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Abstract

Let $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect the Gegenbauer-Sobolev inner product

$$\langle f,g\rangle_S := \int_{-1}^{1} f(x)g(x)(1-x^2)^{\alpha-\frac{1}{2}}dx + \lambda\int_{-1}^{1} f'(x)g'(x)(1-x^2)^{\alpha-\frac{1}{2}}dx,$$

where $\alpha > -\frac{1}{2}$ and $\lambda \geq 0$. In this paper we use a recent result due to B.D. Bojanov and N. Naidenov [3], in order to study the maximization of a local extremum of the kth derivative $\frac{d^k}{dx^k}Q_{n,\lambda}^{(\alpha)}$ in $[-M_{n,\lambda}, M_{n,\lambda}]$, where $M_{n,\lambda}$ is a suitable value such that all zeros of the polynomial $Q_{n,\lambda}^{(\alpha)}$ are contained in $[-M_{n,\lambda}, M_{n,\lambda}]$ and the function $\left|Q_{n,\lambda}^{(\alpha)}\right|$ attains its maximal value at the end-points of such interval. Also, some illustrative numerical examples are presented.

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1 Introduction

Extremal properties for general orthogonal polynomials is an interesting subject in approximation theory and their applications permeate many fields in science and engineering [5, 18, 21, 28, 29]. Although it may seem an old subject from the view point of the standard orthogonality [5, 18, 29], this is not the case neither in the general setting (cf. [11, 12, 13, 14, 20]) nor from the view point of Sobolev orthogonality, where it remains like a partially explored subject [1]. In fact, new results continue to appear in some recent publications [10, 11, 12, 24, 26, 27].

Let $d\mu(x) = (1-x^2)^{\alpha-\frac{1}{2}}dx$ with $\alpha > -\frac{1}{2}$, be the Gegenbauer measure supported on the interval $[-1,1]$. We consider the following Sobolev inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients.

$$\langle f,g\rangle_S := \int_{-1}^{1} f(x)g(x)d\mu(x) + \lambda \int_{-1}^{1} f'(x)g'(x)d\mu(x), \tag{1.1}$$

where $\lambda \geq 0$. Let $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ denote the sequence of monic orthogonal polynomials with respect to (1.1). These polynomials are usually called monic Gegenbauer-Sobolev polynomials [7, 8, 15, 16, 17, 25] and it is known that the zeros of these polynomials are in the real line [15, 16], and therefore they belong to other important class of algebraic polynomials, namely the oscillating polynomials [3, 19].

The main result of [3] allows to guarantee the maximal absolute value of higher derivatives of a symmetric oscillating polynomial on a finite interval are attained in the end-points of such interval, whenever the maximal absolute value of the polynomial is attained in the end-points of that interval. Then, [3, Section 4] contains a brief explanation about applications of previous result to orthogonal polynomials on the real line associated to symmetric weights. We focus our attention on that last part of [3, Section 4] in order to enlarge the range of application of [3, Theorem 1] to the class of Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1).

The paper is structured as follows. Section 2 provides some background about structural properties of the Gegenbauer and Gegenbauer-Sobolev polynomials corresponding to the inner product (1.1), respectively. Section 3 contains some well-known characteristics of the class of oscillating polynomials on a finite interval. We also state there our main result (see Theorem 3.3) and provide some illustrative numerical examples. Throughout this paper, the notation $u_n \cong v_n$ means that the sequence $\left\{\frac{u_n}{v_n}\right\}_n$ converges to 1 as $n \to \infty$. We will denote by $\mathbb{P}_n$ and $\|f\|_\infty$, the space of polynomials of degree at most n and the uniform norm of f on the interval in consideration, respectively. Any other standard notation will be properly introduced whenever needed.

2 Basic facts: Gegenbauer and Gegenbauer-Sobolev orthogonal polynomials

For $\alpha > -\frac{1}{2}$ we denote by $\{\hat{C}_n^{(\alpha)}\}_{n\geq 0}$ the sequence of Gegenbauer polynomials, orthogonal on $[-1,1]$ with respect to the measure $d\mu(x)$ (cf. [29, Chapter IV]), normalized by

$$\hat{C}_n^{(\alpha)}(1) = \frac{\Gamma(n+2\alpha)}{n!\Gamma(2\alpha)}.$$

It is clear that this normalization does not allow α to be zero or a negative integer. Nevertheless, the following limits exist for every $x \in [-1,1]$ (see [29, formula (4.7.8)].)

$$\lim_{\alpha\to 0} \hat{C}_0^{(\alpha)}(x) = T_0(x), \quad \lim_{\alpha\to 0} \frac{\hat{C}_n^{(\alpha)}(x)}{\alpha} = \frac{2}{n} T_n(x),$$

where T_n is the nth Chebyshev polynomial of the first kind. In order to avoid confusing notation, we define the sequence $\{\hat{C}_n^{(0)}\}_{n\geq 0}$ as follows.

$$\hat{C}_0^{(0)}(1) = 1, \quad \hat{C}_n^{(0)}(1) = \frac{2}{n}, \quad \hat{C}_n^{(0)}(x) = \frac{2}{n} T_n(x), \quad n \geq 1.$$

We denote the nth monic Gegenbauer orthogonal polynomial by

$$C_n^{(\alpha)}(x) = (h_n^{\alpha})^{-1} \hat{C}_n^{(\alpha)}(x), \tag{2.1}$$

where the constant h_n^{α} (cf. [29, formula (4.7.31)]) is given by

$$h_n^{\alpha} = \frac{2^n \Gamma(n+\alpha)}{n!\Gamma(\alpha)}, \quad \alpha \neq 0, \tag{2.2}$$

$$h_n^0 = \lim_{\alpha\to 0} \frac{h_n^{\alpha}}{\alpha} = \frac{2^n}{n}, \quad n \geq 1. \tag{2.3}$$

Then for $n \geq 1$, we have $C_n^{(0)}(x) = \lim_{\alpha\to 0}(h_n^{\alpha})^{-1}\hat{C}_n^{(\alpha)}(x) = \frac{1}{2^{n-1}} T_n(x)$.

Proposition 2.1. *Let $\{C_n^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic Gegenbauer orthogonal polynomials. Then the following statements hold.*

1. *Three-term recurrence relation. For every $n \in \mathbb{N}$,*

$$xC_n^{(\alpha)}(x) = C_{n+1}^{(\alpha)}(x) + \gamma_n^{(\alpha)} C_{n-1}^{(\alpha)}(x), \quad \alpha > -\frac{1}{2}, \quad \alpha \neq 0, \tag{2.4}$$

with initial conditions $C_0^{(\alpha)}(x) = 1$, $C_1^{(\alpha)}(x) = x$, and recurrence coefficient $\gamma_n^{(\alpha)} = \frac{n(n+2\alpha-1)}{4(n+\alpha)(n+\alpha-1)}$.

2. *For every $n \in \mathbb{N}$ (see [29, formula (4.7.15)]),*

$$\|C_n^{(\alpha)}\|_{\mu}^2 = \int_{-1}^{1} [C_n^{(\alpha)}(x)]^2 d\mu(x) = \pi 2^{1-2\alpha-2n} \frac{n!\Gamma(n+2\alpha)}{\Gamma(n+\alpha+1)\Gamma(n+\alpha)}. \tag{2.5}$$

3. *Structure relation (see [29, formula (4.7.29)]). For every $n \geq 2$,*

$$C_n^{(\alpha-1)}(x) = C_n^{(\alpha)}(x) + \xi_{n-2}^{(\alpha)} C_{n-2}^{(\alpha)}(x), \tag{2.6}$$

where

$$\xi_n^{(\alpha)} = \frac{(n+2)(n+1)}{4(n+\alpha+1)(n+\alpha)}, \quad n \geq 0. \tag{2.7}$$

4. *For every $n \in \mathbb{N}$ (see [29, formula (4.7.14)]),*

$$\frac{d}{dx} C_n^{(\alpha)}(x) = n C_{n-1}^{(\alpha+1)}(x). \tag{2.8}$$

Some well-known properties of the monic Gegenbauer-Sobolev orthogonal polynomials corresponding to the inner product (1.1) are the following.

Proposition 2.2. *Let $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to* (1.1). *Then the following statements hold.*

1. *The polynomials $Q_{n,\lambda}^{(\alpha)}$ are symmetric, i.e.,*

$$Q_{n,\lambda}^{(\alpha)}(-x) = (-1)^n Q_{n,\lambda}^{(\alpha)}(x). \tag{2.9}$$

2. *The zeros of $Q_{n,\lambda}^{(\alpha)}$ are real and simple, and they interlace with the zeros of the monic Gegenbauer orthogonal polynomials $C_n^{(\alpha)}$. Furthermore, for $\alpha \geq \frac{1}{2}$ they are all contained in the interval* $[-1,1]$ *and for $-\frac{1}{2} < \alpha < \frac{1}{2}$ there is at most a pair of zeros symmetric with respect to the origin outside the interval* $[-1,1]$*, (cf. [15, 16]).*

3. *[15, Lemma 5.1]. For $\alpha \geq \frac{1}{2}$, we have $Q_{n,\lambda}^{(\alpha)}(1) > 0$.*

It is worthwhile to point out that in the case $-\frac{1}{2} < \alpha < \frac{1}{2}$, no global properties about the sign $Q_{n,\lambda}^{(\alpha)}(1)$ can be deduced (cf. [15].)

However, the location of zeros of Sobolev orthogonal polynomials is not a trivial problem. For instance, if we consider (μ_0, μ_1) a vector of compactly supported positive measures on the real line with finite total mass and the following Sobolev inner product on the linear space $\mathbb{P}$ of polynomials with real coefficients.

$$\langle f, g\rangle_{(\mu_0,\mu_1)} := \int f(x)g(x)d\mu_0(x) + \int f'(x)g'(x)d\mu_1(x), \tag{2.10}$$

then, simple examples show that the zeros of these Sobolev orthogonal polynomials do not necessarily remain in the convex hull of the union of the supports of the measures μ_k, $k = 0, 1$, and they can be complex. In this regard some numerical experiments may be found in [9]. In particular, the boundedness of the zeros of Sobolev orthogonal polynomials is an open problem [1, 16], but as was stated in [10], it could be obtained as a consequence of the boundedness of the multiplication operator $Mf(z) = zf(z)$: If M is bounded and $\|M\|$ is its operator norm (induced by (2.10)), then all the zeros of the Sobolev orthogonal polynomials Q_n are contained in the disc $\{z \in \mathbb{C} : |z| \leq \|M\|\}$.

Indeed, if x_0 is a zero of Q_n then $xp(x) = x_0 p(x) + Q_n(x)$ for a polynomial $p \in \mathbb{P}_{n-1}$. Since p and Q_n are orthogonal, we get

$$|x_0|^2 \|p\|^2_{(\mu_0,\mu_1)} = \|xp\|^2_{(\mu_0,\mu_1)} - \|Q_n\|^2_{(\mu_0,\mu_1)} \leq \|xp\|^2_{(\mu_0,\mu_1)} = \|Mp\|^2_{(\mu_0,\mu_1)} \leq \|M\|^2 \|p\|^2_{(\mu_0,\mu_1)},$$

which yields the above result.

Thus, in the last decades the question whether or not the multiplication operator M is bounded has been a topic of interest to investigators in the field, since it turns out to be a key for the location of zeros and for establishing the asymptotic behavior of orthogonal polynomials with respect to diagonal (or non-diagonal) Sobolev inner products (cf. [16, 26, 27] and the references therein.)

From the structure relation (2.6) and [17, formula (3)] (see also [7, Proposition 1]) the following connection formula can be obtained.

Proposition 2.3. *For* $\alpha > -\frac{1}{2}$,

$$C_n^{(\alpha-1)}(x) = Q_{n,\lambda}^{(\alpha)}(x) - d_{n-2}(\alpha) Q_{n-2,\lambda}^{(\alpha)}(x), \quad n \geq 2, \tag{2.11}$$

where

$$d_n(\alpha) = \xi_n^{(\alpha)} \frac{\|C_n^{(\alpha)}\|^2_\mu}{\|Q_{n,\lambda}^{(\alpha)}\|^2_S}. \tag{2.12}$$

Moreover,

$$d_n(\alpha) \cong \frac{1}{16\lambda n^2}. \tag{2.13}$$

3 Maximization of local extremum of the derivatives for families of Gegenbauer-Sobolev polynomials

A polynomial $P \in \mathbb{P}$ is said oscillating (see [2, 3, 4, 19, 22, 23]) if it has all its zeros on the real line $\mathbb{R}$. For example, the classical orthogonal polynomials on subsets of $\mathbb{R}$ (Hermite, Laguerre and Jacobi polynomials [6, 20, 29]), orthogonal polynomials for weights in the Nevai class $M(0,1)$ [21], including whose orthogonal with respect to weights belonging to Levin-Lubinsky class $\hat{\mathcal{W}}$ [13], and a broad class of Sobolev orthogonal polynomials [7, 9, 15, 16, 17, 25] constitute an important family of oscillating polynomials. Usually, when all zeros of a polynomial $P \in \mathbb{P}_n$ with $\deg(P) = n$, are contained in a given finite interval $[a,b]$, it is called oscillating polynomial on $[a,b]$, (see [3, 19].)

We denote by $\mathrm{Osc}(\mathbb{R})$ and $\mathrm{Osc}[a,b]$ the classes of oscillating polynomials on $\mathbb{R}$ and $[a,b]$, respectively. For any $P \in \mathrm{Osc}[a,b]$ with $\deg(P) = n$, we consider the vector $\mathbf{h}(P) = (h_0(P), \ldots, h_n(P))$, where $h_j(P) = |P(t_j)|$, $0 \leq j \leq n$, $t_0 = a$, $t_n = b$, and $t_1 \leq t_2 \leq \cdots \leq t_{n-1}$ are the zeros of P'.

Amongst the main characteristics of the class $\mathrm{Osc}[a,b]$ we list the following.

i) $P' \in \mathrm{Osc}[a,b]$, for all $P \in \mathrm{Osc}[a,b]$.

ii) Any $P \in \mathrm{Osc}[a,b]$ is completely determined by its local extrema and the values at the end-points of the interval $[a,b]$ (cf. [2, Theorem 2], [4, Remark 1].)

iii) For $P \in \mathrm{Osc}[a,b]$ with $\deg(P) = n$, there exists a monotone dependence of the parameters $h_j(P')$ on the parameters $h_0(P),\dots,h_n(P)$ of P (cf. [4, Lemma 3].)

iv) If $P \in \mathrm{Osc}[a,b]$ with $\deg(P) \geq 3$ and $\|P\| = |P(a)|$, then each local extremum of P' from the first half (i.e., with an index less than or equal to $\left\lfloor \frac{n-1}{2} \right\rfloor$, and $\lfloor t \rfloor$ denoting the integer part of t) is smaller in absolute value than $|P'(a)|$.

More precisely, the property iv) was stated in the following theorem.

Theorem 3.1. *([3, Theorem 1]) Let $P \in \mathrm{Osc}[a,b]$ with $\deg(P) \geq 3$. Assume that $\|P\|_\infty = |P(a)| = 1$. Then*

$$|P'(\tau_j)| < |P'(a)|, \text{ for } j = 0,\dots,\left\lfloor \frac{n-1}{2} \right\rfloor, \tag{3.1}$$

where $\tau_1 \leq \cdots \leq \tau_{n-2}$ are the zeros of P''.

Corollary 3.2. *([3, Corollary 1]) Let $P \in \mathrm{Osc}[-1,1]$ be a symmetric polynomial, with $\deg(P) = n$. Assume that $\|P\|_\infty = |P(1)| = 1$. Then,*

$$\|P^{(k)}\|_\infty = |P^{(k)}(1)|, \text{ for } k = 1,\dots,n. \tag{3.2}$$

As a consequence of the combination of Theorem 3.1 (or Corollary 3.2) and the structural properties of the sequence $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ given in the previous section, we can obtain the maximization of local extremum of the derivatives for the sequence $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ as follows.

Let $\{Q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ be the sequence of monic orthogonal polynomials with respect to (1.1). Let us consider $x_{n,\lambda}^{\alpha,[1]} < x_{n,\lambda}^{\alpha,[2]} < \cdots < x_{n,\lambda}^{\alpha,[n]}$ the zeros of the Gegenbauer-Sobolev polynomial $Q_{n,\lambda}^{(\alpha)}$ and N the maximum value attained by $|Q_{n,\lambda}^{(\alpha)}(x)|$ on the interval $[x_{n,\lambda}^{\alpha,[1]}, x_{n,\lambda}^{\alpha,[n]}]$. Then $M_{n,\lambda}$ can be defined to be the minimal real point such that $x_{n,\lambda}^{\alpha,[n]} < M_{n,\lambda}$ and $|Q_{n,\lambda}^{(\alpha)}(M_{n,\lambda})| = N$, i.e., $M_{n,\lambda}$ is the point closest to $x_{n,\lambda}^{\alpha,[n]}$ where the maximal absolute value of the polynomial $Q_{n,\lambda}^{(\alpha)}$ is attained. Notice that $M_{n,\lambda}$ also depends on the parameter α and $Q_{n,\lambda}^{(\alpha)} \in \mathrm{Osc}[-M_{n,\lambda}, M_{n,\lambda}]$. Thus, we can consider the following normalized polynomials

$$q_{n,\lambda}^{(\alpha)}(x) := \frac{Q_{n,\lambda}^{(\alpha)}(x)}{Q_{n,\lambda}^{(\alpha)}(M_{n,\lambda})}, \quad x \in [-M_{n,\lambda}, M_{n,\lambda}], \quad n \geq 0. \tag{3.3}$$

Theorem 3.3. *Let $\{q_{n,\lambda}^{(\alpha)}\}_{n\geq 0}$ be the sequence of orthogonal polynomials given in (3.3). Then $\left|\frac{d^k}{dx^k} q_{n,\lambda}^{(\alpha)}\right|$ attains its maximal value on the interval $[-M_{n,\lambda}, M_{n,\lambda}]$ at the end-points, for $\alpha > -\frac{1}{2}$ and $1 \leq k \leq n$.*

Proof. It suffices to follow the proof of Theorem 3.1 (or Corollary 3.2) given in [3, Theorem 1 (or Corollary 1)] by making the corresponding modifications. □

Notice that from a numerical point of view the value $M_{n,\lambda}$ can be difficult to obtain for n large enough. However, for any value $K>0$ such that $N<|Q_{n,\lambda}^{(\alpha)}(x)|$ for $x<-K$ and $x>K$, the result of Theorem 3.3 remains true on the interval $[-K,K]$.

We finish this section providing some illustrative numerical examples (with the help of MAPLE) about the above result for different values of n, α and λ (see Figure 1 and Figure 2 below).

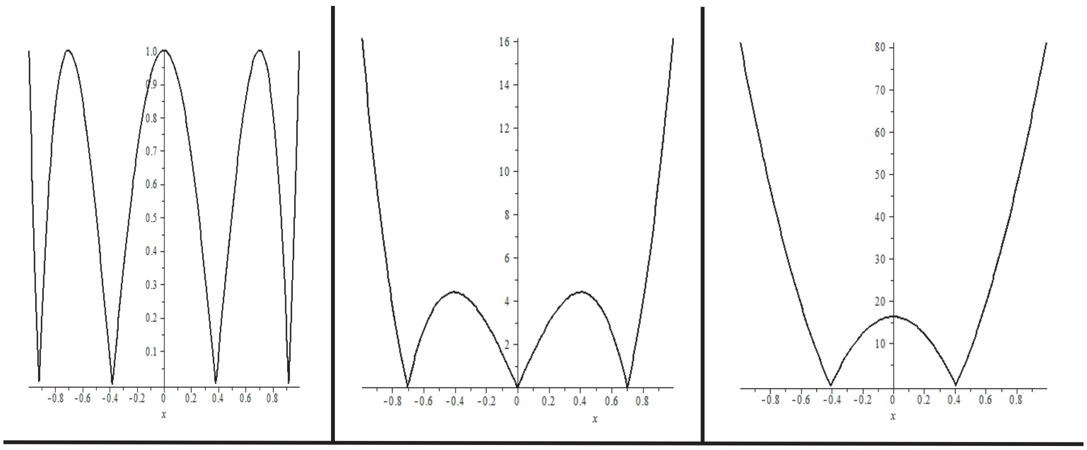

Figure 1. Graphics of $\left|\frac{d^k}{dx^k}q_{n,\lambda}^{(\alpha)}\right|$ for $n=4$, $\alpha=\lambda=1$, $M_{n,\lambda}=0.9926198253$ and $k=0,1,2$, respectively.

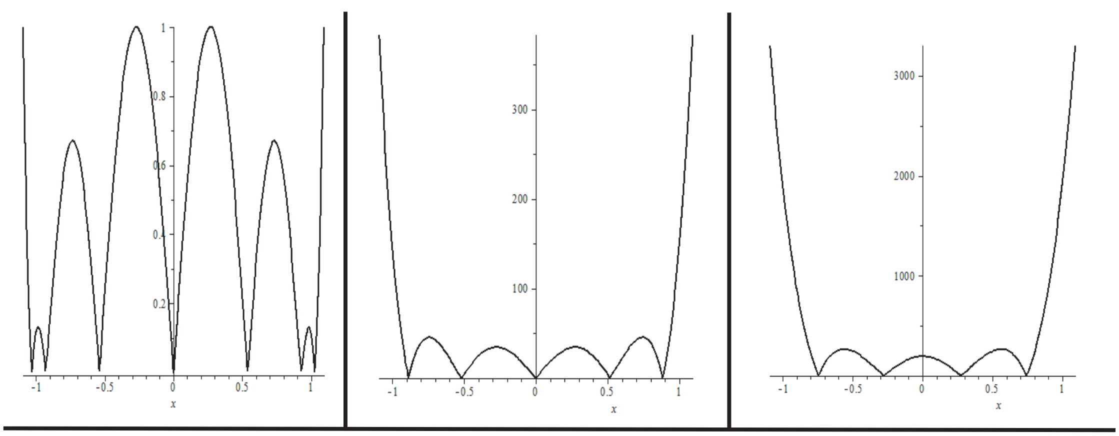

Figure 2. Graphics of $\left|\frac{d^k}{dx^k}q_{n,\lambda}^{(\alpha)}\right|$ for $n=7$, $\alpha=-\frac{1}{4}$, $\lambda=\frac{1}{2}$, $M_{n,\lambda}=1.091516326$ and $k=0,2,3$, respectively.

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