Communications in **M**athematical **A**nalysis
Volume 16, Number 1, pp. 84–101 (2014) www.math-res-pub.org/cma
ISSN 1938-9787

Electromagnetic Waves Propagation from Moving Sources in Waveguides Filled by a Dispersive Dielectric Media

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(Communicated by Marco Squassina)

Abstract

We consider the problem of electromagnetic wave propagation in homogeneous dielectric dispersive waveguides $\Pi = \mathcal{D}\times\mathbb{R}$ where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^2$, produced by non-uniformly moving sources of the form

$$\mathbf{j}(\mathbf{x},t) = \mathbf{A}(t)\delta(\mathbf{x}-\mathbf{x}_0(t)) \tag{0.1}$$

where $\mathbf{j}(\mathbf{x},t)$ is the current density, $\mathbf{A}(t)$ is a vector amplitude, $\mathbf{x} = \mathbf{x}_0(t)$ is a trajectory of the source.

We consider the propagation of TE and TM waves in the waveguide Π, produced by the source (0.1). As example we study the propagation of electromagnetic waves in a waveguides filled by a cold, non magnetized plasma.

AMS Subject Classification: Telecommunications

Keywords: wave, propagation, dispersive media, waveguide, Maxwells equation, moving sources

1 Introduction

We consider the problem of electromagnetic wave propagation in homogeneous dielectric dispersive waveguides $\Pi = \mathcal{D}\times\mathbb{R}$ where $\mathcal{D}$ is a bounded domain in $\mathbb{R}^2$, produced by a

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non-uniformly moving source of the form

$$\mathbf{j}(\mathbf{x},t) = \mathbf{A}(t)\delta(\mathbf{x} - \mathbf{x}_0(t))$$

where $\mathbf{j}(\mathbf{x},t)$ is the current density, $\mathbf{x} = \mathbf{x}_0(t), t \in \mathbb{R}$ is a trajectory of the source, $\mathbf{A}(t)$ is a vector amplitude. The problem under consideration has a well-known theoretical and applied interest in the many branches of theoretical and applied physics, for example, in the satellite communications, nuclear and relativistic physics.

It should be noted that the problem of propagation of electromagnetic waves from the moving particle in a homogeneous space is a classical problem of electrodynamics (see, for instance, [15], [12], [9], [10]). The electromagnetic field produced by uniformly moving source in homogeneous waveguides was considered in the monograph [7], see, also papers devoted the well-known Vavilov-Cherenkov effect in homogeneous waveguides generated by uniformly moving sources (see, for instance, [1], [9] and references cited there).

Our approach is based on the asymptotic analysis of the problem where the large parameter characterizes simultaneously a large distance between the source and receiver, slowly oscillation of the velocity $\mathbf{v}(t)$ and the amplitude $\mathbf{A}(t)$. We apply in the paper the methods which developed earlier under investigation of the problem of underwater wave propagation from moving sources (see [17], [18], [19]) and in the electromagenetic waves propagation in dispersive media in [3], [20]. As an example we consider the waves propagation in the plasma waveguides generated by moving source.

2 Maxwell's equation

Maxwell's equations in the differential representations are (see for instance [24])

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{x},t) &= -\frac{\partial \mathbf{B}(\mathbf{x},t)}{\partial t}, \\ \nabla \times \mathbf{H}(\mathbf{x},t) &= \frac{\partial \mathbf{D}(\mathbf{x},t)}{\partial t} + \mathbf{J}(\mathbf{x},t), \\ \nabla \cdot \mathbf{B}(\mathbf{x},t) = 0,\ \nabla &\cdot \mathbf{D}(\mathbf{r},t) = \rho(\mathbf{x},t), \mathbf{x} = (x_1, x_2, x_3) \end{aligned} \tag{2.1}$$

where

$\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3, t \in \mathbb{R}$,
$\mathbf{E} = (E_1, E_2, E_3)$ is the electric field,
$\mathbf{H} = (H_1, H_2, H_3)$ is the magnetic field,
$\mathbf{D} = (D_1, D_2, D_3)$ is the electric flux,
$\mathbf{B} = (B_1, B_2, B_3)$ is the magnetic flux,
$\mathbf{j}(\mathbf{x},t)$ is the vector of current density,
$\rho(\mathbf{x},t)$ is the charge density.

The current density $\mathbf{j}(\mathbf{x},t)$ and the charge density $\rho(\mathbf{x},t)$ are connected by the continuity equation

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0. \tag{2.2}$$

We will suppose that electromagnetic waves are produced by the moving source for which

$$\mathbf{j}(\mathbf{x},t) = \mathbf{A}(t)\delta(\mathbf{x}-\mathbf{x}^0(t)),$$

where $\mathbf{A}(t)$ is vector-valued amplitude.

For isotropic homogeneous dispersive media:

$$\hat{\mathbf{D}}(\omega,\mathbf{x})=\varepsilon(\omega)\hat{\mathbf{E}}(\omega,\mathbf{x}),\ \hat{\mathbf{B}}(\omega,\mathbf{x})=\mu(\omega)\hat{\mathbf{H}}(\omega,\mathbf{x}), \tag{2.3}$$

where

$$\hat{\mathbf{\Phi}}(\omega,\mathbf{x}) = \int_{\mathbb{R}} \mathbf{\Phi}(t,\mathbf{x})e^{i\omega t}dt$$

is a Fourier transform with respect to $t \in \mathbb{R}$ of the vector-function $\mathbf{\Phi}(t,\mathbf{x})$,$\varepsilon = \varepsilon(\omega)$ is electrical permittivity, $\mu = \mu(\omega)$ is magnetic permittivity depending on the frequency.

We suppose (see [16], Chap. IX) that:

(i) The functions $\varepsilon(\omega), \mu(\omega)$ are limits in the sense of the distributions of analytic bounded in the upper complex half-plane functions;

(ii) $k^2(\omega) = \omega^2\varepsilon(\omega)\mu(\omega)$ has a finite number $\omega_1 < ... < \omega_k$ of simple zeros on $\mathbb{R}$, and

$$\inf_{\omega\in\mathbb{R}\setminus[\omega_1-\epsilon,\omega_k+\epsilon]} k^2(\omega) > 0$$

for small enough $\epsilon > 0$.

(iii) the group velocity $v_g(\omega) = \frac{1}{k'(\omega)} > 0$ for all $\omega \in \mathbb{R}\setminus[\omega_1-\epsilon,\omega_k+\epsilon]$.

After Fourier transform and the standard manipulations (see for instance [24]) the Maxwell's system splits into two independent equations

$$\begin{aligned}\nabla^2\hat{\mathbf{E}}(\omega,\mathbf{x})+k^2(\omega)\hat{\mathbf{E}}(\omega,\mathbf{x})&=\varepsilon^{-1}(\omega)\nabla\hat{\rho}(\omega,\mathbf{x})-i\omega\mu(\omega)\hat{\jmath}(\omega,\mathbf{x}) \\ &= -i\omega\mu(\omega)(\hat{\jmath}(\omega,\mathbf{x})+\frac{1}{k^2(\omega)}\nabla(\nabla\cdot\hat{\jmath})(\omega,\mathbf{x})) = \mathbf{F}(\omega,\mathbf{x})\end{aligned} \tag{2.4}$$

$$\nabla^2\hat{\mathbf{H}}(\omega,\mathbf{x})+k^2(\omega)\hat{\mathbf{H}}(\omega,\mathbf{x})= -\nabla\times\hat{\jmath}(\omega,\mathbf{x})= \mathbf{\Phi}(\omega,\mathbf{x}) \tag{2.5}$$

In what follows we consider the propagation of transverse electric (TE) and transverse magnetic (TM) waves. The electric field of the TE-waves is directed in x_1x_2-plane. We will characterize the $TE-$waves to use the H_{x_3} component of the magnetic field, and suppose that $E_{x_3} = 0$, and TM-waves are characterized by E_{x_3} while $H_{x_3} = 0$. We suppose that the boundary of the waveguide Π is an ideal conductor. It implies the boundary conditions (see

$$E_z\mid_{\partial\Pi}= 0, \tag{2.6}$$

or

$$\frac{\partial H_z}{\partial\nu}\mid_{\partial\Pi}= 0,$$

where $\frac{\partial E_z}{\partial\nu}$ is the normal derivative of E_z at the point of the boundary.

The components $\mathbf{E}' = (E_{x_1},E_{x_2}),\mathbf{H}' = (H_{x_1},H_{x_2})$ of the electric and magnetic fields can be found from the expressions (E_z,H_z) (see for instance [24], page 339-340).

Equations (2.4) and (2.5) implies

$$\Delta\hat{E}_z(\omega,\mathbf{x})+k^2(\omega)\hat{E}_z(\omega,\mathbf{x})=\varepsilon^{-1}(\omega)\frac{\partial\hat{\rho}(\omega,\mathbf{x})}{\partial z}-i\omega\mu(\omega)\hat{\jmath}_z(\omega,\mathbf{x}) \tag{2.7}$$
$$=-i\omega\mu(\omega)(\hat{\jmath}_z(\omega,\mathbf{x})+\frac{1}{k^2(\omega)}\nabla(\nabla\cdot\hat{\jmath})_z(\omega,\mathbf{x}))=\hat{F}_z(\omega,\mathbf{x}),$$
$$\hat{E}_z\,|_{\partial\Pi}=0.$$

$$\mathbf{\Delta}\hat{H}_z(\omega,\mathbf{x})+k^2(\omega)\hat{H}_z(\omega,\mathbf{x})=-\nabla\times\hat{\jmath}(\omega,\mathbf{x})_z=\hat{\Phi}_z(\omega,\mathbf{x}), \tag{2.8}$$
$$\frac{\partial\hat{H}_z}{\partial\nu}\,|_{\partial\Pi}=0.$$

3 Waveguide Green function

First we consider the Green function for Helmholtz equation in the waveguide Π, that is the solution of the equation

$$-\left(\Delta_{\mathbf{x}}+k^2(\omega)\right)g(\omega,\mathbf{x}',\mathbf{x}_0',z)=\delta(\mathbf{x}'-\mathbf{x}_0')\delta(z),\mathbf{x}=(\mathbf{x}',z)\in\Pi \tag{3.1}$$

where $\mathbf{x}_0'\in\mathcal{D}$ with the Dirichlet condition

$$g\,|_{\partial\Pi}=0, \tag{3.2}$$

and the Neumann condition

$$\frac{\partial g}{\partial\nu}\,|_{\partial\Pi}=0. \tag{3.3}$$

We consider the spectral Dirichlet problem in the cross-section $\mathcal{D}$ of Π

$$\mathcal{B}_{\mathcal{D}}\varphi(x')=\begin{cases}-\Delta_{\mathbf{x}'}\varphi(\mathbf{x}')=\alpha^2\varphi(\mathbf{x}'),\mathbf{x}'\in\mathcal{D},\\ \varphi\,|_{\partial\mathcal{D}}=0.\end{cases} \tag{3.4}$$

It is well known that the spectral problem (3.4) has a positive discrete spectrum

$$sp\mathcal{B}_{\mathcal{D}}=\left\{0<\alpha_1^2<\alpha_2^2<\ldots<\alpha_m^2<\ldots<\right\}$$

in the space $L^2(\mathcal{D})$ and the orthonormal base in $L^2(\mathcal{D})$ of the eigenfunctions $\left\{\varphi_j\right\}_{j=1}^{\infty}$

$$\mathcal{B}_{\mathcal{D}}\varphi_j(\mathbf{x}')=\alpha_j^2\varphi_j(\mathbf{x}'),\mathbf{x}'\in\mathcal{D}. \tag{3.5}$$

We will find a solution of the equation (3.1) in the form

$$g(\omega,\mathbf{x}',\mathbf{x}_0',z)=\sum_{j=1}^{\infty}\psi_j(\omega,z,\mathbf{x}_0')\varphi_j(\mathbf{x}'). \tag{3.6}$$

Substituting $g(\omega,\mathbf{x}',\mathbf{x}_0',z)$ in (3.1) we obtain

$$\sum_{j=1}^{\infty}\left[-\frac{d^2}{dz^2}+(\alpha_j^2-k^2(\omega))\right]\psi_j(\omega,z,\mathbf{x}_0')\varphi_j(\mathbf{x}')=\delta(\mathbf{x}'-\mathbf{x}_0')\delta(z). \tag{3.7}$$

Since the system $\{\varphi_j\}_{j=1}^{\infty}$ is orthonormal in the space $L^2(\mathcal{D})$ we obtain

$$\left[-\frac{d^2}{dz^2}+(\alpha_j^2-k^2(\omega))\right]\psi_j(\omega,z,\mathbf{x}_0')=\left(\int_{\mathcal{D}}\delta(\mathbf{x}'-\mathbf{x}_0')\varphi_j(\mathbf{x}')d\mathbf{x}'\right)\delta(z) \qquad (3.8)$$
$$=\varphi_j(\mathbf{x}_0')\delta(z).$$

Solution of the equation (3.8) satisfying the limiting absorption principle is

$$\psi_j(\omega,z,\mathbf{x}_0')=\frac{e^{i\sqrt{k^2(\omega)-\alpha_j^2}|z|}}{2i\sqrt{k^2(\omega)-\alpha_j^2}}\varphi_j(\mathbf{x}_0'). \qquad (3.9)$$

Hence

$$g(\omega,\mathbf{x}',\mathbf{x}_0',z)=\sum_{j=1}^{\infty}\frac{e^{i\sqrt{k^2(\omega)-\alpha_j^2}|z|}}{2i\sqrt{k^2(\omega)-\alpha_j^2}}\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'). \qquad (3.10)$$

In what follows we are of interesting the members of the row (3.10) which are oscillating at infinity, that is the members for which j is such that the condition

$$\alpha_j^2<k^2(\omega) \qquad (3.11)$$

holds. We set

$$\mu_j(\omega)=\sqrt{k^2(\omega)-\alpha_j^2}$$

and we say that the $\Omega_j>0$ is a critical frequency of the waveguide if $\mu_j(\omega)>0$ for all $\omega>\Omega_j$. Note that the function $\mu_j(\omega)$ is monotonically increasing on the segment $[\Omega_j,+\infty)$ because $k(\omega)$ is a monotonically increasing function.

Hence if we are restricted by the propagated modes in the expression for the Green function $g(\omega,\mathbf{x}',\mathbf{x}_0',z)$ we will write the propagated Green function as

$$g_{prop}^{E_z}(\omega,\mathbf{x}',\mathbf{x}_0',z)=\sum_{j=1}^{\infty}\frac{e^{i\mu_j(\omega)|z|}}{2i\mu_j(\omega)}\Theta_{(\Omega_j,+\infty)}(\omega)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'), \qquad (3.12)$$

where $\Theta_{(\Omega_j,+\infty)}(\omega)$ is the Heaviside function

$$\Theta_{(\Omega_j,+\infty)}(\omega)=\begin{cases}1,\omega>\Omega_j\\0,\omega\le\Omega_j\end{cases}.$$

The similar situation holds for the Neumann problem. In this case he eigenvalues are $0\le\beta_1^2<\beta_2^2<...<\beta_n^2<...<$ and we obtain the decomposition of the Green function for the Neumann problem

$$g_{prop}^{H_z}(\omega,\mathbf{x}',\mathbf{x}_0',z)=\sum_{j=1}^{\infty}\frac{e^{i\nu_j(\omega)|z|}}{2i\nu_j(\omega)}\Theta_{(\Omega_j,+\infty)}(\omega)\phi_j(\mathbf{x}')\phi_j(\mathbf{x}_0'), \qquad (3.13)$$

where $\nu_j(\omega)=\sqrt{k^2(\omega)-\beta_j^2}$ where $\phi_j(\mathbf{x}')$ are eigenfunctions of the Neumann problem for the Laplacian Δ in $\mathcal{D}$.

4 Dynamic problem

We consider the equation

$$\left(\Delta + k^2(\omega)\right)\hat{E}_z(\omega,\mathbf{x}) = -\hat{F}_z(\omega,\mathbf{x}), \mathbf{x} \in \Pi, \qquad (4.1)$$
$$\hat{E}_z(\omega,\mathbf{x})\mid_{\partial\Pi} = 0.$$

The solution of the problem (4.1) is given as

$$\hat{E}_z(\omega,\mathbf{x}) = \int_\Pi g(\omega,\mathbf{x}',\mathbf{x}'_0,z-z_0)\hat{F}_z(\omega,\mathbf{x}'_0,z_0)d\mathbf{x}'_0 dz_0. \qquad (4.2)$$

Formula (4.2) implies that

$$E_z(t,\mathbf{x}',z) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{-i\omega(t-\tau)}\left(\int_\Pi g(\omega,\mathbf{x}',\mathbf{x}'_0,z-z_0)F_z(\tau,\mathbf{x}'_0,z_0)d\mathbf{x}'_0 dz_0\right)d\omega d\tau. \qquad (4.3)$$

Let

$$F_z(t,\mathbf{x}) = A(t)\delta(\mathbf{x}'-\mathbf{x}'_0(t))\delta(z-z_0(t)), \qquad (4.4)$$

describes a source moving in the waveguide Π, where $\mathbf{x}'_0(t) = (x_{01}(t),x_{02}(t)) \in \mathcal{D}, z_0(t) \in \mathbb{R}$ for every $t \in \mathbb{R}$. Applying (4.4) we obtain that

$$E_z(t,\mathbf{x}',z) = \frac{1}{2\pi}\iint\limits_{\mathbb{R}^2} e^{-i\omega(t-\tau)} g(\omega,\mathbf{x}',\mathbf{x}'_0(\tau),z_0(\tau))d\omega d\tau.$$

Taking into account the propagated modes only, we obtain

$$E_{z,prop}(t,\mathbf{x}',z) = \sum_{j=1}^{\infty}\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2}\frac{A(\tau)e^{i(\mu_j(\omega)|z-z_0(\tau)|-\omega(t-\tau))}}{2i\mu_j(\omega)}\Theta_{(\Omega_j,+\infty)}(\omega)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}'_0(\tau))d\omega d\tau. \qquad (4.5)$$

The similar way we obtain

$$H_{z,prop}(t,\mathbf{x}',z) = \sum_{j=1}^{\infty}\frac{1}{2\pi}\iint\limits_{\mathbb{R}^2}\frac{A_1(\tau)e^{i(\mu_j(\omega)|z-z_0(\tau)|-\omega(t-\tau))}}{2i\mu_j(\omega)}\Theta_{(\Omega_j,+\infty)}(\omega)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}'_0(\tau))d\omega d\tau, \qquad (4.6)$$

where

$$\Phi_z(t,\mathbf{x}) = A_1(t)\delta(\mathbf{x}'-\mathbf{x}'_0(t))\delta(z-z_0(t)) \qquad (4.7)$$

5 Asymptotic analysis of modes generated by moving source

We introduce a parameter λ as

$$\lambda = \inf_{\tau\in\mathbb{R}}|z-z_0(\tau)|\,\frac{\omega^0}{c_0}, \qquad (5.1)$$

where c_0 is the light speed in the vacuum, $\omega^0 > 0$ is a frequency scale of the problem and suppose that $\lambda >> 1$ is the large dimensionless parameter.

Let

$$A(t) = a(t)e^{-i\omega_0 t}, \tag{5.2}$$

where $\omega_0 \geq 0$ is a support frequency of the source, $a(t) = \tilde{a}(t/\lambda)$, where $\tilde{a}(t)$ is an infinitely differentiable function bounded with all derivatives,

$$z_0(t) = \frac{1}{\lambda} Z_0(\frac{t}{\lambda}), \tag{5.3}$$

where all derivatives of $Z_0(t)$ are bounded,

$$\mathbf{x}_0'(t) = \mathbf{X}_0'(\frac{t}{\lambda}), \tag{5.4}$$

$\mathbf{X}_0'(t)$ is bounded function with all derivatives.

Formulas (5.2), (5.3), (5.49) demonstrate that $a(t)$ and $\mathbf{x}_0'(t)$ are slowly varying because

$$[a(t)]' = \frac{1}{\lambda}\tilde{a}(\frac{t}{\lambda}), \dot{\mathbf{x}}_0'(t) = \frac{1}{\lambda}\frac{d\mathbf{X}_0'}{dt}(\frac{t}{\lambda}), \tag{5.5}$$

and $\dot{z}_0(t)$ is slowly varying because

$$\ddot{z}_0(t) = \frac{1}{\lambda}\frac{d^2 Z_0}{dt^2}(\frac{t}{\lambda}). \tag{5.6}$$

After substitution of (5.2), (5.3), (5.4) in (4.5) and the scale change of variables

$$z = \lambda Z, t = \lambda T, \tau = \lambda \iota.$$

we obtain

$$\tilde{E}^{\lambda}_{prop}(T, \mathbf{x}', Z)$$
$$= \sum_{j=1}^{\infty} \frac{\lambda}{2\pi} \iint_{\mathbb{R}^2} \frac{\tilde{a}(\iota)e^{i\lambda(\mu_j(\omega)|Z-Z_0(\iota)|-\omega(T-\iota)-\omega_0\iota)}}{2i\mu_j(\omega)} \Theta_{(\Omega_j,+\infty)}(\omega)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'(\iota))d\omega d\iota.$$

Let

$$\tilde{E}^{\lambda}_j(T, \mathbf{x}', Z) = \frac{\lambda}{2\pi} \iint_{\mathbb{R}^2} \frac{\tilde{a}(\iota)e^{i\lambda(\mu_j(\omega)|Z-Z_0(\iota)|-\omega(T-\iota)-\omega_0\iota)}}{2i\mu_j(\omega)} \Theta_{(\Omega_j,+\infty)}(\omega)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'(\iota))d\omega d\iota. \tag{5.7}$$

We will investigate the asymptotics of the function $E^{\lambda}_j(T, \mathbf{x}', Z)$ for fix $(T, \mathbf{x}', Z)$ and $\lambda \to +\infty$ applying the 2-dimensional stationary phase method (see [4], [8]). Let

$$\tilde{S}_j(T, Z, \omega, \iota) = \mu_j(\omega)|Z - Z_0(\iota)| - \omega(T - \iota) - \omega_0\iota$$

the phase of the double integral (5.7). The stationary points of the phase are solutions of the system of the equations

$$\frac{\partial \tilde{S}_j(T, Z, \omega, \iota)}{\partial \omega} = \frac{|Z - Z_0(\iota)|}{v^g_j(\omega)} - (T - \iota) = 0, \tag{5.8}$$

$$\frac{\partial \tilde{S}_j(T, Z, \omega, \iota)}{\partial \iota} = -\mu_j(\omega)\dot{Z}_0(\iota)sgn(Z - Z_0(\iota)) + \omega - \omega_0 = 0,$$

where $\mu_j(\omega)$ is a wave number, and $v_j^g(\omega)$ is the group velocity of the mode with number j, that is

$$v_j^g(\omega) = \frac{1}{\mu_j'(\omega)}.$$

Let $(\omega_s^j = \omega_s^j(T,Z), \iota_s^j = \iota_s^j(T,Z))$ be a stationary point, that is a solution of system (5.8). We suppose that this point is non degenerate, that is

$$\det \tilde{S}''(T,Z,\omega_s^j,\iota_s^j) \neq 0,$$

where

$$\tilde{S}_j''(T,Z,\omega,\iota) = \begin{pmatrix} \frac{\partial^2 \tilde{S}_j(T,Z,\omega,\iota)}{\partial \omega^2} & \frac{\partial^2 \tilde{S}_j(T,Z,\omega,\iota)}{\partial \omega \partial \iota} \\ \frac{\partial^2 \tilde{S}_j(T,Z,\omega,\iota)}{\partial \omega \partial \iota} & \frac{\partial^2 \tilde{S}_j(T,Z,\omega,\iota)}{\partial \iota^2} \end{pmatrix}$$

is the Hess matrix of the phase.

Let $sgn \tilde{S}_j''(T,Z,\omega_s,\iota_s)$ be the difference between the number of positive and negative eigenvalues of the matrix $\tilde{S}_j''(T,Z,\omega_s,\iota_s)$. Then according to the two dimensional stationary phase method the contribution of the stationary point (ω_s,ι_s) in the asymptotics of $E_j^\lambda(T,\mathbf{x}',Z)$ is given by the formula

$$\begin{aligned} &\tilde{E}_j^\lambda(T,\mathbf{x}',Z) \qquad (5.9)\\ &= \frac{\tilde{a}(\iota_s) e^{i\lambda \tilde{S}_j(T,Z,\omega_s^j,\iota_s^j) + i\frac{\pi}{4} sgn \tilde{S}_j''(T,Z,\omega_s^j,\iota_s^j)}}{2i\mu_j(\omega_s^j) \left|\det \tilde{S}_j''(T,Z,\omega_s^j,\iota_s^j)\right|^{1/2}} \Theta_{(\Omega_j,+\infty)}(\omega_s^j) \varphi_j(\mathbf{x}') \varphi_j(\mathbf{x}_0'(\iota_s^j)) \\ &\times (1 + O(\frac{1}{\lambda})) \end{aligned}$$

Note that the main term of the asymptotics $E_j^\lambda(T,\mathbf{x}',Z)$ does not equal 0 if $\omega_s^j > \Omega_j$ only.

The asymptotics of the field $E_{prop}^\lambda(T,\mathbf{x}',Z)$ for $\lambda \to +\infty$ is given by the formula

$$\tilde{E}_{prop}^\lambda(T,\mathbf{x}',Z) = \sum_{j=1}^N \tilde{E}_j^\lambda(T,\mathbf{x}',Z),$$

where N is the number of propagated modes, that is N is a maximal number such that $\omega_s^j \in (\Omega_j,+\infty)$, for $j \le 1 \le N$.

For the further calculations it is convenient to come back to the original notations

$$E_{prop}(t,\mathbf{x}',z) \sim \sum_{j=1}^N E_j(t,\mathbf{x}',z), \qquad (5.10)$$

where

$$\begin{aligned} &E_j(t,\mathbf{x}',z) \qquad (5.11)\\ &= \frac{a(\tau_s) e^{i\lambda S_j(t,z,\omega_s^j,\tau_s^j) + i\frac{\pi}{4} sgn S_j''(t,z,\omega_s^j,\tau_s^j)}}{2i\mu_j(\omega_s^j) \left|\det S_j''(t,z,\omega_s^j,\tau_s^j)\right|^{1/2}} \varphi_j(\mathbf{x}') \varphi_j(\mathbf{x}_0'(\iota_s^j)), \end{aligned}$$

and

$$S_j(t,z,\omega,\tau) = \mu_j(\omega)|z - z_0(\tau)| - \omega(t-\tau) - \omega_0\tau, \tag{5.12}$$

the pair (ω_s^j, τ_s^j) is the solution of the system

$$\frac{\partial S_j(t,z,\omega,\tau)}{\partial \omega} = \frac{|z - z_0(\tau)|}{v_j^g(\omega)} - (t-\tau) = 0, \tag{5.13}$$

$$\frac{\partial S_j(t,z,\omega,\tau)}{\partial \tau} = -\mu_j(\omega)\dot{z}_0(\tau) sgn(z - z_0(\tau)) + \omega - \omega_0 = 0,$$

$$S_j''(t,z,\omega,\tau) = \begin{pmatrix} \frac{\partial^2 \tilde{S}_j(t,z,\omega,\tau)}{\partial \omega^2} & \frac{\partial^2 \tilde{S}_j(t,z,\omega,\tau)}{\partial \omega \partial \tau} \\ \frac{\partial^2 \tilde{S}_j(t,z,\omega,\tau)}{\partial \omega \partial \tau} & \frac{\partial^2 \tilde{S}_j(t,z,\omega,\tau)}{\partial \tau^2} \end{pmatrix}$$

is the Hess matrix with respect to (ω, τ).

Note that the stationary points (ω_s^j, τ_s^j) have explicit physical sense: $\omega_s^j = \omega_s^j(t,z)$ is the instantaneous frequency of the wave process defined by the mode $E_j(t, \mathbf{x}', z)$ and $\tau_s^j = \tau_s^j(t,z)$ is the radiation time of a mode arrived to the source at the time t.

5.1 Motion with a constant horizontal velocity

Let the sources move with a constant horizontal velocity, that is

$$\mathbf{x}(t) = (\mathbf{X}'(\frac{1}{\lambda}t), vt).$$

The the system (5.8) is of the form

$$\frac{\partial \tilde{S}_j(T,Z,\omega,\iota)}{\partial \omega} = \frac{|Z - v\iota|}{v_j^g(\omega)} - (T - \iota) = 0, \tag{5.14}$$

$$\frac{\partial \tilde{S}_j(T,Z,\omega,\iota)}{\partial \iota} = \mp\mu_j(\omega)v + \omega - \omega_0 = 0, \tag{5.15}$$

Equations (5.15) are independent of ι and under condition

$$\sup_{\omega > \Omega_j} \frac{|v|}{v_j^g(\omega)} < 1 \tag{5.16}$$

that is the horizontal velocity is smaller then the minimum of the group velocity, has the unique solution which can be find by the method of successive approximations.

Consider now the equation (5.14) where $\omega = \omega_s^j$. Then

$$\frac{|Z - v\iota|}{v_j^g(\omega_s^j)} - (T - \iota) = 0. \tag{5.17}$$

This equation under condition (5.16) has unique solution ι_s^j which also can be find by the method of successive approximations.

Note that $\frac{\partial^2 \tilde{S}_j(T,Z,\omega,\iota)}{\partial\iota^2} = 0$. Hence

$$\left|\det \tilde{S}_j''(T,Z,\omega_s^j,\iota_s^j)\right|^{1/2} = \left|1 \pm \frac{v}{v_j^g(\omega_s^j)}\right|,$$

and

$$sgn \tilde{S}_j''(T,Z,\omega_s^j,\iota_s^j) = 0.$$

Hence

$$\tilde{E}_j^\lambda(T,\mathbf{x}',Z) \tag{5.18}$$
$$= \frac{\tilde{a}(\iota_s)e^{i\lambda\tilde{S}_j(T,Z,\omega_s^j,\iota_s^j)}}{2i\mu_j(\omega_s^j)\left|1 \pm \frac{v}{v_j^g(\omega_s^j)}\right|}\Theta_{(\Omega_j,+\infty)}(\omega_s^j)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'(\iota_s^j))$$
$$\times(1+O(\frac{1}{\lambda})),$$

and coming back to the old coordinates we $(t,\mathbf{x}',z)$ we obtain

$$E_j(t,\mathbf{x}',z) \tag{5.19}$$
$$\sim \frac{a(\tau_s^j)e^{i\lambda S_j(t,z,\omega_s^j,\tau_s^j)}}{2i\mu_j(\omega_s^j)\left|1 \pm \frac{v}{v_j^g(\omega_s^j)}\right|}\Theta_{(\Omega_j,+\infty)}(\omega_s^j)\varphi_j(\mathbf{x}')\varphi_j(\mathbf{x}_0'(\tau_s^j)),$$

where

$$S_j(t,z,\omega,\tau) = \mu_j(\omega)\,|z - v\tau)| - \omega(t-\tau) - \omega_0\tau,$$

ω_s^j are solutions of the equation

$$\mp\mu_j(\omega)v + (\omega-\omega_0) = 0, \tag{5.20}$$

and τ_s^j are solutions of the equation

$$\frac{|z - v\tau|}{v_j^g(\omega_s^j)} - (t-\tau) = 0. \tag{5.21}$$

5.2 Motion in plasma waveguides

We consider a lossless no magnetized plasma whose the collision frequency equals to zero (see for instance [11], [22], [23]). Hence the constitutive parameters in plasma are

$$\varepsilon(\omega) = \varepsilon_0(1 - \frac{\omega_p^2}{\omega^2}), \mu = \mu_0, \tag{5.22}$$

ε_0, μ_0 are the electric and magnetic permittivity of the vacuum,

$$\omega_p = \sqrt{\frac{4\pi e^2 N}{m}} = (5.64\cdot 10^4\sqrt{N/(sm)^3})\frac{1}{\text{sec}} \tag{5.23}$$

is the *plasma frequency*, where N is the *particle density concentration of electrons*, m, e are the mass and charge of the electron.

In the ionosphere

$$10^3 \frac{1}{cm^3} \leq N \leq 3 \cdot 10^6 \frac{1}{cm^3} \tag{5.24}$$

Graphic of N is given in Fig.1. Note that in the accelerator of the particles $N \simeq 10^8 \frac{1}{sm^3}$.

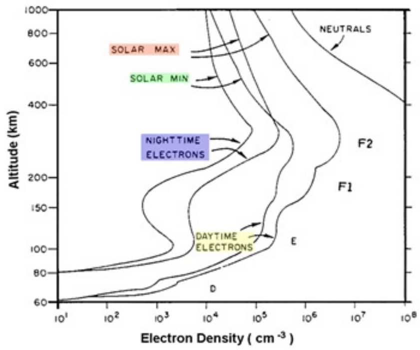

Fig. 1. Graphic of N.

Phase velocity in the plasma is

$$c(\omega) = \frac{c_0}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}, \tag{5.25}$$

where c_0 is the light speed in the vacuum, and the wave-number is

$$k(\omega) = \frac{\sqrt{\omega^2 - \omega_p^2}}{c_0}. \tag{5.26}$$

The group velocity in the plasma

$$v_g(\omega) = \frac{1}{k'(\omega)} = c_0 \sqrt{1 - \frac{\omega_p^2}{\omega^2}} \tag{5.27}$$

where $c_0 = 2.997 \cdot 10^8 \frac{m}{\sec}$ is the light speed in the vacuum. Hence the phase velocity in the plasma larger than c_0, and the group velocity is smaller than c_0 if $\omega > \omega_p$.

The mode wave number in the plasma waveguide is

$$\mu_j(\omega) = \sqrt{\frac{\omega^2 - \omega_p^2}{c_0^2} - \alpha_j^2} = \frac{\sqrt{\omega^2 - \Omega_j^2}}{c_0}, \tag{5.28}$$

where $\Omega_j = \sqrt{\omega_p^2 + \alpha_j^2 c_0^2}$ is the critical frequency of the $j-$mode, and the group velocity of the $j-$mode is

$$v_g^j(\omega) = \frac{1}{\mu_j'(\omega)} = c_0 \sqrt{1 - \frac{\Omega_j^2}{\omega^2}}. \tag{5.29}$$

Hence the phase velocity of the $j-$mode

$$\frac{\omega}{\mu_j(\omega)} = \frac{c_0}{\sqrt{1-\frac{\Omega_j^2}{\omega^2}}} > c_0$$

is larger than the light speed in the vacuum if $\omega > \Omega_j$ while the group velocity $v_g^j(\omega) < c_0$ if $\omega > \Omega_j$.

Note that in the case of a rectangular waveguide

$$\Pi = \mathcal{D}\times\mathbb{R}$$

where

$$\mathcal{D} = [0,h_1]\times[0,h_2]$$

the spectrum of the Dirichlet problem in $\mathcal{D}$ is

$$\alpha_{\mathbf{j}}^2 = \left\{\pi^2(\frac{j_1^2}{h_1^2}+\frac{j_2^2}{h_2^2}), \mathbf{j} = (j_1, j_2) \in \mathbb{N}^2\right\}, \tag{5.30}$$

and normed in $L^2(\mathcal{D})$ eigenfunctions are

$$\varphi_{\mathbf{j}}(x_1,x_2) = 2\,(h_1h_2)^{-\frac{1}{2}} \sin\frac{\pi j_1 x_1}{h_1} \sin\frac{\pi j_2 x_2}{h_2}, \tag{5.31}$$
$$\mathbf{j} = (j_1, j_2) \in \mathbb{N}\times\mathbb{N}$$

We consider the electric waves generated by the moving in the waveguide $\diamond$ source of the form

$$F_z(t,\mathbf{x}) = = a(t)e^{-i\omega_0 t}\delta(\mathbf{x}'-\mathbf{x}_0'(t))\delta(z-vt),$$
$$\delta(\mathbf{x}'-\mathbf{x}_0'(t)) = \delta(x_1-x_{10}(t))\delta(x_2-x_{20}(t))$$

where the functions $x_{10}(t), x_{20}(t)$, and $a(t)$ are slowly varying, and

$$0 < x_{10}(t) < h_1, 0 < x_{20}(t) < h_2.$$

According formulas (5.19) we obtain

$$E_{\mathbf{j}}(t,\mathbf{x}',z) \tag{5.32}$$
$$\sim \frac{a(\tau_s^{\mathbf{j}})e^{i\lambda S_{\mathbf{j}}(t,z,\omega_s^{\mathbf{j}},\tau_s^{\mathbf{j}})}}{2i\mu_{\mathbf{j}}(\omega_s^{\mathbf{j}})\left|1\pm\frac{v}{v_j^g(\omega_s^{\mathbf{j}})}\right|}\Theta_{(\Omega_{\mathbf{j}},+\infty)}(\omega_s^{\mathbf{j}})\varphi_{\mathbf{j}}(\mathbf{x}')\varphi_{\mathbf{j}}(\mathbf{x}_0'(\tau_s^{\mathbf{j}})),$$

where

$$S_{\mathbf{j}}(t,z,\omega,\tau) = \mu_{\mathbf{j}}(\omega)\,|z-v\tau)| - \omega(t-\tau) - \omega_0\tau, \mathbf{j} = (j_1, j_2),$$

$$\mu_j(\omega) = \sqrt{\frac{\omega^2 - \omega_p^2}{c_0^2} - \alpha_j^2} = \frac{\sqrt{\omega^2 - \Omega_j^2}}{c_0}$$

$$\alpha_{\mathbf{j}}^2 = \alpha_{j_1,j_2}^2 = \pi^2\left(\frac{j_1^2}{h_1^2} + \frac{j_2^2}{h_2^2}\right),$$

$$v_g^j(\omega) = \frac{1}{\mu_j'(\omega)} = c_0\sqrt{1 - \frac{\Omega_j^2}{\omega^2}}, \mathbf{j} = (j_1, j_2) \in \mathbb{N}\times\mathbb{N},$$

$(\omega_s^{\mathbf{j}}, \tau_s^{\mathbf{j}})$ are solutions of the system

$$\mp\mu_{\mathbf{j}}(\omega)v + (\omega - \omega_0) = 0, \mathbf{j} = (j_1, j_2) \in \mathbb{N}\times\mathbb{N},$$

$$\frac{|z - v\tau|}{v_{\mathbf{j}}^g(\omega_s^{\mathbf{j}})} - (t - \tau) = 0.$$

6 Numerical Examples and Graphics

Here is a numerical example of motion in plasma waveguides. We graph some equations and parameters like critical frequencies, dispersion curves, group velocities, instant frequencies, doppler effect and the field E.

For the graphs we take the particle density concentration of electrons of $N = 10^4 \frac{1}{cm^3}$ and a total of 4 eigenvalues with $h_1 = h_2 = 5m$ as a dimensions of the rectangular waveguide.

In the Figure 2 we graph the critical frequencies, we have 4 values: $2.3730 * 10^8$, $9.4670 * 10^8$, $2.1298 * 10^9$, and $3.7862 * 10^9$. The plasma frequency ω_p for $N = 10^4 \frac{1}{cm^3}$ is $1.7835 * 10^7$.

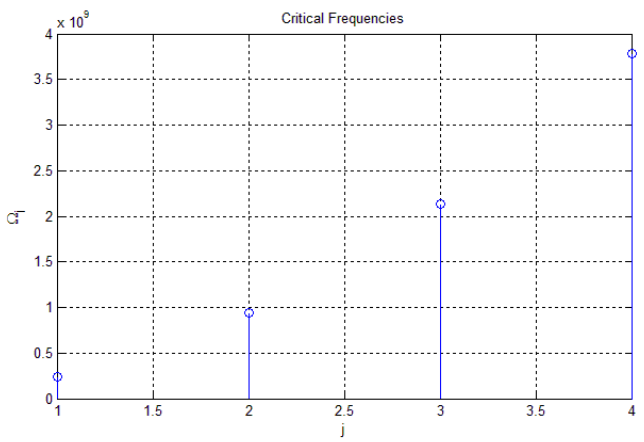

Fig. 2. Critical Frequencies $\Omega_j = \sqrt{\omega_p^2 + \alpha_j^2 c_0^2}$

In the figure 3 we graph the dispersion curves of the equation $\mu_{\mathbf{j}}(\omega) = \frac{\sqrt{\omega^2 - \Omega_j^2}}{c_0}$, each curve corresponds to each eigenvalue and to each critical frequency.

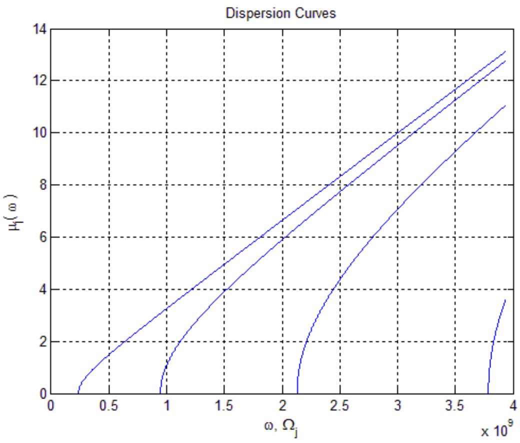

Fig. 3. Dispersion Curves $\mu_{\mathbf{j}}(\omega) = \frac{\sqrt{\omega^2 - \Omega_j^2}}{c_0}$

In the figure 4 we graph the group velocity of the j–mode $v_g^{\mathbf{j}}(\omega) = \frac{1}{\mu'_{\mathbf{j}}(\omega)} = c_0 \sqrt{1 - \frac{\Omega_j^2}{\omega^2}}$ where $\Omega_j = \sqrt{\omega_p^2 + \alpha_j^2 c_0^2}$ is the critical frequency of the j–mode, with $j = 4$ modes, each curve corresponds to its critical frequency mentioned in the figure 2. As we can see the group velocities are near to the light velocity but never are biggest that it.

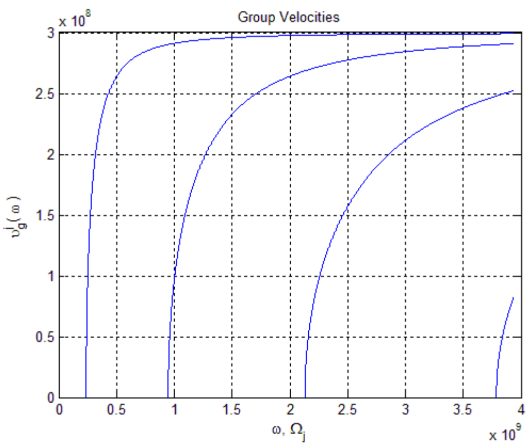

Fig. 4. Group Velocities $v_g^{\mathbf{j}}(\omega) = \frac{1}{\mu'_{\mathbf{j}}(\omega)} = c_0 \sqrt{1 - \frac{\Omega_j^2}{\omega^2}}$

In the figure 5 we graph the instants frequencies $\omega_s^{\mathbf{j}}$ of the wave process as the solution of the equation $\mp\mu_{\mathbf{j}}(\omega)v + (\omega - \omega_0) = 0$ with respect to ω, where each group of 4 curves (color) corresponds to a different ω_0, as we can see we made three groups of graphs with the values of ω_0 of $1.8 * 10^{10}$, $1.9 * 10^{10}$, and $1.7 * 10^{10}$. It shows that there is two branches for each curve because the double sign of the equation $\mp\mu_{\mathbf{j}}(\omega)v + (\omega - \omega_0) = 0$.

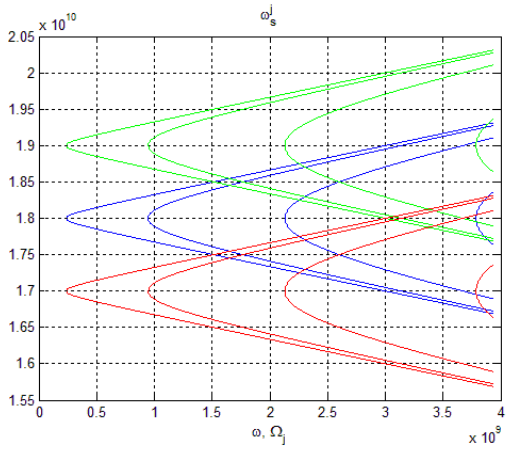

Fig. 5. $\omega_s^{\mathbf{j}}$ as the solution of the equation $\mp\mu_{\mathbf{j}}(\omega)v + (\omega - \omega_0) = 0$ with respect to ω

In the figure 6 we graph the doppler effect as $\omega_s^{\mathbf{j}} - \omega_0$ with $\omega_0 = 1.7 * 10^{10}$, each curve corresponds to each critical frequency of each $j-$mode, and the curves have two branches because the double sign of the equation $\mp\mu_{\mathbf{j}}(\omega)v + (\omega - \omega_0) = 0$.

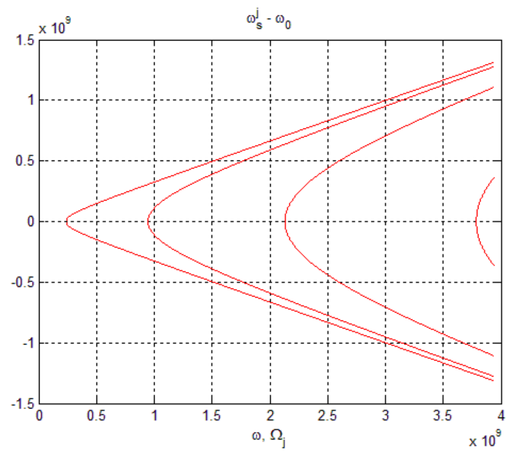

Fig. 6. Doppler effect.

Finally in the figure 7 we graph the field E of the real part of the equation

$$E_{\mathbf{j}}(t,\mathbf{x}',z) \sim \frac{a(\tau_s^{\mathbf{j}})e^{i\lambda S_{\mathbf{j}}(t,z,\omega_s^{\mathbf{j}},\tau_s^{\mathbf{j}})}}{2i\mu_{\mathbf{j}}(\omega_s^{\mathbf{j}})\left|1 \pm \frac{v}{v_j^g(\omega_s^{\mathbf{j}})}\right|}\Theta_{(\Omega_{\mathbf{j}},+\infty)}(\omega_s^{\mathbf{j}})\varphi_{\mathbf{j}}(\mathbf{x}')\varphi_{\mathbf{j}}(\mathbf{x}_0'(\tau_s^{\mathbf{j}})), \tag{6.1}$$

As we can see the field shows an oscillatory behavior and is a sum of the $j-$modes of the field, for the shown graph we have a source velocity of $1*10^8 m/s$ with an $a(\tau_s^{\mathbf{j}})$ sinusoidal.

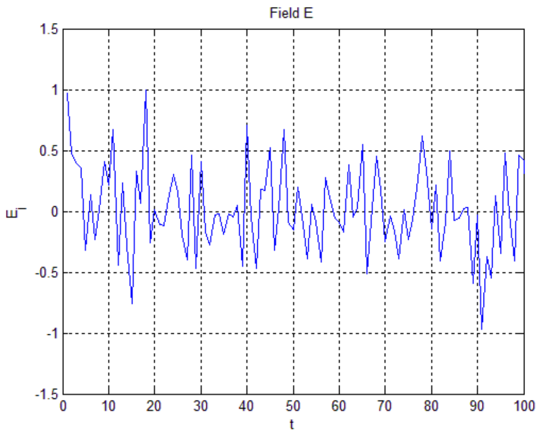

Fig. 7. Field E

Acknowledgments

Thanks to the referees for their careful reading of the manuscript and insightful comments.

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