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Sharp Maximal Function Estimates for Parameterized Littlewood-Paley Operators and Area Integrals

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Abstract

In this paper, we establish sharp maximal function estimates for parameterized Littlewood-Paley operators and area integrals with kernel satisfying the logarithmic type Lipschitz condition. As applications, the weighted L^p bounds and Morrey boundedness of these operators can be obtained.

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1 Introduction

Suppose that S^{n-1} is the unit sphere of $\mathbb{R}^n (n \geq 2)$ equipped with normalized Lebesgue measure. Let Ω be a homogeneous function of degree zero and

$$\int_{S^{n-1}} \Omega(x')d\sigma(x') = 0. \tag{1.1}$$

Then the parameterized area integral $\mu^{\rho}_{\Omega,S}$ and parameterized Littlewood-Paley operator $\mu^{*,\rho}_{\lambda}$ are defined by

$$\mu^{\rho}_{\Omega,S}(f)(x) = \Big(\iint_{\Gamma(x)} \Big|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}},$$

$$\mu^{*,\rho}_{\lambda}(f)(x) = \Big(\iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{t+|x-y|}\Big)^{\lambda n} \Big|\frac{1}{t^{\rho}} \int_{|y-z|<t} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f(z)dz\Big|^2 \frac{dydt}{t^{n+1}}\Big)^{\frac{1}{2}},$$

where $\rho > 0$, $\lambda > 1$ and $\Gamma(x) = \{(t, y) \in \mathbb{R}^{n+1}_+ : |x-y| < t\}$.

We define the Hilbert spaces as follows

$$\mathcal{H}_1 = \Big\{h : \|h\|_{\mathcal{H}_1} = \Big(\int_0^{\infty} \int_{|y|<1} |h(t,y)|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} < \infty\Big\},$$

$$\mathcal{H}_2 = \Big\{h : \|h\|_{\mathcal{H}_2} = \Big(\int_0^{\infty} \int_{\mathbb{R}^n} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} |h(t,y)|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} < \infty\Big\},$$

where $\lambda > 1$. Then

$$\mu^{\rho}_{\Omega,S}(f)(x) = \Big(\int_0^{\infty} \int_{|y|<1} \Big|\int t^{-n}\phi\Big(\frac{x-z}{t} - y\Big) f(z)dz\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} := \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_1},$$

$$\mu^{*,\rho}_{\lambda}(f)(x) = \Big(\int_0^{\infty} \int_{\mathbb{R}^n} \Big(\frac{1}{1+|y|}\Big)^{\lambda n} \Big|\int t^{-n}\phi\Big(\frac{x-z}{t} - y\Big) f(z)dz)\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} := \|\phi_{t,y}(f)(x)\|_{\mathcal{H}_2},$$

where $\phi(x) = \frac{\Omega(x)}{|x|^{n-\rho}} \chi_{\{x:|x|<1\}}$, $\phi_{t,y}(f)(x) = \int t^{-n}\phi(\frac{x-z}{t} - y)f(z)dz$.

For $p > 0$ and $f \in L^1_{loc}(\mathbb{R}^n)$, let

$$M_p f(x) = \sup_{x \in B} \Big(\frac{1}{|B|} \int_B |f(y)|^p dy\Big)^{\frac{1}{p}},$$

$$M^{\sharp}(f)(x) = \sup_{x \in B} \frac{1}{|B|} \int_B |f(y) - f_B| dy,$$

where $f_B = \frac{1}{|B|} \int_B f(y)dy$.

For a general positive function φ on $\mathbb{R}^n \times \mathbb{R}^+$, the generalized Morrey space $L^{p,\varphi}$ with $1 \leq p < \infty$ is defined as follows:

$$L^{p,\varphi} = \{f \in L^p_{loc}(\mathbb{R}^n),\ \|f\|_{L^{p,\varphi}} < +\infty\},$$

where

$$\|f\|_{L^{p,\varphi}} = \sup_{x\in\mathbb{R}^n,\ r>0}\left(\frac{1}{\varphi(x,\ r)}\int_{B(x,r)}|f(y)|^p dy\right)^{\frac{1}{p}}.$$

Inspired by Hörmander's work [4] on the parameterized Marcinkiewicz integral, the parameterized Littlewood-Paley g_λ^* function $\mu_\lambda^{*,\rho}$ and parameterized area integral $\mu_{\Omega,S}^\rho$ were discussed by Sakamoto and Yabuta [9] in 1999. In [9], the authors studied the $L^p(1<p<\infty)$ boundedness with the kernel satisfying the Lip_α condition. In 2002, Ding, Lu and Yabuta [2] proved the $L^p(2\le p<\infty)$ boundedness of $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with the kernel satisfying a weaker $Llog^+L(S^{n-1})$ condition.

Torchinsky and Wang [10] first considered the weighted L^p boundedness of $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with the kernel satisfying the Lip_α condition. In 1999, Ding, Fan and Pan [1] improved Torchinsky and Wang's result in [10] and gave the weighted L^p boundedness of $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ where $\Omega\in L^q(S^{n-1})(q>1)$. In 2002, Duoandikoetxea and Seijo [3] studied the weighed L^p boundedness of $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with rough kernel. In 2004, Xue [11] proved the weighed L^p boundedness of $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with Ω satisfying the L^2-Dini condition.

On the other hand, Lee and Rim [5], in 2004, established the logarithmic type Lipschitz condition

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C}{(\log\frac{2}{|y_2-y_1|})^\alpha}, \tag{1.2}$$

for any $y_1,y_2\in S^{n-1}$, where $\alpha>1$, and proved the type $(L^\infty,\ BMO)$ and $(L^p,\ L^p)$ boundedness of the Marcinkiewicz integral with the kernel satisfying the logarithmic condition. In 2012, Lin, Liu and Gao [6] gave the endpoint estimate and the following L^p result of $\mu_\lambda^{*,\rho}$.

Theorem 1.1. ([6]) *Let $n\ge 2$, $\Omega\in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1) *and there exist constants $C_0>0$ and $\alpha>\frac{3}{2}$, such that*

$$|\Omega(y_1)-\Omega(y_2)|\le\frac{C_0}{(\log\frac{2}{|y_1-y_2|})^\alpha}\ for\ any\ y_1,y_2\in S^{n-1},$$

then for $\rho>n/2$, $\lambda>2$ and $1<p<\infty$, there is

$$\|\mu_\lambda^{*,\rho}(f)\|_p\le C\|f\|_p.$$

Recently, the authors in [7] discussed the operators $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with the kernel satisfying (1.2) on the weak Hardy space.

Inspired by the above results, in this paper, we will establish the shap maximal function esimates for $\mu_\lambda^{*,\rho}$ and $\mu_{\Omega,S}^\rho$ with the kernel satisfying the logarithmic type Lipschitz condition (1.2) and discuss the weighted L^p boundedness and Morrey boundedness of these operators.

2 Some Lemmas

First, we give some necessary lemmas as follows.

Lemma 2.1. *Let $\Omega\in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha>\frac{3}{2}$. Then for $\rho>n/2$ and $1<p<\infty$, there is*

$$\|\mu_{\Omega,S}^\rho(f)\|_p\le C\|f\|_p.$$

Proof.

$$\begin{aligned}\mu^{\rho}_{\Omega,S}(f)(x) &= \left(\iint_{\Gamma(x)}\left(\frac{t}{t+|x-y|}\right)^{\lambda n}\left(\frac{t}{t+|x-y|}\right)^{-\lambda n}\left|\frac{1}{t^{\rho}}\int_{|y-z|<t}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f(z)dz\right|^{2}\frac{dydt}{t^{n+1}}\right)^{\frac{1}{2}}\\ &\le 2^{\lambda n}\mu^{*,\rho}_{\lambda}(f)(x).\end{aligned}$$

By Theorem 1.1, we have

$$||\mu^{\rho}_{\Omega,S}(f)||_{p} \le 2^{\lambda n}||\mu^{*,\rho}_{\lambda}(f)||_{p} \le C||f||_{p}.$$

□

Lemma 2.2. *As for $|y-z| \ge 4r$, there is*

$$\int_{|y-z|}^{\infty}\frac{(\log\frac{t}{r})^{2+2\epsilon}}{t^{2\rho-n+1}}dt \le C\frac{[\log(\frac{|y-z|}{r})]^{2+2\epsilon}}{(|y-z|)^{2\rho-n}},$$

where $r > 0$, $0 < \epsilon < \rho - \frac{n}{2}$ and $\rho > \frac{n}{2}$.

The proof of this lemma is similar to that of Lemma 2.1.2 in [11], so we omit the details.

Lemma 2.3. *There exists a constant $C > 0$ such that for any $z \in (8B^*)^c$, $|y-z| \ge 6r$,*

$$\left|\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}\right| \le \frac{C(1+|\Omega(y-z)|)}{|y-z|^{n-\rho}(\log\frac{|y-z|}{r})^{\alpha}},$$

where Ω satisfies the same conditions as in Theorem1.1; B is a ball with center at $\overline{x}$ and radius r_0; B^* is a ball with center at $\overline{x}$ and radius $r = 2r_0$ and x_0, $\omega \in B$.

The proof of this lemma is similar to that of Lemma 2.2 in [6], so we omit the details.

Lemma 2.4. ([8]) *Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exists $0 < C_0 < 2^n$ such that*

$$\varphi(x,2r) \le C_0\varphi(x,r) \; for \; all \; x \in \mathbb{R}^n, \; r > 0. \tag{2.1}$$

If $1 < p < \infty$, then

$$||Mf||_{L^{p,\varphi}} \le C||f||_{L^{p,\varphi}} \quad and \quad ||Mf||_{L^{p,\varphi}} \le C||f^{\sharp}||_{L^{p,\varphi}},$$

where C is independent of f.

Remark 2.5. As a matter of fact, the conditions of φ are stronger in [8] than here. However, just for the result of Lemma 2.4, the hypothesis here is sufficient.

3 Main Results

Now, we state our main results as follows.

Theorem 3.1. *Let $\Omega \in L^2(S^{n-1})$ be a homogeneous function of degree zero satisfying* (1.1) *and* (1.2) *with $\alpha > \frac{3}{2}$. Then for $\rho > n/2$, $\lambda > 2$ and $f \in L^p(\mathbb{R}^n)(1 < p < \infty)$,*

$$M^\sharp(\mu^\rho_{\Omega,S} f)(x) \le CM_p f(x), \quad for\ all\ x \in \mathbb{R}^n,$$

$$M^\sharp(\mu^{*,\rho}_\lambda f)(x) \le CM_p f(x), \quad for\ all\ x \in \mathbb{R}^n,$$

where C is a constant independent of f.

Applying the sharp maximal function estimates of $\mu^\rho_{\Omega,S}$ and $\mu^{*,\rho}_\lambda$, we can get the weighted L^p-boundedness and Morrey boundedness of these operators as follows.

Theorem 3.2. *Suppose that Ω satisfies the same conditions as in Theorem* 3.1 *and $\omega \in A_p$, then for $\rho > n/2$, $\lambda > 2$ and $1 < p < \infty$, there is*

$$\|\mu^\rho_{\Omega,S}(f)\|_{p,\omega} \le C\|f\|_{p,\omega}, \quad \|\mu^{*,\rho}_\lambda(f)\|_{p,\omega} \le C\|f\|_{p,\omega}.$$

Theorem 3.3. *Suppose that Ω satisfies the same conditions as in Theorem* 3.1. *Let φ be a positive function on $\mathbb{R}^n \times \mathbb{R}^+$ and suppose there exists $0 < C_0 < 2^n$ such that* (2.1) *holds. Then for $\rho > n/2$, $\lambda > 2$ and $1 < p < \infty$, there is*

$$\|\mu^\rho_{\Omega,S}(f)\|_{L^{p,\varphi}} \le C\|f\|_{L^{p,\varphi}}, \quad \|\mu^{*,\rho}_\lambda(f)\|_{L^{p,\varphi}} \le C\|f\|_{L^{p,\varphi}}.$$

4 Proof of Theorems

First, we give the proof of Theorem 3.1.

Proof. (1) First we give the estimate of $M^\sharp(\mu^\rho_{\Omega,S} f)(x)$.

Given $x \in \mathbb{R}^n$, let $B = B(\overline{x}, r_0)$ be a ball centered at $\overline{x}$ and radius r_0 with $x \in B$. Denote B^* a ball with center at $\overline{x}$ and radius $r = 2r_0$. Set

$$f = f\chi_{8B^*} + f(1-\chi_{8B^*}) := f_1 + f_2.$$

Then by Lemma 2.1,

$$\frac{1}{|B|}\int_B \mu^\rho_{\Omega,S}(f_1)(u)du \le \Big(\frac{1}{|B|}\int_B \mu^\rho_{\Omega,S}(f_1)^p(u)du\Big)^{\frac{1}{p}} \le C\Big(\frac{1}{|B|}\int_{\mathbb{R}^n}|f_1(u)|^p du\Big)^{\frac{1}{p}} \le CM_p f(x). \tag{4.1}$$

Since $f \in L^p$, and $\mu^\rho_{\Omega,S}$ is L^p bounded, then

$$\int_B |\mu^\rho_{\Omega,S}(f_2)(u)|du \le |B|^{\frac{1}{p'}}\Big(\int_B |\mu^\rho_{\Omega,S}(f_2)(u)|^p du\Big)^{\frac{1}{p}} \le C|B|^{\frac{1}{p'}}\Big(\int_{\mathbb{R}^n}|f(u)|^p du\Big)^{\frac{1}{p}}.$$

This shows that $\mu^\rho_{\Omega,S}(f_2)(u) < \infty$ a.e. on B, so except a subset E with measure zero, for all $u \in B\setminus E$, $\mu^\rho_{\Omega,S}(f_2)(u) < \infty$. Hence we can take a point $x_0 \in B\setminus E$, such that $\mu^\rho_{\Omega,S}(f_2)(x_0) < \infty$.

For any $\omega \in B\setminus E$, we consider $I = |\mu^{\rho}_{\Omega,S}(f_2)(x_0) - \mu^{\rho}_{\Omega,S}(f_2)(\omega)|$. We have

$$\begin{aligned}
I &= \Big|\|\phi_{t,y}(f_2)(x_0)\|_{\mathcal{H}_1} - \|\phi_{t,y}(f_2)(\omega)\|_{\mathcal{H}_1}\Big| \\
&\le \|\phi_{t,y}(f_2)(x_0) - \phi_{t,y}(f_2)(\omega)\|_{\mathcal{H}_1} \\
&= \Big(\int_0^\infty \int_{|y|<1} \Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big) - \phi\Big(\frac{\omega-z}{t}-y\Big)\Big) f_2(z)dz\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} \qquad (4.2)\\
&\le \Big(\int_0^\infty \int_{|y|<1} \Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\ |\frac{\omega-z}{t}-y|\ge 1}} t^{-n}\phi\Big(\frac{x_0-z}{t}-y\Big) f_2(z)dz\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^\infty \int_{|y|<1} \Big|\int_{\substack{|\frac{x_0-z}{t}-y|\ge 1\\ |\frac{\omega-z}{t}-y|<1}} t^{-n}\phi\Big(\frac{\omega-z}{t}-y\Big) f_2(z)dz\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^\infty \int_{|y|<1} \Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\ |\frac{\omega-z}{t}-y|<1}} t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big) - \phi\Big(\frac{\omega-z}{t}-y\Big)\Big) f_2(z)dz\Big|^2 \frac{dydt}{t}\Big)^{\frac{1}{2}}
\end{aligned}$$

Using the transform $y \to \frac{x_0-y'}{t}$ (we still use y instead y'), then

$$\begin{aligned}
I &\le \Big(\int_0^\infty \int_{|x_0-y|<t} \Big|\int_{\substack{|y-z|<t\\ |\omega-x_0+y-z|\ge t}} \frac{\Omega(y-z)}{|y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^\infty \int_{|x_0-y|<t} \Big|\int_{\substack{|y-z|\ge t\\ |\omega-x_0+y-z|<t}} \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} \\
&\quad + \Big(\int_0^\infty \int_{|x_0-y|<t} \Big|\int_{\substack{|y-z|<t\\ |\omega-x_0+y-z|<t}} \Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}}\Big) f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} \\
&:= I_1 + I_2 + I_3.
\end{aligned}$$

Take $0 < \varepsilon < \min\{\frac{1}{2}, \rho - \frac{n}{2}, \alpha - \frac{3}{2}, \frac{(\lambda-2)}{2}n\}$ (we always restrict ε satisfies this in the whole proof). As for I_1, by the Minkowski inequality we get

$$I_1 \le \int_{(8B^*)^c} |f(z)| \Big[\Big(\iint_{\substack{y\in 2B^*\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\ge t}} + \iint_{\substack{y\in(2B^*)^c\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\ge t}}\Big) \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\Big]^{\frac{1}{2}} \le I_{1.1} + I_{1.2},$$

where

$$I_{1.1} = \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{y\in 2B^*\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\ge t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz,$$

$$I_{1.2} = \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{y\in(2B^*)^c\\ |y-z|<t\\ |x_0-y|<t\\ |\omega-x_0+y-z|\ge t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}} dz.$$

For $I_{1.1}$, since $y \in 2B^*$, $z \in (8B^*)^c$, $|y-z| \sim |x_0-z| \sim |\omega-x_0+y-z|$, $|z-x_0| \sim |z-\overline{x}|$, we have

$$I_{1.1} \le \int_{(8B^*)^c} |f(z)| \Big(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-z|<t\le|\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy\Big)^{\frac{1}{2}} dz$$

$$
\begin{aligned}
&= \int_{(8B^*)^c} |f(z)| \Big(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \Big| \frac{1}{|\omega - x_0 + y - z|^{n+2\rho}} - \frac{1}{|y-z|^{n+2\rho}} \Big| dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} |f(z)| \Big(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} |f(z)| \Big(\int_{y\in 2B^*} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{2\rho-n+1-2\varepsilon}} \frac{1}{|z-x_0|^{2n+2\varepsilon}} dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon}} \Big(\int_{|y-z|>6r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \Big)^{\frac{1}{2}} dz \qquad (4.3) \\
&= C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon}} \Big(\int_{S^{n-1}} |\Omega(z')|^2 d\sigma(z') \int_{6r}^{\infty} \frac{r}{s^{n+1-2\varepsilon}} s^{n-1} ds \Big)^{\frac{1}{2}} dz \\
&\leq C r^{\varepsilon} \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon}} dz \qquad (4.4) \\
&\leq C r^{\varepsilon} \sum_{k=3}^{\infty} \int_{2^k r \leq |z-\overline{x}| < 2^{k+1} r} \frac{|f(z)|}{|z-\overline{x}|^{n+\varepsilon}} dz \\
&\leq C r^{\varepsilon} \sum_{k=3}^{\infty} \frac{1}{(2^k r)^{n+\varepsilon}} \int_{|z-\overline{x}|<2^{k+1} r} |f(z)| dz \\
&\leq C r^{\varepsilon} \sum_{k=3}^{\infty} \frac{1}{(2^k r)^{\varepsilon}} \left(\frac{1}{(2^{k+1} r)^n} \int_{|z-\overline{x}|<2^{k+1} r} |f(z)|^p dz \right)^{\frac{1}{p}} \\
&\leq C M_p f(x).
\end{aligned}
$$

Now we give the estimate of $I_{1.2}$.

$$
\begin{aligned}
I_{1.2} \leq\ & \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t \\ |x_o-y|<t,\ 2|y-z|\geq|z-x_0| \\ |\omega-x_0+y-z|\geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\
& + \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t \\ |x_o-y|<t,\ 2|y-z|<|z-x_0| \\ |\omega-x_0+y-z|\geq t}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\
:=\ & I_{1.2'} + I_{1.2''}.
\end{aligned}
$$

First we give the estimate of $I_{1.2'}$.

$$
\begin{aligned}
I_{1.2'} &\leq \int_{(8B^*)^c} |f(z)| \Big(\int_{\substack{y\in(2B^*)^c \\ 2|y-z|\geq|z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \int_{|y-z|<t\leq|\omega-x_0+y-z|} \frac{dt}{t^{n+2\rho+1}} dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} |f(z)| \Big(\int_{\substack{y\in(2B^*)^c \\ 2|y-z|\geq|z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{n+2\rho+1}} dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} |f(z)| \Big(\int_{\substack{y\in(2B^*)^c \\ 2|y-z|\geq|z-x_0|}} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} \frac{r}{|y-z|^{2\rho-n+1-2\varepsilon}} \frac{1}{|z-x_0|^{2n+2\varepsilon}} dy \Big)^{\frac{1}{2}} dz \\
&\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{n+\varepsilon}} \Big(\int_{2|y-z|\geq|z-x_0|>6r} \frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}} dy \Big)^{\frac{1}{2}} dz
\end{aligned}
$$

$$\leq \ C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|>3r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{\frac{1}{2}}dz.$$

Similarly to the estimate of (4.3), we have $I_{1.2'} \leq CM_pf(x)$.

Then we give the estimate of $I_{1.2''}$.

$$\begin{aligned} I_{1.2''} \ &\leq \ \int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |x_o-y|<t,\ 2|y-z|<|z-x_0|\\ |\omega-x_0+y-z|\geq t,\ |y-z|<2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}}dz \\ &\quad +\int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|<t\\ |x_o-y|<t,\ 2|y-z|<|z-x_0|\\ |\omega-x_0+y-z|\geq t,\ |y-z|\geq 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}}dz \\ &:= \ I_{1.2.1''}+I_{1.2.2''}. \end{aligned}$$

For $I_{1.2.1''}$, since $|y-z|<\frac{|z-x_0|}{2}$, then $|y-x_0|\geq|z-x_0|-|y-z|>\frac{|z-x_0|}{2}$, we have

$$\begin{aligned} I_{1.2.1''} \ &\leq \ \int_{(8B^*)^c}|f(z)|\Big(\int_{\substack{|y-z|<2r\\ |y-x_0|>\frac{|z-x_0|}{2}}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-x_0|}^{\infty}\frac{dt}{t^{n+2\rho+1}}dy\Big)^{\frac{1}{2}}dz \\ &\leq \ C\int_{(8B^*)^c}|f(z)|\Big(\int_{|y-z|<2r}\frac{1}{|y-x_0|^{n+2\rho}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Big)^{\frac{1}{2}}dz \\ &\leq \ C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}\Big(\int_{|y-z|<2r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dy\Big)^{\frac{1}{2}}dz \\ &= \ C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}\Big(\int_{S^{n-1}}|\Omega(z')|^2d\sigma(z')\int_0^{2r}\frac{1}{s^{2n-2\rho}}s^{n-1}ds\Big)^{\frac{1}{2}}dz \\ &\leq \ Cr^{\rho-\frac{n}{2}}\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}}dz. \end{aligned}$$

Similarly to the estimate of (4.4), we have $I_{1.2.1''} \leq CM_pf(x)$.

For $I_{1.2.2''}$, since $t>|y-x_0|\geq|z-x_0|-|y-z|>\frac{|z-x_0|}{2}$, and $|y-z|\sim|\omega-x_0+y-z|$, we have

$$\begin{aligned} I_{1.2.2''} \ &\leq \ \int_{(8B^*)^c}|f(z)|\Big(\int_{\substack{y\in(2B^*)^c\\ 2|y-z|<|z-x_0|\\ |y-z|\geq 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-z|<t\leq|\omega-x_0+y-z|}\frac{dt}{t^{n+2\rho+1}}dy\Big)^{\frac{1}{2}}dz \\ &\leq \ \int_{(8B^*)^c}\frac{|f(z)|}{(\frac{|z-x_0|}{2})^{n+\varepsilon}}\Big(\int_{\substack{y\in(2B^*)^c\\ 2|y-z|<|z-x_0|\\ |y-z|\geq 2r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-z|<t\leq|\omega-x_0+y-z|}\frac{dt}{t^{2\rho-n+1-2\varepsilon}}dy\Big)^{\frac{1}{2}}dz \\ &\leq \ C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|\geq 2r}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{2\rho-n-2\varepsilon+1}}dy\Big)^{\frac{1}{2}}dz \\ &= \ C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|\geq 2r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{\frac{1}{2}}dz. \end{aligned}$$

Similarly to the estimate of (4.3), we have $I_{1.2.2''} \leq CM_pf(x)$.

Combining the estimates of $I_{1.1}$, $I_{1.2'}$, $I_{1.2.1''}$ and $I_{1.2.2''}$, we obtain $I_1 \leq CM_pf(x)$. Similarly as we deal with I_1, we can obtain $I_2 \leq CM_pf(x)$.

So we only need to give the estimate of I_3. Apply the Minkowski inequality to I_3 and divide the region by $|y-z| \geq 6r$, $|y-z| < 6r$. When $|y-z| < 6r$, we have $y \in (2B^*)^c$, so

$$\begin{aligned} I_3 &\leq \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{y\in(2B^*)^c \\ |y-z|<t \\ |x_o-y|<t \\ |y-z|<6r \\ |\omega-x_0+y-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\ &+ \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{|y-z|\geq 6r \\ |y-z|<t \\ |x_o-y|<t \\ |\omega-x_0+y-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\ &:= I_{3.1} + I_{3.2}. \end{aligned}$$

It is easy to see that when $z \in (8B^*)^c$ and $|y-z| < 6r$, there are $|y-x_0| \sim |z-x_0|$ and $|\omega-x_0+y-z| \leq |\omega-x_0| + |y-z| < 8r$. Then

$$\begin{aligned} I_{3.1} &\leq C \int_{(8B^*)^c} |f(z)| \Big(\int_{\substack{y\in(2B^*)^c,\ |y-z|<6r \\ |x_o-y|<t,\ |\omega-x_0+y-z|<8r}} \Big(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} \Big) \\ &\quad \times \int_{|y-x_0|}^{\infty} \frac{dt}{t^{n+2\rho+1}} dy \Big)^{\frac{1}{2}} dz \\ &\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \Big(\int_{\substack{|y-z|<6r \\ |\omega-x_0+y-z|<8r}} \Big(\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} + \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} \Big) dy \Big)^{\frac{1}{2}} dz \\ &\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \Big(\int_{|y-z|<6r} \frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}} dy \Big)^{\frac{1}{2}} dz \\ &\quad + C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{\frac{n}{2}+\rho}} \Big(\int_{|\omega-x_0+y-z|<8r} \frac{|\Omega(\omega-x_0+y-z)|^2}{|\omega-x_0+y-z|^{2n-2\rho}} dy \Big)^{\frac{1}{2}} dz \\ &\leq C r^{\rho-\frac{n}{2}} \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^{\rho+\frac{n}{2}}} dz. \end{aligned}$$

Similarly to the estimate of (4.4), we have $I_{3.1} \leq CM_p f(x)$.

Finally, we give the estimate of $I_{3.2}$. Note that $|z-x_0| \leq |x_0-y| + |y-z| < 2t$, so $t > \frac{|z-x_0|}{2}$.

$$\begin{aligned} I_{3.2} &\leq \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{t>|z-x_0|/2,\ |y-z|<t \\ |y-x_0|<t,\ |y-z|\geq 6r \\ |\omega-x_0+y-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \\ &\quad \times \frac{(log\frac{t}{r})^{2+2\varepsilon} dydt}{t^{2\rho-n+1} t^{2n} (log\frac{t}{r})^{2+2\varepsilon}} \Big)^{\frac{1}{2}} dz \\ &\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^n (log\frac{|z-x_0|}{2r})^{1+\varepsilon}} \Big(\iint_{\substack{t>|z-x_0|/2,\ |y-z|<t \\ |y-x_0|<t,\ |y-z|\geq 6r \\ |\omega-x_0+y-z|<t}} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \right. \\ &\quad \left. - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \frac{(log\frac{t}{r})^{2+2\varepsilon} dydt}{t^{2\rho-n+1}} \Big)^{\frac{1}{2}} dz \\ &\leq C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^n (log\frac{|z-x_0|}{2r})^{1+\varepsilon}} \Big(\int_{|y-z|\geq 6r} \left| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(\omega-x_0+y-z)}{|\omega-x_0+y-z|^{n-\rho}} \right|^2 \end{aligned}$$

$$\times\int_{|y-z|}^{\infty}\frac{(log\frac{t}{r})^{2+2\varepsilon}}{t^{2\rho-n+1}}dtdy\Big)^{\frac{1}{2}}dz. \tag{4.5}$$

By Lemma 2.2 and Lemma 2.3, there is

$$\begin{aligned}
I_{3.2} &\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\frac{(1+|\Omega(y-z)|)^2}{|y-z|^{2n-2\rho}(\log\frac{|y-z|}{r})^{2\alpha}}\frac{(log\frac{|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}}dy\Big)^{\frac{1}{2}}dz\\
&= C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\frac{(1+|\Omega(y-z)|)^2}{|y-z|^{n}(\log\frac{|y-z|}{r})^{2\alpha-2-2\varepsilon}}dy\Big)^{\frac{1}{2}}dz \qquad (4.6)\\
&= C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{S^{n-1}}(1+|\Omega(z')|)^2d\sigma(z')\\
&\quad\times\int_{6r}^{\infty}\frac{s^{n-1}}{s^n(log\frac{s}{r})^{2\alpha-2-2\varepsilon}}ds\Big)^{\frac{1}{2}}dz\\
&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{6}^{\infty}\frac{dt}{t(logt)^{2\alpha-2-2\varepsilon}}\Big)^{\frac{1}{2}}dz\\
&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-\overline{x}|^n(log\frac{|z-\overline{x}|}{r})^{1+\varepsilon}}dz\\
&= C\sum_{k=3}^{\infty}\int_{2^kr\leq|z-\overline{x}|<2^{k+1}r}\frac{|f(z)|}{|z-\overline{x}|^n(log\frac{|z-\overline{x}|}{r})^{1+\varepsilon}}dz\\
&\leq C\sum_{k=3}^{\infty}\frac{1}{(2^kr)^n(log\frac{2^kr}{r})^{1+\varepsilon}}\int_{|z-\overline{x}|<2^{k+1}r}|f(z)|dz\\
&\leq C\sum_{k=3}^{\infty}\frac{1}{k^{1+\varepsilon}}\Big(\frac{1}{(2^{k+1}r)^n}\int_{|z-\overline{x}|<2^{k+1}r}|f(z)|^pdz\Big)^{\frac{1}{p}}\\
&\leq CM_pf(x).
\end{aligned}$$

Combining the estimates of $I_{3.1}$ and $I_{3.2}$, we obtain $I_3\leq CM_pf(x)$. Then

$$I=|\mu^{\rho}_{\Omega,S}(f_2)(x_0)-\mu^{\rho}_{\Omega,S}(f_2)(\omega)|\leq CM_pf(x),\ \text{for all}\ \omega\in B\setminus E.$$

Therefore

$$\frac{1}{|B|}\int_B|\mu^{\rho}_{\Omega,S}(f_2)(x_0)-\mu^{\rho}_{\Omega,S}(f_2)(\omega)|d\omega=\frac{1}{|B|}\int_{B\setminus E}|\mu^{\rho}_{\Omega,S}(f_2)(x_0)-\mu^{\rho}_{\Omega,S}(f_2)(\omega)|d\omega\leq CM_pf(x). \tag{4.7}$$

Since $M^{\sharp}(f)(x)=\sup_{x\in B}\frac{1}{|B|}\int_B|f(y)-f_B|dy\approx\sup_{x\in B}\inf_c\frac{1}{|B|}\int_B|f(y)-c|dy$, we just take $c=\mu^{\rho}_{\Omega,S}(f_2)(x_0)$. Write

$$|\mu^{\rho}_{\Omega,S}(f_1+f_2)(\omega)-\mu^{\rho}_{\Omega,S}(f_2)(x_0)|\leq\mu^{\rho}_{\Omega,S}(f_1)(\omega)+|\mu^{\rho}_{\Omega,S}(f_2)(\omega)-\mu^{\rho}_{\Omega,S}(f_2)(x_0)|.$$

By (4.1) and (4.7), we obtain

$$\frac{1}{|B|}\int_B|\mu^{\rho}_{\Omega,S}(f)(\omega)-\mu^{\rho}_{\Omega,S}(f_2)(x_0)|d\omega$$

$$\begin{aligned}
&\le \frac{1}{|B|}\int_B \mu^{\rho}_{\Omega,S}(f_1)(\omega)d\omega + \frac{1}{|B|}\int_B |\mu^{\rho}_{\Omega,S}(f_2)(x_0)-\mu^{\rho}_{\Omega,S}(f_2)(\omega)|d\omega\\
&\le CM_pf(x).
\end{aligned}$$

Thus $M^{\sharp}(\mu^{\rho}_{\Omega,S}f)(x)\le CM_pf(x)$, for all $x\in\mathbb{R}^n$.

(2) Below we will give the proof for $M^{\sharp}(\mu^{*,\rho}_{\lambda}f)$.

Given $x\in\mathbb{R}^n$, let B, $\bar{x}$, r_0, B^*, r be the same as before, also set

$$f = f\chi_{8B^*} + f(1-f\chi_{8B^*}) := f_1+f_2.$$

Then by Theorem 1.1, we have

$$\frac{1}{|B|}\int_B \mu^{*,\rho}_{\lambda}(f_1)(u)du \le \Big(\frac{1}{|B|}\int_B \mu^{*,\rho}_{\lambda}(f_1)^p(u)du\Big)^{\frac{1}{p}} \le C\Big(\frac{1}{|B|}\int_{\mathbb{R}^n}|f_1(u)|^pdu\Big)^{\frac{1}{p}} \le CM_pf(x). \quad (4.8)$$

By the same reason as we show in the part (1), there exists a measurable set E with measure zero such that $\mu^{*,\rho}_{\lambda}(f_2)(u)<\infty$ for any $u\in B\setminus E$. Now we fixed one point $x_0\in B\setminus E$ and for any $w\in B\setminus E$, we consider

$$\begin{aligned}
J &= |\mu^{*,\rho}_{\lambda}(f_2)(x_0)-\mu^{*,\rho}_{\lambda}(f_2)(w)|\\
&= \Big|\|\phi_{t,y}(f_2)(x_0)\|_{\mathcal{H}_2}-\|\phi_{t,y}(f_2)(w)\|_{\mathcal{H}_2}\Big|\\
&\le \|\phi_{t,y}(f_2)(x_0)-\phi_{t,y}(f_2)(w)\|_{\mathcal{H}_2}\\
&= \Big(\int_0^\infty\int_{\mathbb{R}^n}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}\\
&\le \Big(\int_0^\infty\int_{|y|<1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}\\
&\quad+\Big(\int_0^\infty\int_{|y|\ge1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}\\
&:= J_1+J_2.
\end{aligned}$$

Since $(\frac{1}{1+|y|})^{\lambda n}\le 1$, by the estimate of (4.2), there is $J_1\le I_1+I_2+I_3\le CM_p(f)(x)$.

$$\begin{aligned}
J_2 &\le \Big(\int_0^\infty\int_{|y|\ge1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{w-z}{t}-y|\ge1}} t^{-n}\phi\Big(\frac{x_0-z}{t}-y\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}\\
&\quad+\Big(\int_0^\infty\int_{|y|\ge1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|\ge1\\|\frac{w-z}{t}-y|<1}} t^{-n}\phi\Big(\frac{w-z}{t}-y\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}\\
&\quad+\Big(\int_0^\infty\int_{|y|\ge1}\Big(\frac{1}{1+|y|}\Big)^{\lambda n}\Big|\int_{\substack{|\frac{x_0-z}{t}-y|<1\\|\frac{w-z}{t}-y|<1}} t^{-n}\Big(\phi\Big(\frac{x_0-z}{t}-y\Big)-\phi\Big(\frac{w-z}{t}-y\Big)\Big)f_2(z)dz\Big|^2\frac{dydt}{t}\Big)^{\frac{1}{2}}.
\end{aligned}$$

Using the trasform $y\to\frac{x_0-y'}{t}$ again (we still use y instead y'), we have

$$J_2 \le \Big(\int_0^\infty\int_{|x_0-y|\ge t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|\int_{\substack{|y-z|<t\\|w-x_0+y-z|\ge t}}\frac{\Omega(y-z)}{|y-z|^{n-\rho}}f_2(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac{1}{2}}$$

$$+\Big(\int_0^\infty \int_{|x_0-y|\ge t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|\int_{\substack{|y-z|\ge t\\ |w-x_0+y-z|<t}} \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} f_2(z)dz\Big|^2 \frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}$$
$$+\Big(\int_0^\infty \int_{|x_0-y|\ge t}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}$$
$$\times\Big|\int_{\substack{|y-z|<t\\ |w-x_0+y-z|<t}}\Big(\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big)f_2(z)dz\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}$$
$$:= \quad L_1+L_2+L_3.$$

As for L_1, we claim that $y\in(2B^*)^c$, otherwise if $y\in 2B^*$ then $t\le|x_0-y|<4r$, but $z\in(8B^*)^c$, $t>|y-z|>6r$. Thus by the Minkowski inequality, we get

$$L_1\le\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|x_0-y|\ge t\\ y\in(2B^*)^c\\ |y-z|<t\\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\le L_{1.1}+L_{1.2},$$

where

$$L_{1.1}=\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|<8r,\ |x_0-y|\ge t\\ y\in(2B^*)^c,\ |y-z|<t\\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz,$$

$$L_{1.2}=\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|\ge 8r,\ |x_0-y|\ge t\\ y\in(2B^*)^c,\ |y-z|<t\\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz.$$

As for $L_{1.1}$, since $|y-z|<8r$, $z\in(8B^*)^c$ and $y\in(2B^*)^c$, then $|y-x_0|\sim|z-x_0|$. So

$$\begin{aligned}
L_{1.1} \quad\le\quad& \int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|<8r,\ |x_0-y|\ge t\\ y\in(2B^*)^c,\ |y-z|<t}}\Big(\frac{1}{t+|x_0-y|}\Big)^{2n+2\varepsilon}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n-2n-2\varepsilon}\\
&\times t^{2n+2\varepsilon}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
\le\quad& \int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|<8r\\ |x_0-y|\ge t\\ y\in(2B^*)^c\\ |y-z|<t}}\frac{1}{|x_0-y|^{2n+2\varepsilon}}\frac{t^{2n+2\varepsilon}}{t^{n+2\rho+1}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dydt\Big)^{\frac12}dz\\
\le\quad& C\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|<8r\\ |x_0-y|\ge t\\ y\in(2B^*)^c\\ |y-z|<t}}\frac{1}{|z-x_0|^{2n+2\varepsilon}}\frac{1}{t^{1+2\rho-2\varepsilon-n}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}dydt\Big)^{\frac12}dz\\
\le\quad& C\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{|y-z|<8r\\ |x_0-y|\ge t\\ y\in(2B^*)^c\\ |y-z|<t}}\frac{1}{|z-x_0|^{2n+2\varepsilon}}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}}\frac{dydt}{t^{1-\varepsilon}}\Big)^{\frac12}dz\\
\le\quad& C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}}\Big(\int_{|y-z|<8r}\frac{|\Omega(y-z)|^2}{|z-x_0|^{\varepsilon}|y-z|^{n-\varepsilon}}\Big(\int_0^{|x_0-y|}\frac{1}{t^{1-\varepsilon}}dt\Big)dy\Big)^{\frac12}dz
\end{aligned}$$

$$
\begin{aligned}
&= C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}}\Big(\int_{|y-z|<8r}\frac{|\Omega(y-z)|^2}{|y-z|^{n-\varepsilon}}dy\Big)^{\frac12}dz\\
&\le Cr^{\varepsilon/2}\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon/2}}dz.
\end{aligned}
$$

Similarly to the estimate of (4.4), we have $L_{1.1}\le CM_p(f)(x)$.

$$
\begin{aligned}
L_{1.2} &\le \int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{2|y-z|\ge|z-x_0|\\ |x_0-y|\ge t,\ y\in(2B^*)^c\\ |y-z|<t,\ |y-z|\ge 8r\\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&\quad+\int_{(8B^*)^c}|f(z)|\Big(\int_0^\infty\int_{\substack{2|y-z|<|z-x_0|\\ |x_0-y|\ge t,\ y\in(2B^*)^c\\ |y-z|<t,\ |y-z|\ge 8r\\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&:= L_{1.2'}+L_{1.2''}.
\end{aligned}
$$

As for $L_{1.2'}$,

$$
\begin{aligned}
L_{1.2'} &\le \int_{(8B^*)^c}|f(z)|\Big(\int_{\substack{y\in(2B^*)^c\\ |y-z|\ge 8r\\ 2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\int_{|y-z|<t\le|w-x_0+y-z|}\frac{dt}{t^{n+2\rho+1}}dy\Big)^{\frac12}dz\\
&\le C\int_{(8B^*)^c}|f(z)|\Big(\int_{\substack{y\in(2B^*)^c\\ |y-z|\ge 8r\\ 2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{n+2\rho+1}}dy\Big)^{\frac12}dz\\
&\le C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{\substack{y\in(2B^*)^c\\ |y-z|\ge 8r\\ 2|y-z|\ge|z-x_0|}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{1+2\rho-2\varepsilon-n}}dy\Big)^{\frac12}dz\\
&\le C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|\ge 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{\frac12}dz\\
&\le Cr^{\varepsilon}\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}dz.
\end{aligned}
$$

By the estimate of (4.4), we have $L_{1.2'}\le CM_p(f)(x)$.

As for $L_{1.2''}$, by $|z-x_0|>2|y-z|$, we know that

$$
|y-x_0|\ge|z-x_0|-|y-z|>|z-x_0|/2 \quad\text{and}\quad |y-z|\sim|w-x_0+y-z|.
$$

Thus

$$
\begin{aligned}
L_{1.2''} &\le \int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|\ge 8r\\ |y-x_0|\ge|z-x_0|/2\\ |y-z|<t,\ |w-x_0+y-z|\ge t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n-2n-2\varepsilon}\frac{t^{2n+2\varepsilon}}{|x_0-y|^{2n+2\varepsilon}}\\
&\quad\times\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&\le \int_{(8B^*)^c}\frac{|f(z)|}{(\frac{|z-x_0|}{2})^{n+\varepsilon}}\Big(\iint_{\substack{y\in(2B^*)^c,\ |y-z|\ge 8r\\ |y-x_0|\ge|z-x_0|/2\\ |y-z|<t,\ |w-x_0+y-z|\ge t}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{t^{2n+2\varepsilon}}{t^{n+2\rho+1}}dydt\Big)^{\frac12}dz
\end{aligned}
$$

$$
\begin{aligned}
&= C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{\substack{y\in(2B^*)^c,\ |y-z|\ge 8r\\ |y-x_0|\ge|z-x_0|/2}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\Big(\int_{|y-z|}^{|w-x_0+y-z|}\frac{1}{t^{2\rho-n-2\varepsilon+1}}dt\Big)dy\Big)^{\frac12}dz\\
&\le C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{\substack{y\in(2B^*)^c\\ |y-z|\ge 8r}}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{r}{|y-z|^{2\rho-n-2\varepsilon+1}}dy\Big)^{\frac12}dz\\
&\le C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^{n+\varepsilon}}\Big(\int_{|y-z|\ge 8r}\frac{r|\Omega(y-z)|^2}{|y-z|^{n+1-2\varepsilon}}dy\Big)^{\frac12}dz.
\end{aligned}
$$

Similarly to the estimate of (4.3), we have $L_{1.2''}\le CM_p(f)(x)$.

Combining the estimates of $L_{1.1}, L_{1.2'}$ and $L_{1.2''}$, we obtain $L_1\le CM_p(f)(x)$. Similarly as we deal with L_1, we can obtain $L_2\le CM_p(f)(x)$.

Finally, we deal with the last part L_3. By the Minskowski inequality,

$$
\begin{aligned}
L_3 &\le \int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{|x_0-y|\ge t,\ |y-z|<t\\ |w-x_0+y-z|<t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big|^2\\
&\qquad\times\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&\le \int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{|y-z|<6r,\ |x_0-y|\ge t\\ |y-z|<t,\ |w-x_0+y-z|<t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big|^2\\
&\qquad\times\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&\quad+\int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{|y-z|\ge 6r,\ |x_0-y|\ge t\\ |y-z|<t,\ |w-x_0+y-z|<t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\\
&\qquad\times\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big|^2\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&:= L_{3.1}+L_{3.2}.
\end{aligned}
$$

As for $L_{3.1}$, since $|y-z|<6r$ and $z\in(8B^*)^c$, then $|y-\bar{x}|\ge|z-\bar{x}|-|y-z|>2r$, we can have $y\in(2B^*)^c$, $|w-x_0+y-z|\le|w-x_0|+|y-z|<8r$.

$$
\begin{aligned}
L_{3.1} &\le \int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |x_0-y|\ge t\\ |y-z|<6r,\ |y-z|<t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(y-z)|^2}{|y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&\quad+\int_{(8B^*)^c}|f(z)|\Big(\iint_{\substack{y\in(2B^*)^c,\ |x_0-y|\ge t\\ |w-x_0+y-z|<8r\\ |w-x_0+y-z|<t}}\Big(\frac{t}{t+|x_0-y|}\Big)^{\lambda n}\frac{|\Omega(w-x_0+y-z)|^2}{|w-x_0+y-z|^{2n-2\rho}}\frac{dydt}{t^{n+2\rho+1}}\Big)^{\frac12}dz\\
&:= L_{3.1'}+L_{3.1''}.
\end{aligned}
$$

Using the similar method as we deal with $L_{1.1}$, we easily have

$$L_{3.1'}\le CM_p(f)(x),\ L_{3.1''}\le CM_p(f)(x),$$

thus $L_{3.1}\le CM_p(f)(x)$.

As for $L_{3.2}$,

$$\begin{aligned}
L_{3.2} \leq & \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{2|y-z|\geq|z-x_0| \\ |y-z|\geq 6r,\ |x_0-y|\geq t \\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|} \Big)^{\lambda n} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \\
& \times \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\
& + \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{2|y-z|<|z-x_0| \\ |y-z|\geq 6r,\ |x_0-y|\geq t \\ |y-z|<t,\ |w-x_0+y-z|<t}} \Big(\frac{t}{t+|x_0-y|} \Big)^{\lambda n} \\
& \times \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\
:= & L_{3.2'} + L_{3.2''}.
\end{aligned}$$

For $L_{3.2'}$, there is

$$\begin{aligned}
L_{3.2'} \leq & \int_{(8B^*)^c} |f(z)| \Big(\int_{|y-z|\geq 6r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \\
& \times \Big(\int_{max\{|y-z|,|z-x_0|/2\}}^{\infty} \frac{dt}{t^{n+2\rho+1}} \Big) dy \Big)^{\frac{1}{2}} dz \\
\leq & C \int_{(8B^*)^c} |f(z)| \Big(\int_{|y-z|\geq 6r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \\
& \times \Big(\int_{max\{|y-z|,|z-x_0|/2\}}^{\infty} \frac{(log\frac{t}{r})^{2+2\varepsilon} dt}{t^{2\rho-n+1}|z-x_0|^{2n}(log\frac{|z-x_0|}{2r})^{2+2\varepsilon}} \Big) dy \Big)^{\frac{1}{2}} dz \\
\leq & C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^n (log\frac{|z-x_0|}{2r})^{1+\varepsilon}} \Big(\int_{|y-z|\geq 6r} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \\
& \times \Big(\int_{|y-z|}^{\infty} \frac{(log\frac{t}{r})^{2+2\varepsilon} dt}{t^{2\rho-n+1}} \Big) dy \Big)^{\frac{1}{2}} dz.
\end{aligned}$$

By the estimate of (4.5), we get $L_{3.2'} \leq CM_p(f)(x)$.

For $L_{3.2''}$, denote $C(\varepsilon) = e^{(2+2\varepsilon)/\varepsilon}$. Since $2|y-z| < |z-x_0|$, then $|x_0-y| \geq |z-x_0| - |y-z| > \frac{|z-x_0|}{2}$, thus

$$\begin{aligned}
L_{3.2''} \leq & \int_{(8B^*)^c} |f(z)| \Big(\iint_{\substack{|y-z|\geq 6r,\ |x_0-y|\geq t \\ |x_0-y|>|z-x_0|/2 \\ |y-z|<t}} \frac{t^{\lambda n}(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n+2n}(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}} \\
& \times \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz \\
\leq & C \int_{(8B^*)^c} \frac{|f(z)|}{|z-x_0|^n (log\frac{|z-x_0|}{2r})^{1+\varepsilon}} \Big(\iint_{\substack{|y-z|\geq 6r,\ |x_0-y|\geq t \\ |x_0-y|>|z-x_0|/2 \\ |y-z|<t}} \Big| \frac{\Omega(y-z)}{|y-z|^{n-\rho}} \\
& - \frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}} \Big|^2 \frac{t^{\lambda n}(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}} \frac{dydt}{t^{n+2\rho+1}} \Big)^{\frac{1}{2}} dz
\end{aligned}$$

$$
\begin{aligned}
&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big|^2\\
&\times\Big(\int_{|y-z|}^{|x_0-y|}\frac{t^{\lambda n}(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}\Big)dy\Big)^{\frac12}dz.
\end{aligned}
$$

Notice that the function $G(s)=\frac{(log s)^{2+2\varepsilon}}{s^{\varepsilon}}$ is decreasing when $s\geq e^{(2+2\varepsilon)/\varepsilon}$ and

$$
\frac{t+|y-x_0|+C(\varepsilon)r}{r}\geq\frac{|y-z|+C(\varepsilon)r}{r}\geq C(\varepsilon)=e^{(2+2\varepsilon)/\varepsilon}.
$$

Then

$$
\frac{[log(\frac{t+|y-x_0|+C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{\varepsilon}}=G\Big(\frac{t+|y-x_0|+C(\varepsilon)r}{r}\Big)\leq G\Big(\frac{|y-z|+C(\varepsilon)r}{r}\Big)=\frac{[log(\frac{|y-z|+C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(\frac{|y-z|+C(\varepsilon)r}{r})^{\varepsilon}}.
$$

Since $t+|y-x_0|\sim t+|y-x_0|+C(\varepsilon)r$ and $0<\varepsilon<\min\{\frac12,\frac{(\lambda-2)n}{2},\rho-\frac n2,\alpha-\frac32\}$, then

$$
\begin{aligned}
&\int_{|y-z|}^{|x_0-y|}\frac{t^{\lambda n}(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n}}\frac{dt}{t^{n+2\rho+1}}\\
=&\int_{|y-z|}^{|x_0-y|}\frac{(log\frac{t+|y-x_0|+C(\varepsilon)r}{r})^{2+2\varepsilon}}{(t+|x_0-y|)^{\varepsilon}}\frac{t^{\lambda n-2n-\varepsilon}t^{2n+\varepsilon}}{(t+|x_0-y|)^{\lambda n-2n-\varepsilon}}\frac{dt}{t^{n+2\rho+1}}\\
\leq&\ C\int_{|y-z|}^{\infty}\frac{[log(\frac{|y-z|+C(\varepsilon)r}{r})]^{2+2\varepsilon}}{(|y-z|+C(\varepsilon)r)^{\varepsilon}}\frac{dt}{t^{2\rho-n+1-\varepsilon}}\\
\leq&\ C\frac{[log(\frac{|y-z|+C(\varepsilon)r}{r})]^{2+2\varepsilon}}{|y-z|^{2\rho-n}}.
\end{aligned}
$$

Since $|y-z|\geq 6r$, there exists a constant $l\geq 1$ such that $|y-z|+C(\varepsilon)r\leq 2^l|y-z|$. Hence

$$
\begin{aligned}
L_{3.2''}&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\Big|\frac{\Omega(y-z)}{|y-z|^{n-\rho}}-\frac{\Omega(w-x_0+y-z)}{|w-x_0+y-z|^{n-\rho}}\Big|^2\\
&\quad\times\frac{(log\frac{2^l|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}}dy\Big)^{\frac12}dz\\
&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\frac{(1+|\Omega(y-z)|)^2}{|y-z|^{2n-2\rho}(log\frac{|y-z|}{r})^{2\alpha}}\\
&\quad\times\frac{(log\frac{2^l|y-z|}{r})^{2+2\varepsilon}}{|y-z|^{2\rho-n}}dy\Big)^{\frac12}dz\\
&\leq C\int_{(8B^*)^c}\frac{|f(z)|}{|z-x_0|^n(log\frac{|z-x_0|}{2r})^{1+\varepsilon}}\Big(\int_{|y-z|\geq 6r}\frac{(1+|\Omega(y-z)|)^2}{|y-z|^n(log\frac{|y-z|}{r})^{2\alpha-2-2\varepsilon}}dy\Big)^{\frac12}dz.
\end{aligned}
$$

By the estimate of (4.6), we get $L_{3.2''}\leq CM_p(f)(x)$.

Combining the estimates of $L_{3.1}$ and $L_{3.2}$, we obtain $L_3\leq CM_p(f)(x)$. Then

$$
J\leq J_1+L_1+L_2+L_3\leq CM_p(f)(x),\ \text{for all}\ w\in B\setminus E.
$$

Therefore

$$\frac{1}{|B|}\int_B |\mu_\lambda^{*,\rho}(f_2)(x_0) - \mu_\lambda^{*,\rho}(f_2)(\omega)|d\omega \le CM_p f(x). \tag{4.9}$$

By (4.8) and (4.9), we have

$$\begin{aligned} & \frac{1}{|B|}\int_B |\mu_\lambda^{*,\rho}(f)(\omega) - \mu_\lambda^{*,\rho}(f_2)(x_0)|d\omega \\ \le\ & \frac{1}{|B|}\int_B \mu_\lambda^{*,\rho}(f_1)(\omega)d\omega + \frac{1}{|B|}\int_B |\mu_\lambda^{*,\rho}(f_2)(x_0) - \mu_\lambda^{*,\rho}(f_2)(\omega)|d\omega \\ \le\ & CM_p f(x). \end{aligned}$$

Taking $c = \mu_\lambda^{*,\rho}(f_2)(x_0)$, we can have

$$M^\sharp(\mu_\lambda^{*,\rho} f)(x) \le \sup_{x\in B}\frac{1}{|B|}\int_B |\mu_\lambda^{*,\rho}(f)(\omega) - \mu_\lambda^{*,\rho}(f)(x_0)|dy \le CM_p f(x), \text{ for all } x \in \mathbb{R}^n.$$

Then we finish the proof of Theorem 3.1. □

The proof of Theorem 3.2 is conventional, so we omit the details.

Finally, we give the proof of Theorem 3.3.

Proof. Noticing that $1 < p < \infty$, there exists an s such that $1 < s < p$. By Theorem 3.1 and Lemma 2.4, we have

$$\begin{aligned} \|\mu_{\Omega,S}^{\rho}(f)\|_{L^{p,\varphi}} &\le \|M(\mu_{\Omega,S}^{\rho} f)\|_{L^{p,\varphi}} \le C\|M^\sharp(\mu_{\Omega,S}^{\rho} f)\|_{L^{p,\varphi}} \\ &\le C\|M_s f\|_{L^{p,\varphi}} = C\|M(|f|^s)\|_{L^{p/s,\varphi}}^{1/s} \\ &\le C\||f|^s\|_{L^{p/s,\varphi}}^{1/s} = C\|f\|_{L^{p,\varphi}}. \end{aligned}$$

Similarly, we can obtain $\|\mu_\lambda^{*,\rho}(f)\|_{L^{p,\varphi}} \le C\|f\|_{L^{p,\varphi}}$, which complete the proof of Theorem 3.3. □

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