text{ is closed}\}.$$

The complementary set $\sigma_{asc}(T) = \mathbb{C}\backslash\varrho_{asc}(T)$ is the *ascent spectrum* of T.

The *essential ascent resolvent* and *essential ascent spectrum* for $T \in \varphi(\mathrm{X})$ are defined respectively by

$$\varrho^e_{asc}(T) = \{\lambda \in \mathbb{C} : T_\lambda \in \varphi_\infty(\mathrm{X}), \boldsymbol{a_e}(T_\lambda) < +\infty, \mathrm{Im}(T_\lambda) + \ker[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)}] \text{ is closed}\}$$

and

$$\sigma^e_{asc}(T) = \mathbb{C}\backslash\varrho^e_{asc}(T).$$

It is clear from Lemma 3.3, that

$$\varrho(T) \subseteq \varrho_{asc}(T) \subseteq \varrho^e_{asc}(T).$$

On the other hand, if $\varrho^+_e(T) \neq \emptyset$, then

$$\varrho_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a}(T_\lambda) < +\infty, \mathrm{Im}(T_\lambda) + \ker[(T_\lambda)^{\boldsymbol{a}(T_\lambda)}] \text{ is closed}\}$$

and

$$\varrho^e_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a_e}(T_\lambda) < +\infty, \mathrm{Im}(T_\lambda) + \ker[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)}] \text{ is closed}\}.$$

Example 3.5.

1) Let $T \in \Phi_+(\mathrm{X})$, then $\boldsymbol{a_e}(T) = 0$, $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is closed and $T \in \varphi_\infty(\mathrm{X})$. Hence, $0 \in \varrho^e_{asc}(T)$.

2) Let $T \in \mathcal{A}(\mathrm{X}) \cap \varphi_\infty(\mathrm{X})$ such that $q = \max\{\boldsymbol{a_e}(T), \boldsymbol{d_e}(T)\} < +\infty$, then we have $\beta^1_q(T) = \dim \mathrm{X}/[\mathrm{Im}(T) + \ker(T^q)] < +\infty$. Using [8, Proposition 2.1.1], [8, Proposition 2.2] together with Lemma 3.2 and Lemma 3.3, we infer $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is closed. Hence, $0 \in \varrho^e_{asc}(T)$.

3) Let $T \in \mathcal{A}(\mathrm{X}) \cap \varphi_\infty(\mathrm{X})$ such that $\max\{\boldsymbol{a}(T), \boldsymbol{d_e}(T)\} < +\infty$, then $0 \in \varrho_{asc}(T)$.

Remark 3.6. Let $T \in \varphi_\infty(\mathrm{X})$, such that $\boldsymbol{a_e}(T) < +\infty$.

1) If $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)+n})$ is closed for some $n \in \mathbb{N}$, then $\mathrm{Im}(T^k) + \ker(T^{\boldsymbol{a_e}(T)+n})$ is closed for all $k \in \mathbb{N}$. Indeed, from the proof of Lemma 3.4, we conclude that

$$\begin{array}{rccl} \widetilde{T} : & \mathcal{D}(\widetilde{T}) \subseteq \mathrm{X}/\ker(T^{\boldsymbol{a_e}(T)+n}) & \longrightarrow & \mathrm{X}/\ker(T^{\boldsymbol{a_e}(T)+n}) \\ & \overline{x} & \longmapsto & \overline{Tx} \end{array}$$

is upper semi-Fredholm operator and hence,

$$\mathrm{Im}(\widetilde{T}^k) = [\mathrm{Im}(T^k) + \ker(T^{\boldsymbol{a_e}(T)+n})]/\ker(T^{\boldsymbol{a_e}(T)+n})$$

is closed for all $k \in \mathbb{N}$. Consequently, $\mathrm{Im}(T^k) + \ker(T^{\boldsymbol{a_e}(T)+n})$ is closed.

2) If $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)+n})$ is closed for some $n \in \mathbb{N}$, then $\mathrm{Im}(T^k)$ is closed for all $k \geq \boldsymbol{a_e}(T)$. Indeed, from assertion 1), we know that $\mathrm{Im}(T^k) + \ker(T^{\boldsymbol{a_e}(T)+n})$ is closed and $\dim \mathrm{Im}(T^k) \cap \ker(T^{\boldsymbol{a_e}(T)+n}) = \alpha^{\boldsymbol{a_e}(T)+n}_k(T) < +\infty$. Now the result follows from Lemma 3.2 and [8, Proposition 2.1.1].

Remark 3.7. For the case of Hilbert spaces, if $T \in \mathcal{A}(X)$, from Remark 3.6 and [8, Proposition 2.3.5], we can deduce that :

$$\varrho_{asc}(T) = \{\lambda \in \mathbb{C} : T_\lambda \in \varphi_\infty(X), \boldsymbol{a}(T_\lambda) < +\infty \text{ and } \mathrm{Im}[(T_\lambda)^{\boldsymbol{a}(T_\lambda)+1}] \text{ is closed}\}$$

and

$$\varrho^e_{asc}(T) = \{\lambda \in \mathbb{C} : T_\lambda \in \varphi_\infty(X), \boldsymbol{a_e}(T_\lambda) < +\infty \text{ and } \mathrm{Im}[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)+1}] \text{ is closed}\}.$$

Now, we are ready to state our main result of this section, which represents an improvement of [1, Theorem 2.3] to the class of unbounded closed operators.

Theorem 3.8. *Let $T \in \varphi_\infty(X)$ such that $\boldsymbol{a_e}(T) < +\infty$ and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is closed. For all $j, n \in \mathbb{N}\backslash\{0\}$, there exists $\varepsilon > 0$ such that for every λ with $0 < |\lambda| < \varepsilon$, the following assertions hold :*

1) *$\lambda I - T^j$ is s-regular and upper semi-Fredholm,*

2) *$\alpha[(\lambda I - T^j)^n] = jn\alpha^1_{p_1(T)}(T)$,*

3) *$\boldsymbol{a_e}(\lambda I - T^j) = p_1(\lambda I - T^j) = 0$,*

4) *if $\overline{\mathcal{D}(T)} = X$, then $\beta[(\lambda I - T)^n] \geq n\beta^1_{p_1(T)}(T)$, and equality holds when $T \in \mathcal{A}(X)$.*

Proof. Assertions 1) and 2) can be proved as in [1, Theorem 2.3].
3) Let $p = p_1(T)$. Since $\alpha[(\lambda I - T^j)^n] = nj\alpha^1_p(T) < +\infty$, it follows that

$$\boldsymbol{a_e}(\lambda I - T^j) = 0,$$

$$\begin{aligned} \alpha^1_n(\lambda I - T^j) &= \dim(\ker[(\lambda I - T^j)^{n+1}]\backslash\ker[(\lambda I - T^j)^n]), \\ &= \alpha[(\lambda I - T^j)^{n+1}] - \alpha[(\lambda I - T^j)^n], \\ &= j\alpha^1_p(T). \end{aligned}$$

Hence, $p_1(\lambda I - T^j) = 0$.
4) Let $p = p_1(T)$. We have $\overline{\mathcal{D}(\widetilde{T_p})} = \overline{\mathcal{D}(T)}/\ker(T^p) = X/\ker(T^p)$, so as in the proof of [1, Theorem 2.3], we see that

$$\begin{aligned} \beta[(\lambda I - T)^n] &= \beta[(\lambda I - \widetilde{T_p})^n], \\ &= n \dim X/[\mathrm{Im}(T) + \ker(T^p)], \\ &\geq n \dim \mathrm{Im}(T^p)/\mathrm{Im}(T^{p+1}). \end{aligned}$$

Now suppose that $T \in \mathcal{A}(X)$. We know from Remark 3.6 that $\mathrm{Im}(T^p)$ and $\mathrm{Im}(T^{p+1})$ are both closed. Let us define $\widetilde{T} : \mathcal{D}(\widetilde{T}) \subseteq \mathrm{Im}(T^p) \longrightarrow \mathrm{Im}(T^p)$ by $\widetilde{T}(x) = Tx$. Clearly,

$$\alpha(\widetilde{T}) = \dim \ker(T) \cap \mathrm{Im}(T^p) < +\infty, \tag{1}$$

$$\mathrm{Im}(\widetilde{T}) = \mathrm{Im}(T^{p+1}) \text{ is closed in } \mathrm{Im}(T^p), \tag{2}$$

$$\ker(\widetilde{T}) = \ker(T) \cap \mathrm{Im}(T^p) = \ker(T) \cap \mathrm{Im}(T^{p+n}) \subseteq \mathrm{Im}(T^{p+n}) \subseteq \mathrm{Im}(\widetilde{T}^n). \tag{3}$$

Then combining (1), (2) and (3), we obtain $\widetilde{T}$ is both s-regular and upper semi-Fredholm operator. Thus, there exists $\varepsilon > 0$ such that if $0 < |\lambda| < \varepsilon$, the following facts hold

- $\lambda I-\widetilde{T}$ is both s-regular and upper semi-Fredholm operator,
- $\alpha[(\lambda I-\widetilde{T})^n]=n\alpha(\widetilde{T})=n\alpha_p^1(T)$,
- $\operatorname{ind}[(\lambda I-\widetilde{T})^n]=n\operatorname{ind}(\lambda I-\widetilde{T})=n\operatorname{ind}(\widetilde{T})=n\operatorname{ind}(\widetilde{T})$.

Now, we observe that $\operatorname{Im}[(\lambda I-T)^n]+\operatorname{Im}(T^p)=\mathrm{X}$, for all $\lambda\in\mathbb{C}\backslash\{0\}$. Indeed, since the polynomials $Q_1(\mu)=(\lambda-\mu)^n$ and $Q_2(\mu)=\mu^p$ are relatively prime, then there exist R_1 and R_2 two polynomials of degree n_1 and n_2, respectively, such that

$$1=Q_1(\mu)R_1(\mu)+Q_2(\mu)R_2(\mu),\quad \forall\,\mu\in\mathbb{C}.$$

Now, setting $n_0=\max\{n+n_1,\ p+n_2\}$, then

$$\mathcal{D}[Q_1(T)R_1(T)+Q_2(T)R_2(T)]=\mathcal{D}(T^{n_0}).$$

Let $x\in\mathcal{D}(T^{n_0})$, we have

$$x=Q_1(T)R_1(T)x+Q_2(T)R_2(T)x\in\operatorname{Im}[(\lambda I-T)^n]+\operatorname{Im}(T^p),$$

which implies that

$$\mathcal{D}(T^{n_0})\subseteq\operatorname{Im}[(\lambda I-T)^n]+\operatorname{Im}(T^p).$$

Since $T\in\mathcal{A}(\mathrm{X})$, it follows that

$$\mathrm{X}=\mathcal{D}(T^{n_0})+\operatorname{Im}(T^p)\subseteq\operatorname{Im}[(\lambda I-T)^n]+\operatorname{Im}(T^p).$$

On the other hand, for every $\lambda\in\mathbb{C}\backslash\{0\}$, we have

$$\begin{aligned}\beta[(\lambda I-T)^n] &= \dim \mathrm{X}/\operatorname{Im}[(\lambda I-T)^n],\\ &= \dim[\operatorname{Im}[(\lambda I-T)^n]+\operatorname{Im}(T^p)]/\operatorname{Im}[(\lambda I-T)^n],\\ &= \dim\operatorname{Im}(T^p)/[\operatorname{Im}[(\lambda I-T)^n]\cap\operatorname{Im}(T^p)].\end{aligned}\tag{4}$$

Let us show that

$$\operatorname{Im}[(\lambda I-\widetilde{T})^n]=\operatorname{Im}[(\lambda I-T)^n]\cap\operatorname{Im}(T^p),\ \ \forall\,\lambda\in\mathbb{C}\backslash\{0\}.$$

If $p=0$, the equality above is trivial. If, instead, $p\geq 1$, clearly $\operatorname{Im}[(\lambda I-\widetilde{T})^n]\subseteq\operatorname{Im}[(\lambda I-T)^n]\cap\operatorname{Im}(T^p)$. In order to show the converse inclusion, let $y\in\operatorname{Im}(\lambda I-T)\cap\operatorname{Im}(T^p)$. Then there exist $z,x\in\mathrm{X}$ such that $y=(\lambda I-T)z=T^px$. This implies in particular that $z=\frac{1}{\lambda}(Tz+T^px)\in\operatorname{Im}(T)$. Thus $z\in\operatorname{Im}(T^p)$ and $y=(\lambda I-T)z\in\operatorname{Im}(\lambda I-\widetilde{T})$. Consequently $\operatorname{Im}(\lambda I-\widetilde{T})=\operatorname{Im}(\lambda I-T)\cap\operatorname{Im}(T^p)$. Now, we can prove by induction that

$$\operatorname{Im}[(\lambda I-\widetilde{T})^n]=\operatorname{Im}[(\lambda I-T)^n]\cap\operatorname{Im}(T^p),\ \forall\,n\geq 1.\tag{5}$$

Finally, for every $0<|\lambda|<\varepsilon$, from (4) and (5), it follows that

$$\begin{aligned}\beta[(\lambda I-T)^n] &= \beta[(\lambda I-\widetilde{T})^n]\\ &= \alpha[(\lambda I-\widetilde{T})^n]-\operatorname{ind}[(\lambda I-\widetilde{T})^n]\\ &= n\alpha(\widetilde{T})-n\operatorname{ind}(\widetilde{T})\\ &= n\beta(\widetilde{T})=n\beta_p^1(T),\end{aligned}$$

and this completes the proof of Theorem 3.8. □

As a consequence, we have the following result.

Corollary 3.9. *Let* $T \in \varphi_\infty(\mathrm{X})$.

1) *If* $\boldsymbol{a}(T) < +\infty$ *and* $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a}(T)})$ *is closed, then for all* $j, n \in \mathbb{N}\backslash\{0\}$, *there exists* $\varepsilon > 0$ *such that for every* λ *with* $0 < |\lambda| < \varepsilon$, *the following assertions hold :*

   i) $\lambda I - T^j$ *is injective with closed range,*

   ii) if $\overline{\mathcal{D}(T)} = \mathrm{X}$, *we have* $\beta[(\lambda I - T)^n] \geq n\beta^1_{\boldsymbol{a}(T)}(T)$ *and equality holds when* $T \in \mathcal{A}(\mathrm{X})$.

2) $\sigma_{asc}(T)$ *and* $\sigma^e_{asc}(T)$ *are both closed. Moreover,* $\sigma_{asc}(T)\backslash\sigma^e_{asc}(T)$ *is an open set.*

Proof. The first assertion is clear and we can prove the assertion 2) similarly as in [1, Corollary 2.6]. □

For $T \in \varphi(\mathrm{X})$, we define

$$\sigma_{iso}(T) = \{\lambda \in \sigma(T) \,:\, \lambda \text{ an isolated point}\}$$

and

$$\mathrm{E}(T) = \sigma_{iso}(T) \cap \{\lambda \in \sigma(T) \,:\, \boldsymbol{a}(T_\lambda) = \boldsymbol{d}(T_\lambda) = m,\ \mathrm{Im}[(T_\lambda)^m] \text{ is closed}\}.$$

Let's recall that if $\varrho(T) \neq \emptyset$, (see, [9, Theorem 2.1])

$$\mathrm{E}(T) = \{\lambda \in \sigma(T) : \boldsymbol{a}(T_\lambda) < +\infty \text{ and } \boldsymbol{d}(T_\lambda) < +\infty\}.$$

Theorem 3.10. *Let* $T \in \varphi(\mathrm{X})$ *such that* $\overline{\mathcal{D}(T)} = \mathrm{X}$. *Then*

$$\varrho^e_{asc}(T) \cap \partial\sigma(T) = \varrho_{asc}(T) \cap \partial\sigma(T) = \mathrm{E}(T).$$

Proof. The case $\varrho(T) = \emptyset$ is trivial, so assume that $\varrho(T) \neq \emptyset$. Clearly, the following inclusions hold :

$$\mathrm{E}(T) \subseteq \varrho_{asc}(T) \cap \partial\sigma(T) \subseteq \varrho^e_{asc}(T) \cap \partial\sigma(T).$$

For the reverse inclusions, let $\lambda \in \varrho^e_{asc}(T) \cap \partial\sigma(T)$ and put $p = p_1(T_\lambda)$. We know from Theorem 3.8, that there exists $\varepsilon > 0$ such that

$$\alpha(\mu I - T_\lambda) = \alpha^1_p(T_\lambda),\ \ \beta(\mu I - T_\lambda) \geq \beta^1_p(T_\lambda), \quad \forall\, 0 < |\mu| < \varepsilon.$$

Since $\mathrm{B}(0,\,\varepsilon)\backslash\{0\} \cap \varrho(T_\lambda) \neq \emptyset$, $\alpha^1_p(T_\lambda) = \beta^1_p(T_\lambda) = 0$. This leads to $\boldsymbol{a}(T_\lambda) < +\infty$ and $\boldsymbol{d}(T_\lambda) < +\infty$. Since $\lambda \in \sigma(T)$, then $\lambda \in \mathrm{E}(T)$, and this completes the proof of Theorem 3.10. □

This result represents an improvements of [1, Theorem 2.7] to closed unbounded operators.

As an immediate consequence of Theorem 3.10 we have the following result.

Corollary 3.11. *Let* $T \in \varphi(\mathrm{X})$. *Then the following assertions are equivalent :*

1) $\sigma_{asc}(T) = \emptyset$;

2) $\sigma^e_{asc}(T) = \emptyset$;

3) $\partial\sigma(T) \subseteq \varrho_{asc}(T)$;

4) $\partial\sigma(T) \subseteq \varrho^{e}_{asc}(T)$;

5) $\partial\sigma(T) = \mathrm{E}(T)$ *and in this case if* $\varrho(T) \neq \emptyset$, *then* $\sigma(T) = \mathrm{E}(T)$.

Example 3.12.
Let $\mathrm{X} = L^2([0, 1])$, we define the second-order differential operator T by

$$Tu = u'' \text{ and } \mathcal{D}(T) = \{u \in \mathrm{X}_2 \, : \, u'(0) + u'(1) = 0, \; u(0) = 0\},$$

where X_2 denotes the subspace of X consisting of all functions $u \in C^1([0, 1])$ with u' absolutely continuous on $[0, 1]$ and $u'' \in \mathrm{X}$. In [10, page 30], it is proved that $\sigma(T) = \{\lambda_i\}_{i=1}^{\infty}$, where $\lambda_i = (2i-1)^2\pi^2$, and $\boldsymbol{a}(T_{\lambda_i}) = \boldsymbol{d}(T_{\lambda_i}) = 2$, for $i = 1, 2, \cdots$. Then $\sigma(T) = \mathrm{E}(T)$, and hence $\sigma_{asc}(T) = \sigma^{e}_{asc}(T) = \emptyset$ according to Corollary 3.11.

4 A spectral mapping theorem for essential ascent spectrum

We start this section with the following lemma.

Proposition 4.1. *Let* $T \in \mathcal{B}(\mathrm{X})$, *then the two following assertions are equivalent :*

1) $\boldsymbol{a}(T) < +\infty$ *and* $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a}(T)})$ *is closed,*

2) $\boldsymbol{a}(T) < +\infty$ *and* $\mathrm{Im}(T^{\boldsymbol{a}(T)+1})$ *is closed.*

Proof. The implication "1) $\Longrightarrow$ 2)" is a direct consequence of Remark 3.6 and "2) $\Longrightarrow$ 1)" is trivial, because $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a}(T)}) = T^{-\boldsymbol{a}(T)}[\mathrm{Im}(T^{\boldsymbol{a}(T)+1})]$ is closed. □

For $T \in \mathcal{B}(\mathrm{X})$ and $n \in \mathbb{N}$, we define T_n as the restriction of T to $\mathrm{Im}(T^n)$ viewed as a map from $\mathrm{Im}(T^n)$ into $\mathrm{Im}(T^n)$, in particular $T_0 = T$. If for some integer n the range space $\mathrm{Im}(T^n)$ is closed and T_n is semi-Fredholm operator, then T is called semi-B-Fredholm operator (see [2, 3] for more details).

Proposition 4.2.
Let $T \in \varphi_\infty(\mathrm{X})$. *Then the following assertions are equivalent :*

1) $\mathrm{Im}(T) + \ker(T^n)$ *is closed, for some integer* $n \geq \boldsymbol{a_e}(T)$,

2) *there exists* $k \in \mathbb{N}$ *such that* $\widetilde{T_k} \in \Phi_+(\mathrm{X}/\ker(T^k))$ *and* $\widetilde{T_k}$ *is s-regular,*

3) *there exists* $k \in \mathbb{N}$ *such that* $\widetilde{T_k} \in \Phi_+(\mathrm{X}/\ker(T^k))$.

If additionally $T \in \mathcal{B}(\mathrm{X})$, *then all these assertions are equivalent to :*

4) $\boldsymbol{a_e}(T) < +\infty$ *and* $\mathrm{Im}(T^{\boldsymbol{a_e}(T)+1})$ *is closed,*

5) T *is upper semi-B-Fredholm.*

Proof. "1) $\Longrightarrow$ 2) $\Longrightarrow$ 3)" see the proof of Lemma 3.4.

"3) $\Longrightarrow$ 1)" is clear.

"1) $\Longrightarrow$ 4)" this is an immediate consequence of Remark 3.6.

"4) $\Longrightarrow$ 1)" is clear, because $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)}) = T^{-\boldsymbol{a_e}(T)}[\mathrm{Im}(T^{\boldsymbol{a_e}(T)+1})]$ is closed.

"1) $\Longrightarrow$ 5)" follows from Remark 3.6.

"5) $\Longrightarrow$ 1)" it suffices to remark that $\mathrm{Im}(T^{n+1})$ is closed for some $n \geq \boldsymbol{a_e}(T)$ and consequently $\mathrm{Im}(T)+\ker(T^n) = T^{-n}[\mathrm{Im}(T^{n+1})]$ is closed. □

Corollary 4.3. *Let $T \in \mathcal{B}(\mathrm{X})$ and let f be an analytic function on an open neighborhood of $\sigma(T)$. If f is not identically constant in any connected component of its domain, then*

$$f(\sigma^e_{asc}(T)) = \sigma^e_{asc}(f(T)) \ \ \textit{and} \ \ f(\sigma_{asc}(T)) = \sigma_{asc}(f(T)).$$

Proof. Using Proposition 4.1 together with Proposition 4.2, we infer

$$\varrho_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a}(T_\lambda) < +\infty \ \text{ and } \ \mathrm{Im}[(T_\lambda)^{\boldsymbol{a}(T_\lambda)+1}] \ \text{ is closed}\},$$

$$\varrho^e_{asc}(T) = \{\lambda \in \mathbb{C} : \boldsymbol{a_e}(T_\lambda) < +\infty \ \text{ and } \ \mathrm{Im}[(T_\lambda)^{\boldsymbol{a_e}(T_\lambda)+1}] \ \text{ is closed}\}.$$

The result now follows from [12, page 202]. □

For $T : \mathcal{D}(T) \subseteq \mathrm{X} \longrightarrow \mathrm{X}$, we denote by

$$\mathrm{do}(T) = \inf\{n \in \mathbb{N} : \mathcal{D}(T^n) = \mathcal{D}(T^{n+1})\},$$

where as usual the infimum over the empty set is taken to be ∞. Let's remark that if $\mathrm{do}(T) < +\infty$, then

$$\mathcal{D}(T^{\mathrm{do}(T)}) = \mathcal{D}(T^{\mathrm{do}(T)+n}) \subseteq \mathcal{D}(T^n), \quad \forall\ n \in \mathbb{N}.$$

Assume that T is paracomplete operator such that $q = \mathrm{do}(T)$ and $\overline{\mathcal{D}(T^q)} = \mathcal{D}(T^q)$. It is clear that if P is a complex polynomial, then $P(T)$ is paracomplete and $j = \mathrm{do}(P(T)) \leq q$. Furthermore, if P is a non-constant polynomial, then $\mathcal{D}([P(T)]^j) = \mathcal{D}(T^q)$.

Let $[T] : \mathcal{D}(T^q) \longrightarrow \mathrm{X}$ to be the restriction of T to $\mathcal{D}(T^q)$. By [8, Proposition 2.1.4] and [8, Proposition 2.1.5], we deduce that in fact, $[T]$ is a bounded operator.

Define $\widetilde{T}$ to be the restriction of T to $\mathcal{D}(T^q)$ viewed as a map from $\mathcal{D}(T^q)$ into $\mathcal{D}(T^q)$. Since for all $x \in \mathcal{D}(T^q)$, we have $\|\widetilde{T}x\| = \|[T]x\| \leq \|[T]\|\,\|x\|$, then $\widetilde{T}$ is also a bounded operator. Assume now that P is a non-constant complex polynomial. Hence, taking into account that $\ker[P(T)] \subseteq \mathcal{D}([P(T)]^q) = \mathcal{D}(T^q)$, we conclude that $\ker[P(T)] = \ker[P(\widetilde{T})]$ is closed. Also, we remark that $\mathrm{Im}([P(T)]^n) \subseteq \mathcal{D}(T^q)$, for all $n \geq q$. Indeed, let $y \in \mathrm{Im}([P(T)]^n)$, then there exists $x \in \mathcal{D}([P(T)]^n) = \mathcal{D}(T^q) = \mathcal{D}([P(T)]^{n+q})$ such that $y = [P(T)]^n x$. This leads to $y \in \mathcal{D}(T^q)$.

Define

$$\Gamma(\mathrm{X}) = \{T \in \varphi(\mathrm{X}) : q = \mathrm{do}(T) < +\infty, \mathcal{D}(T^q) \text{ and } \mathrm{Im}(T_\lambda)+\mathcal{D}(T^q) \text{ are both closed}, \forall\ \lambda \in \mathbb{C}\}.$$

Let us show that if $T \in \Gamma(\mathrm{X})$, then $\mathrm{Im}[P(T)] + \mathcal{D}([P(T)]^{\mathrm{do}(P(T))})$ is closed, for every non-constant complex polynomial. Put $q = \mathrm{do}(T)$ and define

$$\begin{array}{rccc} \overline{T}\ : & \mathcal{D}(\overline{T}) \subseteq \mathrm{X}/\mathcal{D}(T^q) & \longrightarrow & \mathrm{X}/\mathcal{D}(T^q) \\ & \overline{x} & \longmapsto & \overline{Tx}. \end{array}$$

Let $\lambda \in \mathbb{C}$ and $\overline{x} \in \ker(\lambda I - \overline{T})$, then $T_\lambda x \in \mathcal{D}(T^q)$. Hence, $x \in \mathcal{D}(T^{q+1}) = \mathcal{D}(T^q)$, which implies that $\overline{x} = 0$. Therefore $\ker(\lambda I - \overline{T}) = \{0\}$.

On the other hand, it is clear that $\mathrm{Im}(\lambda I - \overline{T}) = [\mathrm{Im}(T_\lambda) + \mathcal{D}(T^q)]/\mathcal{D}(T^q)$ is closed. As in the proof of Lemma 3.4, we see that $\lambda I - \overline{T}$ is paracomplete. Hence, applying [8, Proposition 2.2.3], we get $\lambda I - \overline{T} \in \varphi(\mathrm{X}/\mathcal{D}(T^q))$. This leads to $\lambda I - \overline{T} \in \Phi_+(\mathrm{X}/\mathcal{D}(T^q))$. Furthermore, let $P(Z) = (\lambda_1 - Z)^{\alpha_1}(\lambda_2 - Z)^{\alpha_2} \cdots (\lambda_m - Z)^{\alpha_m}$ be a non-constant complex polynomial. Recall that, if $S, L \in \varphi(\mathrm{X})$ such that $L \in \Phi_+(\mathrm{X})$ and $\mathrm{Im}(S)$ is closed, then $LS \in \varphi(\mathrm{X})$ and $\mathrm{Im}(LS)$ is closed. Since, for every $i, j \in \{1, 2, \cdots, m\}$, $\lambda_i I - \overline{T} \in \Phi_+(\mathrm{X}/\mathcal{D}(T^q))$ and $\mathrm{Im}(\lambda_j I - \overline{T})$ is closed, we deduce that $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \varphi(\mathrm{X}/\mathcal{D}(T^q))$ and $\mathrm{Im}[(\lambda_i I - \overline{T})(\lambda_j I - \overline{T})]$ is closed. Consequently, $(\lambda_i I - \overline{T})(\lambda_j I - \overline{T}) \in \Phi_+(\mathrm{X}/\mathcal{D}(T^q))$ because $\ker[(\lambda_i I - \overline{T})(\lambda_j I - \overline{T})] = \{0\}$. Therefore $\mathrm{Im}(P(\overline{T})) = [\mathrm{Im}[P(T)] + \mathcal{D}(T^q)]/\mathcal{D}(T^q)$ is closed, and finally we obtain $\mathrm{Im}[P(T)] + \mathcal{D}([P(T)]^{\mathrm{do}(P(T))}) = \mathrm{Im}[P(T)] + \mathcal{D}(T^q)$ is closed.

The following lemma extends [12, Lemma 12.8] to the case of unbounded closed operators. For simplicity of notation, we denote by $\deg(P)$ the degree of the polynomial P.

Lemma 4.4. *Let $T \in \varphi(\mathrm{X})$ and let P and Q be two relatively prime polynomials. If $A = P(T)$ and $B = Q(T)$, then*

1) $\mathrm{Im}(A^n B^n) = \mathrm{Im}(A^n) \cap \mathrm{Im}(B^n)$, *for all* $n \in \mathbb{N}$,

2) $\ker(A^n B^n) = \ker(A^n) + \ker(B^n)$, *for all* $n \in \mathbb{N}$,

3) $\ker^\infty(A) \subseteq \mathrm{Im}^\infty(B)$ *and* $\ker^\infty(B) \subseteq \mathrm{Im}^\infty(A)$,

4) $\max\{\boldsymbol{a_e}(A), \boldsymbol{a_e}(B)\} \leq \boldsymbol{a_e}(AB) \leq \boldsymbol{a_e}(A) + \boldsymbol{a_e}(B)$ *and* $\boldsymbol{a}(AB) = \max\{\boldsymbol{a}(A), \boldsymbol{a}(B)\}$.

In addition, assume that $T \in \Gamma(\mathrm{X})$,

5) *if* $\max\{\boldsymbol{a_e}(A), \boldsymbol{a_e}(B)\} < +\infty$, *then* $\mathrm{Im}(A) + \ker(A^{\boldsymbol{a_e}(A)})$ *and* $\mathrm{Im}(B) + \ker(B^{\boldsymbol{a_e}(B)})$ *are both closed if and only if* $\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a_e}(AB)}]$ *is closed,*

6) *if* $\max\{\boldsymbol{a}(A), \boldsymbol{a}(B)\} < +\infty$ *then* $\mathrm{Im}(A) + \ker(A^{\boldsymbol{a}(A)})$ *and* $\mathrm{Im}(B) + \ker(B^{\boldsymbol{a}(B)})$ *are both closed if and only if* $\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a}(AB)}]$ *is closed.*

Proof. The proof is trivial when P or Q is a constant. Therefore, we may assume that P and Q are two non-constant polynomials. We note that the first assertion follows from [4, Lemma 2.2].

2) Let $n \in \mathbb{N}\backslash\{0\}$. Since P^n and Q^n are relatively prime, we know that there exist two polynomials P_n and Q_n such that $P^n P_n + Q^n Q_n = 1$. Let p_n (resp. q_n, k, m) be the degree of P_n (resp.

Q_n, P, Q). Then, we have $\alpha_1(n) = \deg(P^n P_n) = nk + p_n$ and $\alpha_2(n) = \deg(Q^n Q_n) = nm + q_n$. Put $\alpha(n) = \max\{\alpha_1(n), \alpha_2(n)\}$, we have

$$\begin{array}{rcl} \mathcal{D}[A^n P_n(T) + B^n Q_n(T)] & = & \mathcal{D}[A^n P_n(T)] \cap \mathcal{D}[B^n Q_n(T)], \\ & = & \mathcal{D}(T^{\alpha_1(n)}) \cap \mathcal{D}(T^{\alpha_2(n)}), \\ & = & \mathcal{D}(T^{\alpha(n)}). \end{array}$$

Hence,

$$A^n P_n(T)x + B^n Q_n(T)x = x, \quad \forall\, x \in \mathcal{D}(T^{\alpha(n)}). \tag{1}$$

First, we note that the inclusion $\ker(A^n) + \ker(B^n) \subseteq \ker(A^n B^n)$ is immediate. For the converse inclusion, let $x \in \ker(A^n B^n)$. Then from [4, Lemma 2.1], we know that $x \in \mathcal{D}(T^{\alpha(n)})$. Hence, taking into account of (1), we conclude that $x \in \ker(A^n) + \ker(B^n)$.

3) We observe from [4, Lemma 2.2], that $\ker(A^n) = B^n(\ker(A^n)) \subseteq \mathrm{Im}(B^n)$, for all $n \in \mathbb{N}$. If $m \geq n$, then $\ker(A^n) \subseteq \ker(A^m) \subseteq \mathrm{Im}(B^m)$, which implies that $\ker(A^n) \subseteq \mathrm{Im}^\infty(B)$. Consequently $\ker^\infty(A) \subseteq \mathrm{Im}^\infty(B)$. The reverse inclusion follows by interchanging A and B.

4) Put $n \in \mathbb{N}$. Let M and N be two subspaces of X such that

$$\ker(A^{n+1}) = \ker(A^n) \dotplus \mathrm{N} \ \text{ and } \ \ker(B^{n+1}) = \ker(B^n) \dotplus \mathrm{M}.$$

Then, we have

$$\begin{array}{rcl} \alpha_n^1(AB) & = & \dim\ker(A^{n+1}B^{n+1})/\ker(A^n B^n), \\ & = & \dim[\ker(A^{n+1}) + \ker(B^{n+1})]/[\ker(A^n) + \ker(B^n)], \\ & = & \dim[\ker(A^n) + \ker(B^n) + \mathrm{M} + \mathrm{N}]/[\ker(A^n) + \ker(B^n)], \\ & \leq & \dim \mathrm{M} + \dim \mathrm{N}, \\ & \leq & \alpha_n^1(A) + \alpha_n^1(B). \end{array}$$

Assume that $\alpha_n^1(AB) < +\infty$, and write $j = \alpha_n^1(AB) + 1$. If $\boldsymbol{a}(A) \leq n$. Then $\alpha_n^1(A) = 0$ and there is nothing to prove. Thus we may assume that $n < \boldsymbol{a}(A)$. Let $x_1, x_2, \cdots, x_j \in \ker(A^{n+1}) \backslash \ker(A^n)$. Since $\ker(A^{n+1}) \subseteq \mathrm{Im}(B^{n+1})$, for every $1 \leq i \leq j$, there exists $y_i \in \mathcal{D}(B^{n+1})$, such that $x_i = B^{n+1}y_i$. Therefore, for every $1 \leq i \leq j$, $y_i \in \ker(A^{n+1}B^{n+1})$. But, since $A^n B^{n+1} y_i \neq 0$, $A^n B^n y_i \neq 0$ and consequently, for every $1 \leq i \leq j$, $y_i \in \ker(A^{n+1}B^{n+1}) \backslash \ker(A^n B^n)$, Hence, for every $1 \leq i \leq j$, there exists $\alpha_i \in \mathbb{C}$, such that $\alpha_k \neq 0$ and $\sum_{i=1}^{j} \alpha_i y_i = 0$, with $k \in \{1, 2, \cdots, j\}$.

Now, taking into account that

$$\sum_{i=1}^{j} \alpha_i x_i = \sum_{i=1}^{j} \alpha_i B^{n+1} y_i = B^{n+1}\Big(\sum_{i=1}^{j} \alpha_i y_i\Big) = 0,$$

one concludes that $\alpha_n^1(A) \leq \alpha_n^1(BA)$. Therefore

$$\max\{\alpha_n^1(A), \alpha_n^1(B)\} \leq \alpha_n^1(AB) \leq \alpha_n^1(A) + \alpha_n^1(B), \quad \forall\, n \in \mathbb{N}.$$

5) If $n \geq \mathrm{do}(T) = q$, then $q \geq \alpha(n)$. First, by (1), it is clear that

$$\mathcal{D}(T^q) \subseteq \mathcal{D}(T^{\alpha(n)}) \subseteq \mathrm{Im}(A^n) + \mathrm{Im}(B^n).$$

Since $\mathrm{Im}(A^n) \subseteq \mathcal{D}(T^q)$ and $\mathrm{Im}(B^n) \subseteq \mathcal{D}(T^q)$, for every $j \in \mathbb{N}$, it follows that

$$\mathcal{D}(T^q) = \mathrm{Im}(A^n) + \mathrm{Im}(B^n) = [\ker(A^j) + \mathrm{Im}(A^n)] + [\mathrm{Im}(B^n) + \ker(B^j)]. \tag{2}$$

Now using [5, Lemma 2.1], we infer

$$\begin{aligned} \ker(A^jB^j) + \mathrm{Im}(A^nB^n) &= \ker(A^j) + \ker(B^j) + \mathrm{Im}(A^n) \cap \mathrm{Im}(B^n), \\ &= [\ker(A^j) + \mathrm{Im}(A^n)] \cap \mathrm{Im}(B^n) + \ker(B^j), \\ &= [\ker(A^j) + \mathrm{Im}(A^n)] \cap [\mathrm{Im}(B^n) + \ker(B^j)]. \end{aligned} \tag{3}$$

Thus, taking into account of Lemma 3.2, [8, Proposition 2.1.1, Proposition 2.1.2, page 183], equality (2) and equalities (3), we deduce that $\ker(A^j) + \mathrm{Im}(A^n)$ and $\ker(B^j) + \mathrm{Im}(B^n)$ are both closed if and only if $\ker(A^jB^j) + \mathrm{Im}(A^nB^n)$ is closed.

On the other hand, suppose that $j \geq \boldsymbol{a_e}(A) + \boldsymbol{a_e}(B)$ and $n > \mathrm{do}(T)$. Define the following maps :

$$\begin{array}{rccl} \theta: & \mathcal{D}(\theta) \subseteq \mathrm{X}/\ker(A^jB^j) & \longrightarrow & \mathrm{X}/\ker(A^jB^j) \\ & \overline{x} & \longmapsto & \overline{ABx}, \end{array}$$

$$\begin{array}{rccl} \phi: & \mathcal{D}(\phi) \subseteq \mathrm{X}/\ker(B^j) & \longrightarrow & \mathrm{X}/\ker(B^j) \\ & \overline{x} & \longmapsto & \overline{Bx}, \end{array}$$

$$\begin{array}{rccl} \psi: & \mathcal{D}(\psi) \subseteq \mathrm{X}/\ker(A^j) & \longrightarrow & \mathrm{X}/\ker(A^j) \\ & \overline{x} & \longmapsto & \overline{Ax}, \end{array}$$

$$\begin{array}{rccl} \pi: & \mathrm{X} \times \mathrm{X} & \longrightarrow & (\mathrm{X}/\ker(A^jB^j)) \times (\mathrm{X}/\ker(A^jB^j)) \\ & (x, y) & \longmapsto & (\overline{x}, \overline{y}). \end{array}$$

Let $\mathrm{G}(\theta)$ denote the graph of θ. Since $\mathrm{G}(\theta) = \pi(\mathrm{G}(AB))$, from [8, Proposition 2.1.3] and [8, Proposition 2.1.4], we deduce that $\mathrm{G}(\theta)$ is paracomplete, which prove that θ is paracomplete. Now, if $\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed, by Lemma 3.3, $\mathrm{Im}(AB) + \ker(A^jB^j)$ is closed. Thus, $\mathrm{Im}(\theta)$ is closed. Since $\alpha(\theta) < +\infty$, by [8, Proposition 2.2.3], θ is closed and consequently, $\theta \in \Phi_+(\mathrm{X}/\ker(A^jB^j))$. This shows that $\theta^n \in \Phi_+(\mathrm{X}/\ker(A^jB^j))$, and hence $\mathrm{Im}(A^nB^n) + \ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed according to Lemma 3.3.

Assume now that $\mathrm{N} = \mathrm{Im}(A^nB^n) + \ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed. Since $n > \mathrm{do}(T)$, then $\mathcal{D}(B^{n-1}A^{n-1}) = \mathcal{D}(T^q)$ and $\mathrm{Im}(B^{n-1}A^{n-1}) \subseteq \mathcal{D}(T^q)$. On the other hand, it is clear that

$$\begin{array}{rccl} \widehat{BA}: & \mathcal{D}(T^q) & \longrightarrow & \mathcal{D}(T^q) \\ & x & \longmapsto & B^{n-1}A^{n-1}x, \end{array}$$

is well-defined and $\widehat{BA} = [P(\widetilde{T})Q(\widetilde{T})]^{(n-1)}$ is a bounded operator. We also remark that

$$\begin{aligned} (\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a_e}(AB)+n-1}]) \cap \mathcal{D}(T^q) &= A^{-(n-1)}B^{-(n-1)}(\mathrm{N}) \\ &= \{x \in \mathcal{D}(T^q) : A^{n-1}B^{n-1}x \in \mathrm{N}\} \\ &= \{x \in \mathcal{D}(T^q) : \widehat{AB}x \in \mathrm{N} \subseteq \mathcal{D}(T^q)\} \\ &= \widehat{AB}^{-1}(\mathrm{N}) \text{ is closed.} \end{aligned}$$

Since $(\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a_e}(AB)+n-1}]) + \mathcal{D}(T^q) = \mathrm{Im}(AB) + \mathcal{D}[(AB)^{\mathrm{do}(AB)}]$ is closed, from Lemma 3.2 and [8, Proposition 2.1.1], it follows that $\mathrm{Im}(AB) + \ker[(AB)^{\boldsymbol{a_e}(AB)+n-1}]$ is closed.

Thus, by Lemma 3.3, $\mathrm{Im}(AB)+\ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed. In the same way, we obtain the following equivalences :

$$\begin{array}{lcl} \mathrm{Im}(A)+\ker(A^{\boldsymbol{a_e}(A)}) \text{ is closed} & \Longleftrightarrow & \mathrm{Im}(A^n)+\ker(A^{\boldsymbol{a_e}(A)}) \text{ is closed,} \\ \mathrm{Im}(B)+\ker(B^{\boldsymbol{a_e}(B)}) \text{ is closed} & \Longleftrightarrow & \mathrm{Im}(B^n)+\ker(B^{\boldsymbol{a_e}(A)}) \text{ is closed.} \end{array}$$

Consequently, $\mathrm{Im}(A)+\ker(A^{\boldsymbol{a_e}(A)})$ and $\mathrm{Im}(B)+\ker(B^{\boldsymbol{a_e}(B)})$ are both closed if and only if $\mathrm{Im}(AB)+\ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed.

6) Finally, note, by Lemma 3.3 and assertion 5), we can see that the following facts are equivalent :

(i) $\mathrm{Im}(A)+\ker(A^{\boldsymbol{a}(A)})$ is closed and $\mathrm{Im}(B)+\ker(B^{\boldsymbol{a}(B)})$ is closed,

(ii) $\mathrm{Im}(A)+\ker(A^{\boldsymbol{a_e}(A)})$ is closed and $\mathrm{Im}(B)+\ker(B^{\boldsymbol{a_e}(B)})$ is closed,

(iii) $\mathrm{Im}(AB)+\ker[(AB)^{\boldsymbol{a_e}(AB)}]$ is closed,

(iv) $\mathrm{Im}(AB)+\ker[(AB)^{\boldsymbol{a}(AB)}]$ is closed,

and the proof of the lemma is complete. □

Lemma 4.5. *Let* $T:\mathcal{D}(T)\subseteq \mathrm{X}\longrightarrow \mathrm{X}$ *be a paracomplete operator such that* $\mathrm{do}(T)<+\infty$ *and* $\overline{\mathcal{D}(T^{\mathrm{do}(T)})}=\mathcal{D}(T^{\mathrm{do}(T)})$. *If* $\boldsymbol{a_e}(T)<+\infty$ *and* $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ *is closed, then* $T\in\varphi_\infty(\mathrm{X})$.

Proof. Let $n\in\mathbb{N}$ and put $k\geq \boldsymbol{a_e}(T)$. First, from the proof of Lemma 3.4, we know that $\widetilde{T_k}\in\Phi_+(\mathrm{X}/\ker(T^k))$. Thus, $\widetilde{T_k}^n\in\Phi_+(\mathrm{X}/\ker(T^k))$. Let $\lambda_n\in\mathbb{C}\backslash\{0\}$ such that $\lambda_n I-\widetilde{T_k}^n$ is upper semi-Fredholm. Then $\lambda_n I-T^n$ is also upper semi-Fredholm, and hence $T^n\in\varphi(\mathrm{X})$. This completes the proof. □

Lemma 4.6. *Let* $T\in\Gamma(\mathrm{X})$ *and let* $m\in\mathbb{N}\backslash\{0\}$. *Then*

$$0\in\varrho^e_{asc}(T)\Longleftrightarrow 0\in\varrho^e_{asc}(T^m)$$

and

$$0\in\varrho_{asc}(T)\Longleftrightarrow 0\in\varrho_{asc}(T^m).$$

Proof. First, by Lemma 2.1, $\boldsymbol{a_e}(T)<+\infty$ if and only if $\boldsymbol{a_e}(T^m)<+\infty$. Let $k\geq\max\{\boldsymbol{a_e}(T),\boldsymbol{a_e}(T^m)\}$. Thus, by Lemma 3.3,

$$\begin{array}{lcl} \mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)}) \text{ is closed} & \Longrightarrow & \mathrm{Im}(T)+\ker(T^k) \text{ is closed,} \\ & \Longrightarrow & \widetilde{T_k}\in\Phi_+(\mathrm{X}/\ker(T^k)), \\ & \Longrightarrow & \widetilde{T_k}^m\in\Phi_+(\mathrm{X}/\ker(T^k)), \\ & \Longrightarrow & \mathrm{Im}(T^m)+\ker(T^k) \text{ is closed,} \\ & \Longrightarrow & \mathrm{Im}(T^m)+\ker(T^{mk}) \text{ is closed,} \\ & \Longrightarrow & \mathrm{Im}(T^m)+\ker(T^{m\boldsymbol{a_e}(T^m)}) \text{ is closed.} \end{array}$$

Now, assume that $\mathrm{Im}(T^m)+\ker(T^{m\boldsymbol{a_e}(T^m)})$ is closed and let $n>\max\{\boldsymbol{a_e}(T),\boldsymbol{a_e}(T^m),\mathrm{do}(T)\}$. If we put $A=T^m$, then A is a paracomplete operator with $\boldsymbol{a_e}(A)<+\infty$ and $\mathrm{Im}(A)+\ker(A^{\boldsymbol{a_e}(A)})$ is closed. Thus, $\widetilde{A_n}$ is upper semi-Fredholm. In particular, we deduce that $\mathrm{Im}(\widetilde{A_n}^n)$ is closed.

Furthermore, let $\mathrm{Z} = \mathrm{Im}(T^{nm}) + \ker(T^{nm})$, then $\mathrm{Z} \subseteq \mathcal{D}(T^q)$ and Z is closed. It now follows from $\mathcal{D}(T^{mn-1}) = \mathcal{D}(T^q)$, that

$$\begin{aligned}[\mathrm{Im}(T) + \ker(T^{2mn-1})] \cap \mathcal{D}(T^q) &= T^{-(mn-1)}(\mathrm{Z}) \\ &= \{x \in \mathcal{D}(T^q) : T^{mn-1}x \in \mathrm{Z}\} \\ &= \{x \in \mathcal{D}(T^q) : \widetilde{T}^{mn-1}x \in \mathrm{Z} \subseteq \mathcal{D}(T^q)\} \\ &= \widetilde{T}^{-(mn-1)}(\mathrm{Z}) \text{ is closed.}\end{aligned}$$

Since $[\mathrm{Im}(T) + \ker(T^{2mn-1})] + \mathcal{D}(T^q) = \mathrm{Im}(T) + \mathcal{D}(T^q)$ is closed, by Lemma 3.2 and [8, Proposition 2.1.1], $\mathrm{Im}(T) + \ker(T^{2mn-1})$ is closed. Thus, from Lemma 3.3, $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a}_e(T)})$ is closed and by Lemma 4.5, we see that

$$0 \in \varrho^e_{asc}(T) \Longleftrightarrow 0 \in \varrho^e_{asc}(T^m).$$

On the other hand, since for every $m \in \mathbb{N}\backslash\{0\}$, $\boldsymbol{a}(T) < +\infty$ if and only if $\boldsymbol{a}(T^m) < +\infty$, we deduce

$$\begin{aligned}\mathrm{Im}(T) + \ker(T^{\boldsymbol{a}(T)}) \text{ is closed} &\Longleftrightarrow \mathrm{Im}(T) + \ker(T^{\boldsymbol{a}_e(T)}) \text{ is closed,} \\ &\Longleftrightarrow \mathrm{Im}(T^m) + \ker(T^{m\boldsymbol{a}_e(T^m)}) \text{ is closed,} \\ &\Longleftrightarrow \mathrm{Im}(T^m) + \ker(T^{m\boldsymbol{a}(T^m)}) \text{ is closed.}\end{aligned}$$

Finally, using Lemma 4.5, we obtain

$$0 \in \varrho_{asc}(T) \Longleftrightarrow 0 \in \varrho_{asc}(T^m).$$

This completes the proof. □

Using Lemma 4.4 and Lemma 4.5, one proves the following result.

Theorem 4.7. *Let $T \in \Gamma(\mathrm{X})$. If A and B are defined as in Lemma* 4.4, *then*

$$0 \in \varrho^e_{asc}(AB) \Longleftrightarrow 0 \in \varrho^e_{asc}(A) \cap \varrho^e_{asc}(B)$$

and

$$0 \in \varrho_{asc}(AB) \Longleftrightarrow 0 \in \varrho_{asc}(A) \cap \varrho_{asc}(B).$$

As a consequence, we have the following result.

Corollary 4.8. *Let $T \in \Gamma(\mathrm{X})$ and let $P(Z) = (\lambda_1 - Z)^{m_1}(\lambda_2 - Z)^{m_2} \cdots (\lambda_n - Z)^{m_n}$ be a complex polynomial such that $m_i \neq 0$ for all $i = 1, 2, \cdots, n$. Then*

$$0 \in \varrho^e_{asc}(P(T)) \Longleftrightarrow \lambda_i \in \varrho^e_{asc}(T), \quad \forall\, 1 \leq i \leq n$$

and

$$0 \in \varrho_{asc}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{asc}(T), \quad \forall\, 1 \leq i \leq n.$$

Proof. By Lemma 4.6 and Theorem 4.7, we obtain

$$\begin{aligned}0 \in \varrho^e_{asc}(P(T)) &\Longleftrightarrow 0 \in \bigcap_{1\leq i\leq n} \varrho^e_{asc}[(\lambda_i I - T)^{m_i}], \\ &\Longleftrightarrow 0 \in \bigcap_{1\leq i\leq n} \varrho^e_{asc}(\lambda_i I - T), \\ &\Longleftrightarrow \lambda_i \in \varrho^e_{asc}(T), \quad \forall\, 1 \leq i \leq n.\end{aligned}$$

In the same way, we see

$$0 \in \varrho_{asc}(P(T)) \Longleftrightarrow \lambda_i \in \varrho_{asc}(T), \quad \forall\, 1 \le i \le n.$$

This completes the proof. □

Theorem 4.9. *Let $T \in \Gamma(\mathrm{X})$ and let P be a non-constant complex polynomial. Then*

$$P(\sigma^e_{asc}(T)) = \sigma^e_{asc}(P(T))$$

and

$$P(\sigma_{asc}(T)) = \sigma_{asc}(P(T)).$$

Proof. By Theorem 4.7 and Corollary 4.8, we see that

$$\begin{aligned} \lambda \in P(\sigma^e_{asc}(T)) &\Longleftrightarrow \lambda = P(\mu), \text{ where } \mu \in \sigma^e_{asc}(T), \\ &\Longleftrightarrow \lambda - P(Z) = (\mu - Z)^k Q(Z), \text{ where } Q(\mu) \neq 0, \\ &\Longleftrightarrow \lambda \in \sigma^e_{asc}(P(T)). \end{aligned}$$

In the same way, we prove that $P(\sigma_{asc}(T)) = \sigma_{asc}(P(T))$. This completes the proof. □

5 Decomposition Theorems

We start this section with the following definition.

Definition 5.1. For $T \in \varphi(\mathrm{X})$ such that $n = \boldsymbol{a_e}(T) < +\infty$ and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is closed, the a-essential index, $\mathrm{ind}_{\boldsymbol{a_e}}(T)$, is defined by

$$\mathrm{ind}_{\boldsymbol{a_e}}(T) = \alpha^1_n(T) - \beta^1_n(T) \in \mathbb{Z} \cup \{-\infty\}.$$

Let $T \in \varphi_\infty(\mathrm{X})$ such that $n_0 = \boldsymbol{a_e}(T) < +\infty$ and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is closed. From Remark 3.6, we know that $\mathrm{Im}(T^n)$ is closed for every $n \ge n_0$. Furthermore, if T_n is the restriction of T to $\mathrm{Im}(T^n)$ viewed as a map from $\mathrm{Im}(T^n)$ into $\mathrm{Im}(T^n)$, then T_n is upper semi-Fredholm for all $n \ge n_0$ and $\mathrm{ind}_{\boldsymbol{a_e}}(T) = \mathrm{ind}(T_{n_0})$.

Assume that $T \in \mathcal{A}(\mathrm{X})$, by Lemma 2.2, $S^1_{n_0}(T) = \alpha^1_{n_0}(T) - \alpha^1_{n_0+1}(T)$. Thus from Lemma 2.5, we obtain

$$\beta^1_{n_0}(T) = S^1_{n_0}(T) + \beta^1_{n_0+1}(T) = \alpha^1_{n_0}(T) - \alpha^1_{n_0+1}(T) + \beta^1_{n_0+1}(T).$$

This implies that

$$\mathrm{ind}_{\boldsymbol{a_e}}(T) = \mathrm{ind}(T_{n_0}) = \alpha^1_{n_0}(T) - \beta^1_{n_0}(T) = \alpha^1_{n_0+1}(T) - \beta^1_{n_0+1}(T) = \mathrm{ind}(T_{n_0+1}).$$

Therefore

$$\mathrm{ind}_{\boldsymbol{a_e}}(T) = \mathrm{ind}(T_n), \quad \forall\ n \ge n_0.$$

Lemma 5.2. *Let $T \in \varphi(\mathrm{X})$ and let $n, k \in \mathbb{N}$. Then $\mathrm{Im}(T^k) + \ker(T^n)$ is topological complemented in X if and only if $\mathrm{Im}(T^k) + \ker(T^n)$ is paracomplete complemented in X.*

Proof. The direct implication is obvious. Let us prove the converse implication. First, by Lemma 3.2 and [8, Proposition 2.2], we have $\mathrm{Im}(T^k)+\ker(T^n)$ is a paracomplete subspace of X. Let M be a paracomplete complement of $\mathrm{Im}(T^k)+\ker(T^n)$ in X, thus

$$\mathrm{X} = [\mathrm{Im}(T^k)+\ker(T^n)] \dot{+} \mathrm{M}.$$

Now from [8, Proposition 2.1.1], we conclude that $\mathrm{Im}(T^k)+\ker(T^n)$ and M are both closed in X. This completes the proof. □

Lemma 5.3. *Let $T \in \varphi(\mathrm{X})$ such that $\boldsymbol{a_e}(T) < +\infty$ and let $k \in \mathbb{N}$. Then the following assertions are equivalent :*

1) $\mathrm{Im}(T^k)+\ker(T^n)$ *is topological complemented in* X *for some* $n \geq \boldsymbol{a_e}(T)$,

2) $\mathrm{Im}(T^k)+\ker(T^n)$ *is topological complemented in* X *for all* $n \geq \boldsymbol{a_e}(T)$.

Proof. Only the implication "1) $\Longrightarrow$ 2)" requires a proof. We put $n_0 \geq \boldsymbol{a_e}(T)$ and let M be a topological complement of $\mathrm{Im}(T^k)+\ker(T^{n_0})$ in X. Thus,

$$\mathrm{X} = [\mathrm{Im}(T^k)+\ker(T^{n_0})] \oplus \mathrm{M}. \tag{1}$$

First, we prove that $\mathrm{Im}(T^k)+\ker(T^{n_0+1})$ is topological complemented in X. From

$$[\mathrm{Im}(T^k)+\ker(T^{n_0+1})]/[\mathrm{Im}(T^k)+\ker(T^{n_0})] \approx \ker(T^{n_0+1})/[\mathrm{Im}(T^k)+\ker(T^{n_0})]\cap\ker(T^{n_0+1})$$

and

$$\ker(T^{n_0}) \subseteq [\mathrm{Im}(T^k)+\ker(T^{n_0})]\cap\ker(T^{n_0+1}) \subseteq \ker(T^{n_0+1}),$$

it follows that

$$\dim[\mathrm{Im}(T^k)+\ker(T^{n_0+1})]/[\mathrm{Im}(T^k)+\ker(T^{n_0})] \leq \alpha^1_{n_0}(T) < +\infty.$$

Hence, there exists a finite dimensional subspace Z of X such that

$$\mathrm{Im}(T^k)+\ker(T^{n_0+1}) = [\mathrm{Im}(T^k)+\ker(T^{n_0})] \oplus \mathrm{Z}.$$

Let P be the projection of X onto M associated with the decomposition (1). Thus

$$\mathrm{Z} \subseteq (I-P)(\mathrm{Z}) \oplus P(\mathrm{Z}),\ \ P(\mathrm{Z}) \subseteq \mathrm{M}\ \text{ and }\ (I-P)(\mathrm{Z}) \subseteq \mathrm{Im}(T^k)+\ker(T^{n_0}).$$

We now write

$$\alpha = \dim\{[\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus P(\mathrm{Z})\}/[\mathrm{Im}(T^k)+\ker(T^{n_0})] = \dim P(\mathrm{Z}),$$

$$\beta = \dim\{[\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus P(\mathrm{Z})\}/\{[\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus \mathrm{Z}\},$$

$$\kappa = \dim\{[\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus \mathrm{Z}\}/[\mathrm{Im}(T^k)+\ker(T^{n_0})] = \dim \mathrm{Z}.$$

Then, clearly $\alpha = \beta + \kappa < +\infty$, which implies that $\beta = \alpha - \kappa = \dim P(\mathrm{Z}) - \dim \mathrm{Z} = 0$. Therefore

$$\mathrm{Im}(T^k)+\ker(T^{n_0+1}) = [\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus \mathrm{Z} = [\mathrm{Im}(T^k)+\ker(T^{n_0})]\oplus P(\mathrm{Z}).$$

On the other hand, let W be a closed subspace of M such that $\mathrm{M} = P(\mathrm{Z}) \oplus \mathrm{W}$. Clearly,

$$\begin{aligned} \mathrm{X} &= [\mathrm{Im}(T^k) + \ker(T^{n_0})] \oplus [P(\mathrm{Z}) \oplus \mathrm{W}] \\ &= [\mathrm{Im}(T^k) + \ker(T^{n_0+1})] \dotplus \mathrm{W}. \end{aligned}$$

Hence, by Lemma 5.2, we obtain $\mathrm{Im}(T^k) + \ker(T^{n_0+1})$ and W are both closed. If $n_0 = \boldsymbol{a_e}(T)$, the proof is complete by induction.

To finish the lemma, it is enough to prove that $\mathrm{Im}(T^k) + \ker(T^{n_0-1})$ is topological complemented in X, when $n_0 > \boldsymbol{a_e}(T)$. Since

$$\dim[\mathrm{Im}(T^k) + \ker(T^{n_0})]/[\mathrm{Im}(T^k) + \ker(T^{n_0-1})] \le \alpha^1_{n_0-1}(T) < +\infty,$$

there exists a finite dimensional subspace $\mathrm{N} \subseteq \mathrm{X}$ such that

$$\mathrm{Im}(T^k) + \ker(T^{n_0}) = [\mathrm{Im}(T^k) + \ker(T^{n_0-1})] \dotplus \mathrm{N},$$

and hence

$$\mathrm{X} = [\mathrm{Im}(T^k) + \ker(T^{n_0})] \oplus \mathrm{M} = [\mathrm{Im}(T^k) + \ker(T^{n_0-1})] \dotplus [\mathrm{N} + \mathrm{M}].$$

However, because M is closed and $\dim \mathrm{N} < +\infty$, $\mathrm{N} + \mathrm{M}$ is closed. Finally, by Lemma 5.2, the subspace $\mathrm{Im}(T^k) + \ker(T^{n_0-1})$ is topological complemented in X. This completes the proof. □

Now notice that if $\boldsymbol{a_e}(T) < +\infty$ and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is topological complemented in X, then T is a quasi-Fredholm operator (see, [8, Definition 3.1.2] and [8, Remark page 206]).

In the following result we prove a decomposition theorem for $T \in \varphi(\mathrm{X})$, with $\boldsymbol{a_e}(T)$ finite and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is topological complemented in X.

Theorem 5.4. *Let $T \in \varphi(\mathrm{X})$. Then the following assertions are equivalent :*

1) *there exists $n \in \mathbb{N}$ such that $\boldsymbol{a_e}(T) \le n$ and* $\mathrm{Im}(T) + \ker(T^n)$ *is topological complemented in* X,

2) *there exist $d \in \mathbb{N}$ and two closed subspaces* M *and* N *such that :*

   (*i*) $\mathrm{X} = \mathrm{M} \oplus \mathrm{N}$;

   (*ii*) $T(\mathrm{M} \cap \mathcal{D}(T)) \subseteq \mathrm{M}$, $\mathrm{N} \subseteq \ker(T^d) \subseteq \mathcal{D}(T)$, *and* $T(\mathrm{N}) \subseteq \mathrm{N}$ (*therefore* $\mathrm{Im}(T^d) \subseteq \mathrm{M}$);

   (*iii*) $S = T_{|\mathrm{M}} \in \Phi_+(\mathrm{M})$, *$S$ is a s-regular and* $\mathrm{Im}(S)$ *is topological complemented in* M.

In this case, we have $\mathrm{ind}(T_{|\mathrm{M}}) = \mathrm{ind}(\frac{1}{k}I - T)$, *for sufficiently large k.*

Proof. "1) $\Longrightarrow$ 2)" First, from the proof of [8, Theorem 3.2.1], we know that there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that $\mathrm{X} = \mathrm{M} \oplus \mathrm{N}$, $T(\mathrm{M} \cap \mathcal{D}(T)) \subseteq \mathrm{M}$, $\mathrm{N} \subseteq \ker(T^d) \subseteq \mathcal{D}(T)$, $T(\mathrm{N}) \subseteq \mathrm{N}$ and $S = T_{|\mathrm{M}}$ is s-regular. Put $m = \max\{d, n\}$. Thus, since S is a s-regular, we must have

$$\ker(T) \cap \mathrm{Im}(T^m) = \ker(S) \cap \mathrm{Im}(S^m) = \ker(S).$$

This implies that $\alpha(S) = \alpha_m^1(T) < +\infty$, and hence $S \in \Phi_+(\mathrm{M})$. Consequently, by Lemma 5.3, $\mathrm{Im}(T) + \ker(T^m)$ is topological complemented in X. Furthermore, since $\mathrm{Im}(T) + \ker(T^m) = \mathrm{Im}(S) \oplus \mathrm{N}$, it follows that

$$\mathrm{X} = [\mathrm{Im}(T) + \ker(T^m)] \oplus \mathrm{W} = [\mathrm{Im}(S) \oplus \mathrm{N}] \oplus \mathrm{W} = \mathrm{Im}(S) \dotplus [\mathrm{N} \oplus \mathrm{W}].$$

Thus, by [8, Proposition 2.1.1] and [8, Proposition 2.1.2], we conclude that $\mathrm{N}\oplus\mathrm{W}$ is closed. Therefore

$$\mathrm{M} = [\mathrm{Im}(S) \oplus (\mathrm{N} \oplus \mathrm{W})] \cap \mathrm{M} = \mathrm{Im}(S) \oplus [\mathrm{N} \oplus \mathrm{W}] \cap \mathrm{M}.$$

"2) $\Longrightarrow$ 1)" Assume that $T = S \oplus A$, where $S = T_{|\mathrm{M}}$, $A = T_{|\mathrm{N}}$, thus

$$\ker(T^d) = \ker(S^d) \oplus \mathrm{N} \;\text{ and }\; \ker(T^{d+1}) = \ker(S^{d+1}) \oplus \mathrm{N},$$

from which follows that

$$\begin{aligned} \alpha_d^1(T) &= \dim[\ker(S^{d+1}) \oplus \mathrm{N}]/[\ker(S^d) \oplus \mathrm{N}] \\ &= \dim \ker(S^{d+1})/\ker(S^d) \\ &\leq \dim \ker(S^{d+1}) < +\infty. \end{aligned}$$

Hence, $\boldsymbol{a_e}(T) \leq d$.

On the other hand, we have

$$\mathrm{Im}(T) = \mathrm{Im}(S) \dotplus \mathrm{Im}(A)$$

and

$$\mathrm{Im}(T) + \ker(T^d) = [\mathrm{Im}(S) + \ker(S^d)] \dotplus [\mathrm{N} + \mathrm{Im}(A)] = \mathrm{Im}(S) \oplus \mathrm{N}.$$

Let W be a topological complement of $\mathrm{Im}(S)$ in M, thus

$$\mathrm{Im}(T) + \ker(T^d) + \mathrm{W} = [\mathrm{Im}(S) \oplus \mathrm{W}] \oplus \mathrm{N} = \mathrm{M} \oplus \mathrm{N} = \mathrm{X}$$

and

$$[\mathrm{Im}(T) + \ker(T^d)] \cap \mathrm{W} = [\mathrm{Im}(S) \oplus \mathrm{N}] \cap \mathrm{W} = \{0\},$$

from which one deduces that $\mathrm{Im}(T) + \ker(T^d)$ is topological complemented in X.

Furthermore, by Lemma 5.3, $\mathrm{Im}(T) + \ker(T^n)$ is topological complemented in X for all $n \geq \boldsymbol{a_e}(T)$. To conclude the proof, we remark that for sufficiently large k,

$$\frac{1}{k}I - T = (\frac{1}{k}I - S) \oplus (\frac{1}{k}I - A), \;\; \frac{1}{k}I - S \in \Phi_+(\mathrm{M}) \;\text{ and }\; \frac{1}{k}I - A \;\text{ is invertible.}$$

Therefore $\mathrm{ind}(\frac{1}{k}I - T) = \mathrm{ind}(\frac{1}{k}I - S) = \mathrm{ind}(S)$. This completes the proof. □

Example 5.5.

Let $T \in \Phi_+(\mathrm{X})$ such that $\mathrm{Im}(T)$ is topological complemented in X (in particular T is quasi-Fredholm). From [7, Theorem 4], we know that there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :

(1) $\mathrm{X} = \mathrm{M} \oplus \mathrm{N}$;

(2) $T(\mathrm{M} \cap \mathcal{D}(T)) \subseteq \mathrm{M}$, $\mathrm{N} \subseteq \ker(T^d) \subseteq \mathcal{D}(T)$ and $T(\mathrm{N}) \subseteq \mathrm{N}$ (therefore $\mathrm{Im}(T^d) \subseteq \mathrm{M}$);

(3) $S = T_{|\mathrm{M}}$ is a s-regular.

It is clear that $\alpha(S) \leq \alpha(T) < +\infty$, and hence $S \in \Phi_+(\mathrm{M})$. Let W be a topological complement of $\mathrm{Im}(T)$ in X. Therefore

$$\mathrm{X} = \mathrm{Im}(T) \oplus \mathrm{W} = [\mathrm{Im}(S) \oplus \mathrm{Im}(A)] \oplus \mathrm{W} = \mathrm{Im}(S) \dotplus [\mathrm{Im}(A) \dotplus \mathrm{W}].$$

Taking into account of [8, Proposition 2.1.1], [8, Proposition 2.2] and Lemma 3.2, we deduce that $\mathrm{Im}(A) + \mathrm{W}$ is closed. This leads to

$$\mathrm{M} = (\mathrm{Im}(S) \oplus [\mathrm{Im}(A) + \mathrm{W}]) \cap \mathrm{M} = \mathrm{Im}(S) \oplus [\mathrm{Im}(A) + \mathrm{W}] \cap \mathrm{M}.$$

Consequently $S \in \Phi_+(\mathrm{M})$, S is s-regular and $\mathrm{Im}(S)$ is topological complemented in M.

As an application of the decomposition theorem we have the following result.

Corollary 5.6. *Let $T \in \varphi(\mathrm{X})$ such that $\boldsymbol{a_e}(T) \leq n$ and $\mathrm{Im}(T) + \ker(T^n)$ is topological complemented in X. Then there exists $\varepsilon > 0$ such that $T_\lambda \in \Phi_+(\mathrm{X})$, T_λ is s-regular and $\lambda \longmapsto \mathrm{ind}(T_\lambda)$ is constant for every λ with $0 < |\lambda| < \varepsilon$.*

Theorem 5.7. *Let $T \in \varphi(\mathrm{X})$. There exists $n \in \mathbb{N}$ such that $\boldsymbol{a}(T) = n < \infty$ and $\mathrm{Im}(T) + \ker(T^n)$ is topological complemented if and only if there exist $d \in \mathbb{N}$ and two closed subspaces M and N such that :*

(*i*) $\mathrm{X} = \mathrm{M} \oplus \mathrm{N}$;

(*ii*) $T(\mathrm{M} \cap \mathcal{D}(T)) \subseteq \mathrm{M}$, $\mathrm{N} \subseteq \ker(T^d) \subseteq \mathcal{D}(T)$ *and* $T(\mathrm{N}) \subseteq \mathrm{N}$;

(*iii*) $S = T_{|\mathrm{M}}$ *is injective and* $\mathrm{Im}(S)$ *is topological complemented in* M.

Proof. Let remark that by Lemma 5.3, if $\boldsymbol{a}(T) = n < +\infty$ and $\mathrm{Im}(T) + \ker(T^n)$ is topological complemented in X, then $\boldsymbol{a_e}(T) \leq n$ and $\mathrm{Im}(T) + \ker(T^n)$ is topological complemented in X. Now the proof follows from Theorem 5.4. □

Example 5.8.
Let $T \in \mathcal{A}(\mathrm{X})$ such that $\max\{\boldsymbol{a}(T), \boldsymbol{d}(T)\} < +\infty$. It is clear that if $m = \boldsymbol{a}(T) = \boldsymbol{d}(T)$, then $\mathrm{X} = \mathrm{Im}(T) + \ker(T^m)$, $\mathrm{X} = \mathrm{Im}(T^m) \oplus \ker(T^m)$ and $S = T_{|\mathrm{Im}(T^m)}$ is bijective.

6 Essential ascent and perturbation

For $T \in \varphi(\mathrm{X})$, the algebraic core $\mathrm{Co}(T)$ is defined to be the greatest subspace M of X for which $T(\mathcal{D}(T) \cap \mathrm{M}) = \mathrm{M}$. Trivially, if $T \in \varphi(\mathrm{X})$ is surjective then $\mathrm{Co}(T) = \mathrm{X}$, and for every $T \in \varphi(\mathrm{X})$, we have $\mathrm{Co}(T) = T^n(\mathrm{Co}(T)) \subseteq T^n(\mathrm{X})$, for all $n \in \mathbb{N}$. From which it follows that $\mathrm{Co}(T) \subseteq \mathrm{Im}^\infty(T)$. For the reader's convenience, we recall the following lemma which will be used to prove Proposition 6.2.

Lemma 6.1 ([11])**.** *Let $T \in \varphi(\mathrm{X})$ be a s-regular operator.*

1) $\mathrm{Co}(T) = \mathrm{Im}^\infty(T)$ *is closed.*

2) *If* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \gamma(T)$, *then* T_λ *is s-regular.*

3) *If* $\lambda \in \mathbb{C}$ *with* $|\lambda| < \gamma(T)$, *then* $\mathrm{Co}(T_\lambda) = \mathrm{Co}(T)$.

Proposition 6.2. *Let* $T \in \varphi(\mathrm{X})$ *such that* $\boldsymbol{a_e}(T) \leq n$ *and* $\mathrm{Im}(T) + \ker(T^n)$ *is topological complemented in* X. *Then* $\mathrm{Im}^\infty(T) = \mathrm{Co}(T)$ *and there exists* $\varepsilon > 0$ *such that*

$$\mathrm{Im}^\infty(T_\lambda) = \mathrm{Co}(T_\lambda) = \ker^\infty(T) + \mathrm{Co}(T), \quad \forall\ 0 < |\lambda| < \varepsilon.$$

Proof. Let N and S as in Theorem 5.4. Thus, by Lemma 6.1,

$$\mathrm{Im}^\infty(T) = \mathrm{Im}^\infty(S) = \mathrm{Co}(S) \subseteq \mathrm{Co}(T) \subseteq \mathrm{Im}^\infty(T).$$

Let $\lambda \in \mathbb{C}\backslash\{0\}$, then

$$\mathrm{Im}^\infty(\lambda I - S) \oplus \mathrm{N} \subseteq \mathrm{Im}^\infty(\lambda I - S) + \ker^\infty(T) \subseteq \mathrm{Im}^\infty(T_\lambda) + \ker^\infty(T) \subseteq \mathrm{Im}^\infty(T_\lambda)$$

and

$$\mathrm{Im}^\infty(T_\lambda) = \mathrm{Im}^\infty(\lambda I - S) \oplus \mathrm{N} \subseteq \mathrm{Im}^\infty(\lambda I - S) + \ker^\infty(T).$$

This leads to

$$\mathrm{Im}^\infty(T_\lambda) = \mathrm{Im}^\infty(\lambda I - S) + \ker^\infty(T).$$

Hence, by Lemma 6.1, we deduce that there exists $\varepsilon > 0$ such that $\lambda I - S$ is semi-regular and

$$\mathrm{Co}(S) = \mathrm{Co}(\lambda I - S) = \mathrm{Im}^\infty(\lambda I - S), \quad \forall\, |\lambda| < \varepsilon.$$

On the other hand, let $A = T_{|\mathrm{N}}$, since $\lambda I - A$ is invertible for every $\lambda \neq 0$, then T_λ is semi-regular, for every $0 < |\lambda| < \varepsilon$, from which one deduces that

$$\begin{aligned}
\mathrm{Co}(T_\lambda) = \mathrm{Im}^\infty(T_\lambda) &= \mathrm{Im}^\infty(\lambda I - S) + \ker^\infty(T) \\
&= \mathrm{Co}(S) + \ker^\infty(T) \\
&= \mathrm{Co}(T) + \ker^\infty(T).
\end{aligned}$$

This completes the proof of the proposition. □

Now we recall the following definition [5, Definition 2.5, Theorem 3.2].

Definition 6.3. Let $T \in \varphi(\mathrm{X})$ and $d \in \mathbb{N}$. Then T has an uniform descent for $n \geq d$ if $\mathrm{Im}(T) + \ker(T^n) = \mathrm{Im}(T) + \ker(T^d)$ for all $n \geq d$.
If in addition $\mathrm{Im}(T) + \ker(T^d)$ is closed, then T is said to have a topological uniform descent for $n \geq d$.

Proposition 6.4. *Let* $T \in \varphi(\mathrm{X})$. *The following statements are equivalent :*

1) $\boldsymbol{a_e}(T) < +\infty$ *and* $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ *is closed;*

2) $\boldsymbol{a_e}(T) < +\infty$ *and* T *has a topological uniform descent for* $n \geq p_1(T)$.

Proof. "1) $\Longrightarrow$ 2)" Is a direct consequence of Lemma 2.3 and Lemma 3.3.

"2) $\Longrightarrow$ 1)" By the definition of the descent uniform topological, we clearly have $\mathrm{Im}(T)+\ker(T^{p_1(T)})$ is closed. Thus, by Lemma 3.3, $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ is closed in X (because $\boldsymbol{a_e}(T)\leq p_1(T)$). This completes the proof. $\square$

Theorem 6.5. *Let* $T,S\in\mathcal{B}(\mathrm{X})$ *such that* $TS=ST$, $\max\{\boldsymbol{a_e}(T),\ \boldsymbol{a_e}(S)\}<+\infty$, $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ *and* $\mathrm{Im}(S)+\ker(S^{\boldsymbol{a_e}(S)})$ *are both closed.*

1) *If* $\|S-T\|$ *is sufficiently small, then*

$$\mathrm{ind}_{\boldsymbol{a_e}}(T)=\mathrm{ind}_{\boldsymbol{a_e}}(S),\ \ \overline{\ker^{\infty}(T)}=\overline{\ker^{\infty}(S)}\quad\text{and}\quad \mathrm{Im}^{\infty}(T)=\mathrm{Im}^{\infty}(S).$$

If in addition $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ *and* $\mathrm{Im}(S)+\ker(S^{\boldsymbol{a_e}(S)})$ *are topological complemented subspaces of* X, *then* $\mathrm{Im}^{\infty}(S)=\mathrm{Co}(T)=\mathrm{Co}(S)$.

2) *If* $S-T$ *is a compact operator, then* $\mathrm{ind}_{\boldsymbol{a_e}}(T)=\mathrm{ind}_{\boldsymbol{a_e}}(S)$.

Proof. 1) Let $n\geq d=\max\{p_1(T),\ p_1(S)\}$. First, from [5, Theorem 4.6], we have

$$\mathrm{Im}^{\infty}(T)=\mathrm{Im}^{\infty}(S),\ \overline{\ker^{\infty}(T)}=\overline{\ker^{\infty}(S)}$$

and

$$\beta_n^1(S)=\beta_n^1(T),\quad \alpha_n^1(S)=\alpha_n^1(T),$$

from which follows that

$$\mathrm{ind}_{\boldsymbol{a_e}}(T)=\mathrm{ind}(T_n)=\mathrm{ind}(S_n)=\mathrm{ind}_{\boldsymbol{a_e}}(S).$$

Now, if $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ and $\mathrm{Im}(S)+\ker(S^{\boldsymbol{a_e}(S)})$ are topological complemented subspaces of X, by Proposition 6.2, we deduce that

$$\mathrm{Co}(T)=\mathrm{Im}^{\infty}(T)=\mathrm{Im}^{\infty}(S)=\mathrm{Co}(S).$$

2) From [5, Theorem 5.8], we conclude that $\mathrm{ind}(T_n)=\mathrm{ind}(S_n)$, for sufficiently large n. Hence,

$$\mathrm{ind}_{\boldsymbol{a_e}}(T)=\mathrm{ind}(T_n)=\mathrm{ind}(S_n)=\mathrm{ind}_{\boldsymbol{a_e}}(S).$$

This completes the proof. $\square$

Now we write $\varphi_d(\mathrm{X})=\{T\in\varphi(\mathrm{X}):\overline{\mathcal{D}(T)}=\mathrm{X}\}$.

Theorem 6.6. *Let* $T\in\varphi(\mathrm{X})$ *and let* $F\in\mathcal{B}(\mathrm{X})$ *such that* $\mathrm{Im}(F)\subseteq\mathcal{D}(T)$ *and* $\dim\mathrm{Im}(F^n)<+\infty$ *for some* $n\in\mathbb{N}$. *Assume that* $TF(x)=FT(x)$, *for every* $x\in\mathcal{D}(T)$, *one has*

1) $\boldsymbol{a_e}(T)<+\infty$ *if and only if* $\boldsymbol{a_e}(T+F)<+\infty$.

2) *If* $T\in\varphi_\infty(\mathrm{X})\cap\varphi_d(\mathrm{X})$ *with* $\boldsymbol{a_e}(T)<+\infty$, *then* $\mathrm{Im}(T)+\ker(T^{\boldsymbol{a_e}(T)})$ *is closed if and only if* $\mathrm{Im}(T+F)+\ker[(T+F)^{\boldsymbol{a_e}(T+F)}]$ *is closed.*

Proof. The first assertion can be proved in the same way as [1, Theorem 3.1]. To prove the second assertion, let $p \geq \boldsymbol{a_e}(T)$, we know that

$$\begin{array}{rrcl} \widetilde{T}\ : & \mathcal{D}(\widetilde{T}) \subseteq \mathrm{X}/\ker(T^p) & \longrightarrow & \mathrm{X}/\ker(T^p) \\ & \overline{x} & \longmapsto & \overline{Tx} \end{array}$$

is upper semi-Fredholm operator and $\dim \mathrm{Im}(\widetilde{F}^n) < +\infty$, where

$$\begin{array}{rrcl} \widetilde{F}\ : & \mathrm{X}/\ker(T^p) & \longrightarrow & \mathrm{X}/\ker(T^p) \\ & \overline{x} & \longmapsto & \overline{Fx}. \end{array}$$

Hence, $\widetilde{T}^n - (-\widetilde{F})^n$ is upper semi-Fredholm operator. Since

$$\widetilde{T}^n - (-\widetilde{F})^n = (\widetilde{T} + \widetilde{F})(\widetilde{T}^{n-1} + \widetilde{T}^{n-2}(-\widetilde{F}) + \cdots + (-\widetilde{F})^{n-1}) \in \Phi_+(\mathrm{X}/\ker(T^p)),$$

it follow that

$$(\widetilde{T}^{n-1} + \widetilde{T}^{n-2}(-\widetilde{F}) + \cdots + (-\widetilde{F})^{n-1})^*(\widetilde{T} + \widetilde{F})^* \in \Phi_-([\mathrm{X}/\ker(T^p)]^*).$$

Consequently, $(\widetilde{T} + \widetilde{F})^* \in \Phi_-([\mathrm{X}/\ker(T^p)]^*)$. This proves that $\widetilde{T} + \widetilde{F} \in \Phi_+(\mathrm{X}/\ker(T^p))$. Now we can finish the proof as [1, Theorem 3.1]. □

Question 1. Is it possible to remove the hypothesis $\overline{\mathcal{D}(T)} = \mathrm{X}$ in Theorem 6.6?

For $T \in \varphi(\mathrm{X})$, an operator $V \in \mathcal{B}(\mathrm{X})$ is called T-reducing-space operator, if it satisfies the following condition :

$$\text{if M is a subspace of X reduce } T \Longrightarrow \text{M reduce } V.$$

For $T \in \varphi(\mathrm{X})$, define

$$\Upsilon_T = \{V \in \mathcal{B}(\mathrm{X}) : V \text{ is } T\text{-reducing-space operator}\}.$$

Theorem 6.7. *Let $T \in \varphi(\mathrm{X})$ with $\boldsymbol{a_e}(T) < +\infty$ and $\mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)})$ is topological complemented in* X *and let $V \in \mathcal{B}(\mathrm{X})$. If $V \in \Upsilon_T$ is invertible such that $V(\mathcal{D}(T)) \subseteq \mathcal{D}(T)$, $VT(x) = TV(x)$ for all $x \in \mathcal{D}(T)$ and $\|V\|$ is sufficiently small, then $0 \in \varrho^e_{asc}(T + V)$.*

Proof. First, from Theorem 5.4, we know that there exist M and N two closed subspaces of X such that

- $\mathrm{X} = \mathrm{M} \oplus \mathrm{N}$;
- $T(\mathrm{M} \cap \mathcal{D}(T)) \subseteq \mathrm{M}$, $T(\mathrm{N}) \subseteq \mathrm{N}$;
- $A = T_{|\mathrm{M}} \in \Phi_+(\mathrm{M})$ and $B = T_{|\mathrm{N}}$ is a nilpotent operator of degree d.

Write $S = V_{|\mathrm{M}}$ and $F = V_{|\mathrm{N}}$. In fact, $B + F$ is invertible operator. Indeed, since

$$F^d = F^d - (-B)^d = (F + B)(F^{d-1} + (-B)F^{d-2} + \cdots + (-B)^{d-1}),$$

it follows that

$$x = (F + B)[F^{d-1} + (-B)F^{d-1} + \cdots + (-B)^{d-1}]F^{-d}x, \quad \forall\, x \in \mathrm{N}.$$

Hence, by using the fact that $\|S\| \leq \|V\| < \gamma(A)$, we infer that $A + S \in \Phi_+(\mathrm{M})$. Thus, $T + V \in \Phi_+(\mathrm{X})$ and consequently, $0 \in \varrho^e_{asc}(T + V)$, which completes the proof of Theorem 6.7. □

Lemma 6.8. *Let $T \in \varphi(X)$ such that $\boldsymbol{a_e}(T) < +\infty$. Then*

$$\boldsymbol{a}(T) < +\infty \Longleftrightarrow \mathrm{Im}^{\infty}(T) \cap \ker^{\infty}(T) = \{0\}.$$

Proof. We note that the first assertion of [14, Proposition 1.6] remains valid for $T \in \varphi(X)$. If we put $n = \boldsymbol{a_e}(T)$ and $T_n = T_{|\mathrm{Im}(T^n)}$, then $\alpha(T_n) < +\infty$. Hence, by [14, Proposition 1.6], we deduce that

$$\begin{aligned} \boldsymbol{a}(T) < +\infty &\Longleftrightarrow \boldsymbol{a}(T_n) < +\infty, \\ &\Longleftrightarrow \mathrm{Im}^{\infty}(T_n) \cap \ker^{\infty}(T_n) = \{0\}, \\ &\Longleftrightarrow \mathrm{Im}^{\infty}(T) \cap \ker^{\infty}(T) = \{0\}. \end{aligned}$$

This completes the proof. □

Define the following sets :

$$\begin{aligned} p\Phi_+(X) &= \{T \in \varphi_\infty(X) : \boldsymbol{a_e}(T) < +\infty \text{ and } \mathrm{Im}(T) + \ker(T^{\boldsymbol{a_e}(T)}) \text{ is closed}\}, \\ p\Psi_+(X) &= \{T \in \varphi_\infty(X) : \boldsymbol{a}(T) < +\infty \text{ and } \mathrm{Im}(T) + \ker(T^{\boldsymbol{a}(T)}) \text{ is closed}\}, \\ \mathcal{P}(p\Phi_+(X)) &= \{L \in \mathcal{B}(X) : L + S \in p\Phi_+(X),\ \forall\, S \in p\Phi_+(X) \cap \{L\}'\}, \end{aligned}$$

where $\{L\}' = \{T \in \varphi(X) : L(\mathcal{D}(T)) \subseteq \mathcal{D}(T),\ \ LTx = TLx,\ \forall\ x \in \mathcal{D}(T)\}$.

Recall that by Theorem 6.6, if $F \in \mathcal{B}(X)$ with $\dim \mathrm{Im}(F^n) < +\infty$ for some $n \in \mathbb{N}$, then $F \in \mathcal{P}(p\Phi_+(X))$.

We conclude the paper with the following result.

Theorem 6.9. *Let $T, K \in \mathcal{B}(X)$ such that $TK = KT$. If $T \in p\Psi_+(X)$ and $K \in \mathcal{P}(p\Phi_+(X))$, then $T + K \in p\Psi_+(X)$ and $\beta^1_n(T) = \beta^1_m(T+K)$, for every $n \geq \boldsymbol{a}(T)$, and for every $m \geq \boldsymbol{a}(T+K)$.*

Proof. First, it is clair, for all $\lambda \in \mathbb{C}$, that $T + \lambda K \in p\Phi_+(X)$. By Theorem 6.5, for $\lambda, \mu \in \mathbb{C}$ such that $|\lambda - \mu|$ is sufficiently small, we conclude that

$$\overline{\ker^{\infty}(T+\lambda K)} = \overline{\ker^{\infty}(T+\mu K)},\ \ \mathrm{Im}^{\infty}(T+\lambda K) = \mathrm{Im}^{\infty}(T+\mu K), \tag{1}$$

$$\mathrm{ind}_{\boldsymbol{a_e}}(T+\lambda K) = \mathrm{ind}_{\boldsymbol{a_e}}(T+\mu K). \tag{2}$$

Since $[0, 1]$ is a compact subset of the complex plane, there exist $n \in \mathbb{N}\backslash\{0\}$, $\lambda_i \in [0, 1]$, and a non-negative real number ε_i sufficiently small $(1 \leq i \leq n)$, such that

$$0 = \lambda_1 < \cdots < \lambda_n = 1,\ [0, 1] \subseteq \bigcup_{i=1}^{n} \mathrm{D}(\lambda_i,\ \varepsilon_i),$$

$$\mathrm{D}(\lambda_i,\ \varepsilon_i) \cap \mathrm{D}(\lambda_{i+1},\ \varepsilon_{i+1}) \neq \emptyset,\ \ \forall\, i \in \{1, \cdots, n-1\},$$

where $\mathrm{D}(\lambda_i,\ \varepsilon_i)$ denotes the open disk with center at λ_i and radius ε_i. Hence, taking into account of (1) and (2), for all $i \in \{1, 2, \cdots, n\}$, we deduce that

$$\overline{\ker^{\infty}(T+\lambda_i K)} = \overline{\ker^{\infty}(T)},\ \mathrm{Im}^{\infty}(T+\lambda_i K) = \mathrm{Im}^{\infty}(T) \text{ and } \mathrm{ind}_{\boldsymbol{a_e}}(T+\lambda_i K) = \mathrm{ind}_{\boldsymbol{a_e}}(T).$$

This leads to

$$\ker^{\infty}(T+K) \cap \mathrm{Im}^{\infty}(T+K) = \{0\}.$$

Thus, by Lemma 6.8, $\boldsymbol{a}(T+K) < +\infty$. However, if $L \in \mathcal{A}(X) \cap p\Phi_+(X)$, then $\mathrm{ind}_{\boldsymbol{a}_e}(L) = \mathrm{ind}(L_n)$, for all $n \geq \boldsymbol{a_e}(L)$. Consequently, if $\boldsymbol{a_e}(T) \leq \boldsymbol{a}(T) \leq n$ and $\boldsymbol{a_e}(T+K) \leq \boldsymbol{a}(T+K) \leq m$, then

$$\mathrm{ind}_{\boldsymbol{a}_e}(T+K) = -\beta^1_m(T+K) \quad \text{and} \quad \mathrm{ind}_{\boldsymbol{a}_e}(T) = -\beta^1_n(T).$$

This complete the proof. □

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