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On Intersections of Cantor Sets: Self-Similarity

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Abstract

Let C be a Cantor subset of the real line. For a real number t, let $C+t$ be the translate of C by t. We say two real numbers s,t are *translation equivalent*, if the intersection of C and $C+s$ is a translate of the intersection of C and $C+t$. We consider a class of Cantor sets determined by similarities with one fixed positive contraction ratio. For this class of Cantor set, we show that an "initial segment" of the intersection of C and $C+t$ is a self-similar set with contraction ratios that are powers of the contraction ratio used to describe C as a self-similar set if and only if t is translation equivalent to a rational number. Many of our results are new even for the middle thirds Cantor set.

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1 Introduction

In this paper we study self-similarity properties of the intersections of certain Cantor sets with their translates. The simplest form of our results is Theorem 1.1. Cantor sets may appear to be rather special. However, they occur in mathematical models involving fractals, iterated function systems, and self-similar measures), they further play a role in number theory, in signal processing, in ergodic theory, and in limit-theorems from probability. Cantor sets are also rooted in the theory of dynamical systems. See, for instance, Palis and

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Takens [26]. The literature in this subject, neighboring areas, and applications is vast. Below we limit ourselves to a small sample of the literature: Cabrelli, Mendivil, Molter, and Shonkwiler[2], Dai and Tian [6], Davis and Hu [3], Duan, Liu, and Tang [5], Furstenberg, [8], Garcia, Molter, and Scotto [9], Hare, Mendivil, and Zuberman [11], Kraft [15], [16], Li and Nekka [17], Li, Yao, and Zhang [18], McClure [21], Moreira [22], Peres and Solomyak [25], Pedersen and Phillips [23], [24], Williams [27], and Zhang and Gu [28].

Let $n \geq 3$ be an integer. Any real number $t \in [0,1]$ has at least one *n-ary representation*

$$t = 0._n t_1 t_2 \cdots = \sum_{k=1}^{\infty} \frac{t_k}{n^k}$$

where each t_k is one of the digits $0,1,\ldots,n-1$. Deleting some elements from the full digit set $\{0,1,\ldots n-1\}$ we get a set of *digits* $D := \{d_k \mid k = 1,2,\ldots,m\}$ with $d_k < d_{k+1}$ for all $k = 1,2,\ldots,m-1$. Assuming $2 \leq m < n$ we get a corresponding *deleted digits Cantor set*

$$C = C_{n,D} := \left\{ \sum_{k=1}^{\infty} \frac{x_k}{n^k} \,\middle|\, x_k \in D \text{ for all } k \in \mathbb{N} \right\}. \tag{1.1}$$

We say that D is *uniform,* if $d_{k+1} - d_k$, $k = 1,2,\ldots,m-1$ is constant and ≥ 2. We say D is *regular,* if D is a subset of a uniform digit set. Finally, we say that D is *sparse,* if $|\delta - \delta'| \geq 2$ for all $\delta \neq \delta'$ in the set of differences

$$\Delta := D - D = \left\{ d_j - d_k \mid d_j, d_k \in D \right\}.$$

Clearly, a uniform set is regular and a regular set is sparse. The set $D = \{0,5,7\}$ is sparse and not regular. We will abuse the terminology and say $C_{n,D}$ is uniform, regular, or sparse provided D has the corresponding property. The middle thirds Cantor set is obtained by setting $n = 3$ and $D = \{0,2\}$. In particular, the middle thirds Cantor set is a uniform set.

In this paper we investigate self-similarity properties of the intersections $C \cap (C+t)$ of C with its translates $C + t := \{x + t \mid x \in C\}$, for sparse Cantor sets C. Using a geometric approach, we investigate the class of real numbers $t \in [0,1]$ for which the intersection $C \cap (C+t)$ can be expressed as the finite, disjoint union of self-similar sets. Since the problem is invariant under translation, we will assume $d_1 = 0$.

Compared to previous studies, e.g., [4], [13], and [18], of self-similarity properties of $C \cap (C+t)$ we allow a greater class of digits sets, sparse sets as compared to uniform sets and we study self-similarity of a subset of $C \cap (C+t)$ instead of self-similarity of all of $C \cap (C+t)$.

Previous studies focused on the class of strongly periodic rational values t, see section 6.1. We expand the study of uniform sets to include all rational values of t and give a method for constructing values $t \in C$ for which the intersection $C \cap (C+t)$ is not the finite, disjoint union of self-similar sets. These results, as well as Theorem 1.1, Theorem 1.3, and Theorem 1.4 are new even for the middle thirds Cantor set.

1.1 Statement of results

Fix a real number t. If $C \cap (C+t)$ is non-empty, let

$$C(t) := (C \cap (C+t)) - \inf(C \cap (C+t)),$$

otherwise, let $C(t)$ be the empty set. We say that two real numbers s and t are *translation equivalent*, if $C(s) = C(t)$. Clearly, s and t are translation equivalent if and only if $C\cap(C+s)$ is a translate of $C\cap(C+t)$. We show that a real number t is translation equivalent to a rational if and only if some initial segment of $C(t)$ has a self-similarity property. More precisely, we show:

Theorem 1.1. *Let D be sparse and $x \in [0,1]$ such that $C(x)$ is not empty. Then x is translation equivalent to a rational $t \in [0,1]$ if and only if there exists $\varepsilon > 0$, such that $C(x)\cap[0,\varepsilon]$ is a self-similar set generated by a finite set of similarities $f_j(y) = r_j y + b_j$, where $r_j = n^{-q_j}$ for some $q_j \in \mathbb{N}$.*

Theorem 1.1 only requires that the segment of $C(x)$ in a neighborhood surrounding zero is self-similar. However, if x is translation equivalent to a rational, then the intersection $C(x)\cap[\varepsilon,1]$ cannot be arbitrary, see Theorem 1.3.

Let $\Delta^+ := \Delta\cap[0,\infty)$ and let $F(\Delta^+)$ be the set of α in the interval $[0,1]$ such that

$$\alpha = \sum_{k=1}^{\infty} \alpha_k n^{-k}, \text{ for some } \alpha_k \in \Delta^+.$$

Then $F(\Delta^+)$ is a subset of the set F^+ of all $t \in [0,1]$ such that $C\cap(C+t)$ is non-empty, and any t in F^+ is translation equivalent to some α in $F(\Delta^+)$, see Section 4.

Let δ be an integer. If $D\cap(D+\delta)$ is nonempty, let

$$D_\delta := D\cap(D+\delta) - \min\left(D\cap(D+\delta)\right),$$

otherwise, let D_δ be the empty set. It follows from Lemma 4.12, that α in $F(\Delta^+)$ is translation equivalent to a rational if and only if there are integers $k \geq 0$ and $q > 0$, such that

$$D_{\alpha_j} \subseteq D_{\alpha_{j+q}} \text{ for all } j > k. \tag{1.2}$$

We say α is *strongly periodic* if there are sets $\widetilde{D}_{\alpha_j}$ and $q > 0$, such that

$$D_{\alpha_j} + \widetilde{D}_{\alpha_j} = D_{\alpha_{j+q}} \text{ for all } j > 0.$$

Note this implies (1.2). We show in Section 6.1 that our notion of strong periodicity is consistent with the one in [4], [18], [13], and [29], when D is uniform.

Theorem 1.2. *Let D be sparse and $\alpha = 0._n\alpha_1\alpha_2\ldots$ be an element in $F(\Delta^+)$. Then α is strongly periodic if and only if $C(\alpha)$ is a self-similar set generated by a finite set of similarities $f_j(x) = n^{-q}x + b_j$, where $q \in \mathbb{N}$.*

If D is uniform, this was established in [4], [18] when $d_m = n-1$. After we completed this manuscript we received the preprint [14], this preprint contains a generalization of Theorem 1.2, see Remark 4.16.

One part of Theorem 1.1 is a consequence of a structure theorem for $C\cap(C+x)$, when x is rational. This structure is summarized in the following result.

Theorem 1.3. *Let D be sparse and $t \in [0,1]$ such that $C \cap (C+t)$ is not empty. If $t = 0._n t_1 \cdots t_{k-p} \overline{t_{k-p+1} \cdots t_k}$ for some period p and integer $k \geq p$, then there exists a sparse digits set $E = \left\{0 \leq e_1 < e_2 < \cdots < e_r < n^{2p}\right\}$ and corresponding deleted digits Cantor set $C_{n^{2p},E}$ such that $C(t)$ consists of a finite number of translates of $\frac{1}{n^k} C_{n^{2p},E}$, the translates of $\frac{1}{n^k} C_{n^{2p},E}$ are disjoint, in fact, the translates of the convex hull of $\frac{1}{n^k} C_{n^{2p},E}$ are disjoint.*

Let $\dim_{\mathrm{H}}(C \cap (C+t))$ denote the Hausdorff measure of $C \cap (C+t)$. We showed in [24] that there are uncountably many t such that the $\dim_{\mathrm{H}}(C \cap (C+t))$-dimensional Hausdorff measure of $C \cap (C+t)$ is zero or infinity. For such t, the set $C \cap (C+t)$ is not a finite union of translates of a self-similar set. In particular, not all real numbers are translation equivalent to a rational number. We provide a method for constructing real numbers which are not translation equivalent to any rational, and thus are not a finite, disjoint union of self-similar sets. In particular, we show that if D is uniform and $t \in C_{n,D}$ is irrational, then t is not translation equivalent to any rational.

The structure of uniform deleted digits Cantor sets allows us to prove the following special case of Theorem 1.1:

Theorem 1.4. *Let D be uniform and $x \in [0,1]$ such that $C \cap (C+x)$ is not empty. There exists a rational $t \in [0,1]$ such that $C(x) = C(t)$ if and only if $C(x)$ is the finite, disjoint union of self-similar sets.*

We show in Section 6.2 that our results extend to a class of β-expansions with non-uniform digits sets. The papers [29] and [13] consider β-expansions with uniform digit sets, but they allow a different class of β's than we do.

We refer the reader to [7] for background information on Hausdorff dimension, Hausdorff measure and self-similar sets.

1.2 Outline

In Section 2 we summarize the construction of $C \cap (C+t)$ in our analysis. More details can be found in [24] where this construction was used to investigate the Hausdorff measure of $C \cap (C+t)$. A related construction was used in [23] to investigate the Hausdorff dimension of $C \cap (C+t)$.

In Section 3 we investigate some aspects of translation equivalence leading to a proof of Theorem 1.3. We calculate the Hausdorff measure of $C \cap (C+t)$ for some C and t and apply our methods to situations when D is not sparse.

In Section 4 we resume our analysis of translation equivalence leading to a proof of Theorem 1.1 and to a proof of Theorem 1.2

In Section 5 we associate an uncountable family of irrationals that are not translation equivalent to a rational to any t such that $C(t)$ is not finite.

Finally, in Section 6, we focus on uniform sets and discuss the relationship between strong periodicity and translation equivalence. We extend the definition of strongly periodic rationals to an arbitrary digits set D and show, if D is uniform, then Theorem 1.1 holds with $r_j = n^{-q_j}$ replaced by $r_j > 0$. We prove that our results hold for certain β-expansions with non-uniform digits sets.

2 A Construction of $C \cap (C+t)$

In this section we assume $n \geq 3$ is given and that $D = \{d_k \mid k = 1,2,\ldots,m\}$, $2 \leq m < n$ is a digits set. We demonstrate a natural method for constructing $C = C_{n,D}$, which forms the basis for our analysis of $C \cap (C+t)$. The results in this section are proven in [24], but we summarize the relevant parts of [24] here for the convenience of the reader.

In order to avoid trivial cases where $C \cap (C+t)$ is empty, define

$$F := \{t \mid C \cap (C+t) \neq \varnothing\}.$$

It is easy to see that $F = C - C = \{x - y \mid x, y \in C\}$ and consequently, F is compact. Since $C \cap (C-t)$ is a translate of $C \cap (C+t)$ it is sufficient to consider $t \geq 0$ and $F = (-F^+) \cup F^+$ where $F^+ := F \cap [0,\infty)$.

The middle thirds Cantor set is often constructed by beginning with the closed interval $C_0 = [0,1]$ and, inductively, for $k \geq 0$, obtaining C_{k+1} from C_k by removing the open middle of each interval in C_k. In general, $C = C_{n,D}$ can be constructed in a similar manner. The *refinement* of an interval $[a,b]$ is the set

$$\bigcup_{j=1}^{m}\left[a + \frac{d_j}{n}(b-a), a + \frac{d_j+1}{n}(b-a)\right].$$

Let C_0 be the closed unit interval $[0,1]$ and inductively, for $k \geq 0$, obtain C_{k+1} from C_k by refining each n-ary interval in C_k. Then, $C_k := \left\{0._n x_1 x_2 \ldots \mid x_j \in D \text{ for } 1 \leq j \leq k\right\}$, $C_{k+1} \subset C_k$ for all k, and

$$C = C_{n,D} = \bigcap_{k=0}^{\infty} C_k = \left\{0._n x_1 x_2 \ldots \mid x_j \in D \text{ for all } 1 \leq j\right\}. \tag{2.1}$$

For any integer h, we say that an interval $J^{(h)} = \frac{1}{n^k}(C_0 + h)$ is an *n-ary interval* of length $\frac{1}{n^k}$. We will simply say n-ary interval when k is understood from the context. In particular, if U is a compact set, the phrase *an n-ary interval of U* refers to an n-ary interval of length $\frac{1}{n^k}$ contained in U where k is the smallest such k. Note, C_k consists of m^k disjoint n-ary intervals.

For a fixed $t = 0._n t_1 t_2 \ldots$ in $[0,1]$, our analysis of $C \cap (C+t)$ has three ingredients: (i) It follows from (2.1) that

$$C \cap (C+t) = \bigcap_{k=0}^{\infty} (C_k \cap (C_k + t)). \tag{2.2}$$

(ii) There is a relationship, see Lemma 2.5, between $C_k \cap (C_k + t)$ and $C_k \cap (C_k + \lfloor t \rfloor_k)$, where $\lfloor t \rfloor_k = 0._n t_1 t_2 \ldots t_k$. ($iii$), the structure of the set $C_k \cap (C_k + \lfloor t \rfloor_k)$ is related to the structure of the set $C_{k+1} \cap (C_{k+1} + \lfloor t \rfloor_{k+1})$, see the definition of σ_t and Lemma 2.3, below.

Since $\lfloor t \rfloor_k = \frac{h}{n^k}$ for some integer h, then $C_k + \lfloor t \rfloor_k$ also consists of m^k disjoint n-ary intervals. Thus, an n-ary interval $J^{(h)} \subset C_k$ may interact with $C_k + \lfloor t \rfloor_k$ in combinations of only four cases: we say $J^{(h)}$ is in the *interval case* if $J^{(h)}$ is also an interval of $C_k + \lfloor t \rfloor_k$, the *potential interval case* if $J^{(h)} + \frac{1}{n^k}$ is an interval of $C_k + \lfloor t \rfloor_k$, the *potentially empty case* if $J^{(h)} - \frac{1}{n^k}$ is an interval of $C_k + \lfloor t \rfloor_k$, and the *empty case* if $J^{(h)} \cap (C_k + \lfloor t \rfloor_k)$ is empty.

Remark 2.1. According to Theorem 3.1 of [24], if D is any digits set and $t \in [0,1]$ admits a finite n-ary representation, then $C \cap (C+t) = A \cup B$ where A is either empty or a finite, disjoint collection of sets of the form $\frac{1}{n^k}(C+h)$ for some integers k and h, and B is either empty or a finite collection of points. For these reasons, we focus on real numbers which do not admit finite n-ary representation.

It is important to note that only interval and potential interval cases can contribute points to $C \cap (C+t)$ whenever t does not admit a finite n-ary representation.

Lemma 2.2. *Suppose $0 < t - \lfloor t \rfloor_k < \frac{1}{n^k}$ for some k. If J is an n-ary interval in C_k and J is either in the potentially empty or the empty case, then $J \cap (C_k + t)$ is empty. In particular, the intersection $J \cap C \cap (C+t)$ is empty.*

It is possible for $J^{(h)}$ to be both in the interval case and potentially empty case, or to be both in the potential interval case and in the potentially empty case. However, the intersections corresponding to the potentially empty cases do not contribute points to $C \cap (C+t)$ when $0 < t - \lfloor t \rfloor_k$ and we will not identify these cases with special terminology.

We introduce a function whose values tell us whether $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains interval cases, potential interval cases, both, or neither. Since $C_0 \cap (C_0 + \lfloor t \rfloor_0) = [0,1]$ consists of a single interval case, then we can examine $C_k \cap (C_k + \lfloor t \rfloor_k)$ using induction. Let $i := \sqrt{-1}$ and let $\xi : \{0, \pm 1, i\} \times \{0, \pm 1, \ldots, \pm(n-1)\} \to \{0, \pm 1, \pm i\}$ be determined by

$$\xi(0,h) := 0$$

$$\xi(1,h) := \begin{cases} 1 & \text{if } |h| \text{ is in } \Delta \text{ but not in } \Delta - 1 \\ -1 & \text{if } |h| \text{ is in } \Delta - 1 \text{ but not in } \Delta \\ i & \text{if } |h| \text{ is both in } \Delta \text{ and } \Delta - 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi(-1,h) := \begin{cases} -1 & \text{if } |h| \text{ is in } n-\Delta \text{ but not in } n-\Delta-1 \\ 1 & \text{it } |h| \text{ is in } n-\Delta-1 \text{ but not in } n-\Delta \\ -i & \text{if } |h| \text{ is both in } n-\Delta \text{ and in } n-\Delta-1 \\ 0 & \text{otherwise} \end{cases}$$

$$\xi(i,h) := \begin{cases} -i & \text{if } |h| \text{ is in } \Delta \cup (n-\Delta) \text{ but not in } (\Delta-1) \cup (n-\Delta-1) \\ i & \text{if } |h| \text{ is in } (\Delta-1) \cup (n-\Delta-1) \text{ but not in } \Delta \cup (n-\Delta) \\ 1 & \text{if } |h| \text{ is both in } \Delta \cup (n-\Delta) \text{ and in } (\Delta-1) \cup (n-\Delta-1) \\ 0 & \text{otherwise.} \end{cases}$$

The function $\xi(z,h)$ is completely determined by D and n. Let $\sigma_t : \mathbb{N}_0 \to \{0, \pm 1, i\}$ be determined by

$$\sigma_t(0) := 1 \text{ and inductively}$$
$$\sigma_t(k+1) := \xi(\sigma_t(k), t_{k+1}) \cdot \sigma_t(k) \text{ for } k \geq 0.$$

By construction of ξ we have $\sigma_t(k) \in \{0, \pm 1, i\}$ for all $k \geq 0$. Compared to [24] the present definition of ξ uses $|h|$ in place of h. This is to anticipate a variant needed in Section 4.

Lemma 2.3. *Let $t = 0._n t_1 t_2 \cdots$ be some point in $[0,1]$. Then $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains interval cases but no potential interval cases iff $\sigma_t(k) = 1$, potential interval cases but no interval cases iff $\sigma_t(k) = -1$, both interval and potential interval cases iff $\sigma_t(k) = i$, and neither interval cases nor potential interval cases iff $\sigma_t(k) = 0$.*

Lemma 2.3 allows us to describe F in terms of σ_t, when D is sparse.

Lemma 2.4. *Let $C = C_{n,D}$ be a deleted digits Cantor set. Then D is sparse iff*

$$F^+ = \{t \in [0,1] \mid \sigma_t(k) = \pm 1 \text{ for all } k \in \mathbb{N}\}.$$

Let $\#E$ denote the number of elements in a finite set E. Define $\mu_t(0) := 1$ and inductively

$$\mu_t(k+1) = \begin{cases} \mu_t(k) \cdot \#(D - t_{k+1}) \cap (D \cup (D+1)) & \text{if } \sigma_t(k) = 1 \\ \mu_t(k) \cdot \#(D - n + t_{k+1}) \cap (D \cup (D-1)) & \text{if } \sigma_t(k) = -1 \end{cases}$$

The function μ_t also depends on n and D, but we suppress this dependence in the notation. The function μ_t provides a method for counting the number of intervals contained in $C_k \cap (C_k + t)$.

Lemma 2.5. *Let $C = C_{n,D}$ be given. Suppose $t \in F^+$ does not admit a finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \geq 0$. Then $C_k \cap (C_k + t)$ is a union of $\mu_t(k)$ intervals, each of length*

$$\ell_k := \begin{cases} \frac{1}{n^k} - (t - \lfloor t \rfloor_k) & \text{when } \sigma_t(k) = 1 \\ t - \lfloor t \rfloor_k & \text{when } \sigma_t(k) = -1 \end{cases}.$$

While $\mu_t(k)$ provides an upper bound to the number of intervals of length ℓ_k required to cover $C_k \cap (C_k + t)$, it is important to know that each of these intervals contains points in $C \cap (C + t)$.

Lemma 2.6. *Let $C = C_{n,D}$ be given. Suppose $t \in F^+$ does not admit a finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \geq 0$. For each k, every n-ary interval of C_k in the interval or potential interval case contains points of $C \cap (C + t)$.*

If D is not sparse, then some interval or potential interval cases may not lead to points in $C \cap (C + t)$, see Example 3.13.

3 Real Values *t*.

In this section, we prove Theorem 1.3. Part of Theorem 1.1 is an immediate consequence. The other part of Theorem 1.1 is proved in Section 4.

As in Section 2, many of the results of this section only require that $t \in F$ and $\sigma_t(k) = \pm 1$ for all k. This allows us to apply our results when D is not sparse for specific values of t, see Section 3.4. On the other hand, if D is sparse the condition $\sigma_t(k) = \pm 1$ follows immediately from Lemma 2.4 for all t in F.

3.1 Translation Equivalence of n-ary representations.

We begin by investigating the structure of $C\cap(C+t)$ for an arbitrary value t in F. Lemma 3.1 describes how the structure of $C\cap(C+t)$ is related to the n-ary representation $t = 0._n t_1 t_2 \ldots$.

Lemma 3.1. *Let $C = C_{n,D}$ be given and $t \in F^+$ such that t does not admit finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \in \mathbb{N}_0$. Then $C\cap(C+t)$ is a union of $\mu_t(k)$ disjoint copies of*

$$\frac{1}{n^k}\left[C\cap\left(C+n^k\left(t-\lfloor t\rfloor_k\right)\right)\right] \text{ when } \sigma_t(k)=1$$

and of $\mu_t(k)$ disjoint copies of

$$\frac{1}{n^k}\left[C\cap\left(C-1+n^k\left(t-\lfloor t\rfloor_k\right)\right)\right] \text{ when } \sigma_t(k)=-1.$$

Proof. Any n-ary interval in C_k is of the form $J^{(h)} = \frac{1}{n^k}(C_0+h)$ for some $h\in\mathbb{Z}$. Let $J_j^{(h)} := \frac{1}{n^k}\left(C_j+h\right)$. Then

$$\frac{1}{n^k}\left(C\cap\left(C+n^k\left(t-\lfloor t\rfloor_k\right)\right)\right)+\frac{h}{n^k} = \bigcap_{j=1}^{\infty}\left(J_j^{(h)}\cap\left(J_j^{(h)}+\left(t-\lfloor t\rfloor_k\right)\right)\right) \text{ and}$$
$$\frac{1}{n^k}\left(C\cap\left(C-1+n^k\left(t-\lfloor t\rfloor_k\right)\right)\right)+\frac{h}{n^k} = \bigcap_{j=1}^{\infty}\left(J_j^{(h)}\cap\left(J_j^{(h)}-\frac{1}{n^k}+\left(t-\lfloor t\rfloor_k\right)\right)\right)$$

Now $J^{(h)}\subseteq C_k$ implies $J_j^{(h)}\subset C_{k+j}$ for all j, since $J_j^{(h)}$ is obtained from $J^{(h)}$ by repeated refinement.

According to Lemma 2.2, only n-ary intervals in C_k that are in the interval case or the potential intervals case can have points in common with $C\cap(C+t)$. By Lemma 2.5 there are $\mu_t(k)$ n-ary intervals $J^{(h)}\subseteq C_k$ in the interval or the potential interval case.

Suppose $\sigma_t(k)=1$. Then $J^{(h)}$ is in the interval case by Lemma 2.3. Hence $J^{(h)}\subset C_k+\lfloor t\rfloor_k$ and therefore $J^{(h)}+(t-\lfloor t\rfloor_k)\subset C_k+t$. By repeated refinement $J_j^{(h)}+(t-\lfloor t\rfloor_k)\subset C_{k+j}+t$. Consequently,

$$\begin{aligned}\bigcap_{j=1}^{\infty}\left(J_j^{(h)}\cap\left(J_j^{(h)}+\left(t-\lfloor t\rfloor_k\right)\right)\right) &\subseteq \bigcap_{j=1}^{\infty}\left(C_{k+j}\cap\left(C_{k+j}+t\right)\right)\\ &= C\cap(C+t).\end{aligned}$$

Suppose $\sigma_t(k)=-1$. Then $J^{(h)}$ is in the potential interval case by Lemma 2.3. Hence $J^{(h)}-\frac{1}{n^k}\subset C_k+\lfloor t\rfloor_k$ and therefore $J^{(h)}-\frac{1}{n^k}+(t-\lfloor t\rfloor_k)\subset C_k+t$. By repeated refinement $J_j^{(h)}-\frac{1}{n^k}+(t-\lfloor t\rfloor_k)\subset C_{k+j}+t$. Consequently,

$$\begin{aligned}\bigcap_{j=1}^{\infty}\left(J_j^{(h)}\cap\left(J_j^{(h)}-\frac{1}{n^k}+\left(t-\lfloor t\rfloor_k\right)\right)\right) &\subseteq \bigcap_{j=1}^{\infty}\left(C_{k+j}\cap\left(C_{k+j}+t\right)\right)\\ &= C\cap(C+t).\end{aligned}$$

Conversely, suppose $x \in C \cap (C+t) = \bigcap_{k=1}^{\infty} C_k \cap (C_k + t)$. Let $k \in \mathbb{N}_0$ be arbitrary and $J^{(h)} \subset C_k$ denote the n-ary interval such that $x \in J^{(h)}$ for some h by Lemma 2.2 and $J^{(h)}$ is in interval or potential interval case. Let

$$I_k := \left\{ i \mid J^{(i)} \subset C_k \text{ is } n\text{-ary and in the interval or potential interval case} \right\}.$$

If $J^{(j)} = J^{(i)} - \frac{1}{n^k}$ for some $j, i \in I_k$ then either $J^{(i)}$ or $J^{(j)}$ is in both the interval and potential interval cases, which contradicts that $\sigma_t(k) = \pm 1$. Thus, any $J^{(i)} \subset C_k \cap (C_k + \lfloor t \rfloor_k)$ such that $i \neq j$ is at least a distance of $\frac{1}{n^k}$ from $J^{(j)}$ and $J^{(j)} \cap J^{(i)} = \varnothing$.

Suppose $\sigma_t(k) = 1$ so that $J^{(i)}$ is in the interval case for all $i \in I_k$. Then $C_k \cap (C_k + t) = \bigcup_{i \in I} J^{(i)} \cap \left(J^{(i)} + (t - \lfloor t \rfloor_k) \right)$ so that $x \in J^{(h)} \cap C_k \cap (C_k + t) = J^{(h)} \cap \left(J^{(h)} + (t - \lfloor t \rfloor_k) \right)$. Furthermore, $J^{(h)} \cap C_{k+j} = J_j^{(h)}$ for each $j > 0$ by construction of C_k so that

$$x \in \bigcap_{j=1}^{\infty} \left(J^{(h)} \cap C_{k+j} \cap \left(C_{k+j} + t \right) \right) = \bigcap_{j=1}^{\infty} \left(J_j^{(h)} \cap \left(J_j^{(h)} + (t - \lfloor t \rfloor_k) \right) \right).$$

Since x is arbitrary, then $C \cap (C+t)$ is a subset of the disjoint union

$$\bigcup_{h \in I_k} \left(\bigcap_{j=1}^{\infty} \left(J_j^{(h)} \cap \left(J_j^{(h)} + (t - \lfloor t \rfloor_k) \right) \right) \right).$$

The case $\sigma_t(k) = -1$ is obtained by replacing $J_j^{(h)} + (t - \lfloor t \rfloor_k)$ by $J_j^{(h)} - \frac{1}{n^k} + (t - \lfloor t \rfloor_k)$ above. This completes the proof. □

According to Lemma 2.5, if $\sigma_t(j) = \pm 1$ for all j then $C_k \cap (C_k + \lfloor t \rfloor_k)$ is the disjoint union of $\mu_t(k)$ n-ary intervals of length $\frac{1}{n^k}$. Using the definition I_k from the proof of Lemma 3.1, $C_k \cap (C_k + \lfloor t \rfloor_k) = \bigcup_{h \in I_k} J^{(h)}$ and, for each $h \in I_k$, the corresponding interval $J^{(h)}$ refines to a "small" Cantor set intersected with its real translate. Lemma 3.1 shows that, for each k, the translation value directly depends on digits t_j for $j > k$, and the spacing of intervals $J^{(h)}$ depends on $\lfloor t \rfloor_k = 0._n t_1 t_2 \ldots t_k$. The requirement that $\sigma_t(k) = \pm 1$ guarantees that the intervals $J^{(h)}$ are disjoint. These results follow from the analysis in Section 2.

The next few lemmas establish some properties of translation equivalence. The first of these result allows us to calculate limits of sequences of the form $C \cap \left(C + x_j \right)$.

Let $\mathcal{H}^s(K)$ denote the s-dimensional Hausdorff measure of a set K. If D is sparse and $0 < \beta < 1$, $0 < y < \infty$ are arbitrary real numbers, then the set

$$F_{\beta,y} := \left\{ t \mid m^{-2\beta} y \leq \mathcal{H}^{\beta \log_n(m)} (C \cap (C+t)) \leq y \right\}$$

is dense in F, see [24]. Thus, the mapping $t \mapsto \mathcal{H}^s(C \cap (C+t))$ is everywhere discontinuous on F. In general, if $\left\{ x_j \right\}_{j=0}^{\infty}$ is a sequence in F^+ which converges to x, then the limit $\lim_{j \to \infty} \left(C \cap \left(C + x_j \right) \right)$ need not equal $C \cap (C+x)$, even when the limit exists with respect to the Hausdorff metric.

Example 3.2. Let $C = C_{3,\{0,2\}}$ denote the Middle Thirds Cantor set. Choose $x := \frac{1}{3}$ and $x_j := \sum_{i=1}^{j} \frac{2}{3^{i+1}} = 0._3 02 \cdots 2\bar{0}$ so that x_j converges to x. For each j,

$$C \cap \left(C + x_j \right) = \left(\frac{1}{3} - \frac{1}{3^{j+1}} C \right) \bigcup \left(1 - \frac{1}{3^{j+1}} C \right),$$

so that $\lim_{j\to\infty}\left(C\cap\left(C+x_j\right)\right)=\left\{\frac{1}{3},1\right\}$, however $C\cap(C+x)=\left\{\frac{1}{3},\frac{2}{3},1\right\}$.

However, with suitable restrictions on the sequence $\left\{x_j\right\}$ we show that the sets $C\cap\left(C+x_j\right)$ do converge.

Lemma 3.3. *Let $C=C_{n,D}$ and x_0 be given. Suppose $\left\{x_j\right\}_{j=0}^{\infty}$ is a sequence in F^+ converging to some real number x. If*

1. *$\sigma_{x_j}(k)=\pm1$ for all $j,k\in\mathbb{N}_0$,*
2. *for each j there exists $c_j\in\mathbb{R}$ such that $C\cap\left(C+x_j\right)=(C\cap(C+x_0))+c_j$ and*
3. *the sequence c_j converges to $c\in\mathbb{R}$,*

then $C\cap(C+x)=(C\cap(C+x_0))+c$.

Proof. Note that $x\in F^+$ by compactness. Furthermore, the sequence of compact sets $(C\cap(C+x_0))+c_j$ converges in the Hausdorff metric so that

$$\lim_{j\to\infty}\left(C\cap\left(C+x_j\right)\right)=(C\cap(C+x_0))+c.$$

We must show that $C\cap(C+x)=(C\cap(C+x_0))+c$. The result is trivial if $x=x_j$ for some j, so suppose $x\neq x_j$ for all j.

Let $y\in(C\cap(C+x_0))+c$ be arbitrary. For each j, choose $y_j\in C\cap\left(C+x_j\right)$ such that y_j converges to y. Thus, y_j is a sequence of C so that $y\in C$ and $\lim_{j\to\infty}\left\{C+x_j\right\}=C+x$ converges in the Hausdorff metric so that $y\in(C+x)$. Hence $(C\cap(C+x_0))+c\subseteq C\cap(C+x)$.

Let $y\in C\cap(C+x)$ be arbitrary. For each $j\in\mathbb{N}$, choose $N_j\in\mathbb{N}$ such that $\left(\frac{1}{n}\right)^{N_j+1}\leq\left|x-x_j\right|<\left(\frac{1}{n}\right)^{N_j}$. Thus, $\lfloor x\rfloor_{N_j}=\left\lfloor x_j\right\rfloor_{N_j}$ and $C_{N_j}\cap\left(C_{N_j}+\lfloor x\rfloor_{N_j}\right)=C_{N_j}\cap\left(C_{N_j}+\left\lfloor x_j\right\rfloor_{N_j}\right)$. Let J be the n-ary interval of C_{N_j} which contains y. According to Lemma 2.6, J contains points of $C\cap\left(C+x_j\right)$ so choose $y_j\in J\cap C\cap\left(C+x_j\right)$. Since J has length $\left(\frac{1}{n}\right)^{N_j}$ then $\left|y-y_j\right|\leq\left(\frac{1}{n}\right)^{N_j}$.

Thus, we can construct a sequence $\left\{y_j\right\}$ such that $y_j\in C\cap\left(C+x_j\right)$ for each j. Since $x_j\to x$, then $N_j\to\infty$ and y_j converges to y. Hence, $y\in\lim_{j\to\infty}\left\{C\cap\left(C+x_j\right)\right\}$ and $C\cap(C+x)=(C\cap(C+x_0))+c$. □

Corollary 3.4. *Let $\left\{x_j\right\}_{j=0}^{\infty}$ be a sequence in F^+ such that x_j converges to x, $\sigma_{x_j}(k)=\pm1$ for all k, $C\cap\left(C+x_j\right)=(C\cap(C+x_0))+c_j$ for each j, and the sequence c_j converges to c. Then $c_j\in F$ for all $j\in\mathbb{N}_0$.*

Proof. Let j be arbitrary. Then $C\cap\left(C+x_j\right)=\left(C+c_j\right)\cap\left(C+x_0+c_j\right)$ so that any element $y\in C\cap\left(C+x_j\right)$ is contained in both C and $\left(C+c_j\right)$. Thus $c_j\in F$ for all j and $c\in F$ by compactness. □

We now show that when t is rational with period p, then σ_t is also periodic with period p or $2p$.

Lemma 3.5. *Let $C_{n,D}$ be given. Suppose $t \in F^+$ does not admit finite n-ary representation, $\sigma_t(k) = \pm 1$ for all $k \in \mathbb{N}$, and $t = 0._n t_1 \cdots t_k \overline{t_{k+1} \cdots t_{k+p}}$ for some integer $k \geq 0$ and period p. Then σ_t has period p or $2p$.*

Proof. Suppose $\sigma_t(k+1) = \sigma_t(k+p+1)$. By induction, for any $j > k$,

$$\begin{aligned}\sigma_t(j+p+1) &= \xi\Big(\sigma_t(j+p), t_{j+p+1}\Big) \cdot \sigma_t(j+p) \\ &= \xi\Big(\sigma_t(j), t_{j+1}\Big) \cdot \sigma_t(j) \\ &= \sigma_t(j+1).\end{aligned}$$

Therefore, $\sigma_t(j) = \sigma_t(j+p)$ for all $j > k$ and $\sigma_t(k)$ has period p.

Suppose $\sigma_t(k+1) = -\sigma_t(k+p+1)$. If $\sigma_t(k+p+1) = \sigma_t(k+2p+1)$ then it follows that $\sigma_t(j) = \sigma_t(j+p)$ for $j > k+p$ and σ_t has period p by the argument above. Otherwise, $\sigma_t(k+1) = \sigma_t(k+2p+1)$. By induction, for any $j > k$,

$$\begin{aligned}\sigma_t(j+2p+1) &= \xi\Big(\sigma_t(j+2p), t_{j+2p+1}\Big) \cdot \sigma_t(j+2p) \\ &= \xi\Big(\sigma_t(j), t_{j+1}\Big) \cdot \sigma_t(j) \\ &= \sigma_t(j+1).\end{aligned}$$

Hence, $\sigma_t(j)$ has period $2p$. □

It follows from the next lemma that, if D is sparse, then any t in F^+ is translation equivalent to an s in F^+ such that $\sigma_s(k) = 1$ for all k. That is, all intervals in $C_k \cap (C_k + \lfloor s \rfloor_k)$ are in the interval case. We need Lemma 3.5 to show that, if t is rational, then the s we construct is also rational.

Lemma 3.6. *Let $C = C_{n,D}$ be given. Suppose $t \in F^+$ does not admit finite n-ary representation and $\sigma_t(k) = \pm 1$ for all $k \in \mathbb{N}$. Then there exists $y \in F^+$ such that $\sigma_y(k) = 1$ for all $k \in \mathbb{N}$ and $C \cap (C+t) = (C \cap (C+y)) + c$ for some $c \in \mathbb{R}$. If t is rational, then y is also rational.*

Proof. Let $t \in F$ be arbitrary. For a real sequence $\{x_i\}_{i=0}^{\infty}$, let $x_i := 0._n x_{i1} x_{i2} \ldots$ denote the n-ary representation. Let $x_0 := t$ so that $x_{0j} = t_j$ and $\sigma_{x_0}(j) = \pm 1$ for all $j \in \mathbb{N}$.

We will construct sequences $\{x_i\}_{i=0}^{\infty}$ and $\left\{\sum_{j=0}^{i} c_{k_j}\right\}_{i=0}^{\infty}$ which satisfy Lemma 3.3 and then show that $y := \lim_{i\to\infty} x_i$ satisfies our conditions. Let $c_{k_0} := 0$ so that $C \cap (C + x_0) = (C \cap (C+t)) + c_{k_0}$ and the translation condition is true for $i = 0$. Suppose $C \cap (C + x_i) = (C \cap (C + x_0)) + \sum_{j=0}^{i} c_{k_j}$ for some $i \in \mathbb{N}_0$ and $\sigma_{x_i}(j) = \pm 1$ for all $j \in \mathbb{N}_0$.

Let $P_i := \{h \mid \sigma_{x_i}(h) = -1\}$ be a subset of $\mathbb{N}$. By assumption, P_i is empty iff $\sigma_{x_i}(k) = 1$ for all k and we can choose $y := x_i$.

Suppose P_i is nonempty and let $k_{i+1} \in P_i$ be the minimal element. Thus, $\sigma_{x_i}(k_{i+1} - 1) = 1$ so that $x_{i,k_{i+1}} \in \Delta - 1$ by definition of σ. Therefore, $C \cap (C+t)$ consists of $\mu_{x_i}(k_{i+1})$ copies of $\frac{1}{n^{k_{i+1}}}\Big(C \cap \Big(C - 1 + n^{k_{i+1}}\Big(x_i - \lfloor x_i \rfloor_{k_{i+1}}\Big)\Big)\Big)$ by Lemma 3.1. Since

$$C \cap (C - 1 + z) = (C \cap (C + 1 - z)) - (1 - z)$$

for any real z, then

$$\begin{aligned}
&\frac{1}{n^{k_{i+1}}}\left(C\cap\left(C-1+n^{k_{i+1}}\left(x_i-\lfloor x_i\rfloor_{k_{i+1}}\right)\right)\right)\\
&\quad=\frac{1}{n^{k_{i+1}}}\left(C\cap\left(C+1-n^{k_{i+1}}\left(x_i-\lfloor x_i\rfloor_{k_{i+1}}\right)\right)\right)-c_{k_{i+1}}\\
&\quad=\frac{1}{n^{k_{i+1}}}\left(C\cap\left(C+n^{k_{i+1}}\left(\sum_{j=1+k_{i+1}}^{\infty}\frac{n-1-x_{ij}}{n^j}\right)\right)\right)-c_{k_{i+1}}.
\end{aligned}$$

where $c_{k_{i+1}}:=\frac{1}{n^{k_{i+1}}}\left(1-n^{k_{i+1}}\left(x_i-\lfloor x_i\rfloor_{k_{i+1}}\right)\right)$. Since $0\le n^{k_{i+1}}\left(x_i-\lfloor x_i\rfloor_{k_{i+1}}\right)\le 1$, then $0\le c_{k_{i+1}}\le \frac{1}{n^{k_{i+1}}}$. Choose x_{i+1} such that

$$x_{(i+1)j}=\begin{cases}x_{ij} & \text{for } 1\le j\le k_{i+1}-1\\ x_{ij}+1 & \text{for } j=k_{i+1}\\ n-1-x_{ij} & \text{for } j>k_{i+1}.\end{cases}$$

Thus, $\sigma_{x_{i+1}}(j)=\sigma_{x_i}(j)=1$ for all $1\le j<k_{i+1}$. It is by definition of σ that $\sigma_{x_{i+1}}(k_{i+1})=1$ since $x_{(i+1)k_{i+1}}\in\Delta$. Also, $\sigma_{x_{i+1}}(j)=\pm1$ for any $j>k_{i+1}$ since $\sigma_{x_i}(j)=\pm1$ by assumption and

$$\begin{aligned}
&x_{(i+1)j}\in\Delta && \text{iff } x_{ij}\in n-\Delta-1\\
&x_{(i+1)j}\in\Delta-1 && \text{iff } x_{ij}\in n-\Delta\\
&x_{(i+1)j}\in n-\Delta && \text{iff } x_{ij}\in\Delta-1\\
&x_{(i+1)j}\in n-\Delta-1 && \text{iff } x_{ij}\in\Delta.
\end{aligned}$$

In particular, $\sigma_{x_{i+1}}(j)=-\sigma_{x_i}(j)$ for any $j\ge k_{i+1}$. Therefore, $C\cap(C+x_{i+1})$ is not empty by Lemma 2.6 so that $x_{i+1}\in F^+$. Since each potential interval $J\subset C_{k_{i+1}}\cap\left(C_{k_{i+1}}+\lfloor x_i\rfloor_{k_{i+1}}\right)$ is an interval case in $C_{k_{i+1}}+\lfloor x_{i+1}\rfloor_{k_{i+1}}$ by Lemma 3.1 then $C\cap(C+x_i)=(C\cap(C+x_{i+1}))-c_{k_{i+1}}$. Hence,

$$C\cap(C+x_{i+1})=(C\cap(C+x_0))+\sum_{j=0}^{i+1}c_{k_j}.$$

By induction, $\{x_i\}$ is a sequence in F^+ such that $C\cap(C+x_i)=(C\cap(C+x_0))+\sum_{j=0}^{i}c_{k_j}$ and $\sigma_{x_i}(j)=\pm1$ for all $i,j\in\mathbb{N}_0$.

By construction, $0\le c_{k_i}\le\frac{1}{n^{k_i}}$ and $c_{k_0}=0$ so that $\sum_{j=0}^{i}c_{k_j}\le\sum_{j=1}^{i}\frac{1}{n^{k_j}}\le\sum_{j=1}^{k_i}\frac{1}{n^j}\le\frac{1}{n-1}$ for all $i\in\mathbb{N}_0$. Since the sequence $\left\{\sum_{j=0}^{i}c_{k_j}\right\}$ is increasing and bounded above, let $c:=\lim_{j\to\infty}\left\{\sum_{j=0}^{i}c_{k_j}\right\}$.

Let $\varepsilon>0$ be given. Choose $N\in\mathbb{N}$ such that $\varepsilon>\left(\frac{1}{n}\right)^{k_N}>0$ and let $N\le i<j$. Since x_i and x_j have been constructed so that the first k_i digits are equal, then $\left|x_i-x_j\right|\le\left(\frac{1}{n}\right)^{k_i}<\varepsilon$. Therefore, $\{x_i\}$ is a Cauchy sequence of F^+ and $y:=\lim_{i\to\infty}(x_i)$ is also in F^+. By construction, $y=0._n y_1y_2\ldots$ is the unique value such that

$$y_j=\begin{cases}t_j & \text{if } \sigma_t(j-1)=1 \text{ and } \sigma_t(j)=1\\ t_j+1 & \text{if } \sigma_t(j-1)=1 \text{ and } \sigma_t(j)=-1\\ n-1-t_j & \text{if } \sigma_t(j-1)=-1 \text{ and } \sigma_t(j)=-1\\ n-t_j & \text{if } \sigma_t(j-1)=-1 \text{ and } \sigma_t(j)=1.\end{cases}\tag{3.1}$$

Hence, the sequence $\{x_i\}$ satisfies the conditions of Lemma 3.3 so that

$$C\cap(C+t)=(C\cap(C+y))-c.$$

Furthermore, for each $i\in\mathbb{N}$ there exists $k_j>i$ such that y shares the first k_j digits of x_j and $\sigma_y(h)=\sigma_{x_j}(h)=1$ for all $0\le h\le k_j$. Thus $\sigma_y(h)=1$ for all $h\in\mathbb{N}$.

It remains to show that y is rational whenever t is rational. Suppose $k\ge p>0$ and $t=0._n t_1\cdots t_{k-p}\overline{t_{k-p+1}\cdots t_k}$. Let q denote a period of $\sigma_t(k)$ by Lemma 3.5. Since $t_{j+q}=t_j$ and $\sigma_t(j+q)=\sigma_t(j)$ for any $j>k$, then $y_j=y_{j+q}$ by equation (3.1) so that y has period q. $\square$

Remark 3.7. Define the function $\psi:[0,1]\to[0,1]$ according to equation (3.1) so that $\psi(t)=y$. For example, if $D=\{0,2,7,9\}$, $n=10$, and $t=0._n5\overline{44728}$, then $\psi(t)=0._n5\overline{55272}$. In example 4.4, we choose D, n, and $t,t'\in F$ such that $C(t)=C(t')$, yet $\psi(t)\neq\psi(t')$.

3.2 Proof of Theorem 1.3.

We have now developed the machinery necessary to prove the first half of Theorem 1.1. In fact, Theorem 1.3 is a special case of the following result.

Theorem 3.8. *Let $C_{n,D}$ be given and $z\in F$ be arbitrary. Suppose there exists $t\in F^+$ such that $C(z)=C(t)$, $t=0._n t_1\cdots t_{k-p}\overline{t_{k-p+1}\cdots t_k}$ for some period p and integer $k\ge p$, and $\sigma_t(j)=\pm1$ for all $j\in\mathbb{N}_0$. If q denotes a period of $\sigma_t(j)$ then there exists a digits set $E=\{0\le e_1<e_2<\cdots<e_r<n^q\}$ and corresponding deleted digits Cantor set $B=C_{n^q,E}$ such that $C\cap(C+t)$ consists of $\mu_t(k)$ disjoint copies of $\frac{1}{n^k}B$. If D is sparse then E is also sparse.*

Proof. Let $y:=\psi(t)\in F^+$ according to Lemma 3.6 so that $y=0._n y_1\cdots y_k\overline{x_1x_2\cdots x_q}$ does not admit finite n-ary representation, $\sigma_y(j)=1$ for all j, and $C\cap(C+t)=(C\cap(C+y))+c$ for some $c\in\mathbb{R}$. Define $x:=n^k(y-\lfloor y\rfloor_k)=0._n\overline{x_1\cdots x_q}$ so that $C\cap(C+y)$ consists of $\mu_y(k)$ disjoint copies of $\frac{1}{n^k}(C\cap(C+x))$ by Lemma 3.1. We will construct E and show that $C\cap(C+x)=B$.

Let $\{S_d\}_{d\in D}$ be the similarity mappings which generate C. Let $S^1(a):=\bigcup_{d\in D}S_d(a)$ so that $C=S^1(C)$ by definition and let $S^j(a)=\left(S^{j-1}\circ S^1\right)(a)$ for all $j\in\mathbb{N}$. Thus, $C\cap(C+x)=S^q(C)\cap(S^q(C)+x)$. For each $\boldsymbol{u}=\left(u_1,u_2,\cdots,u_q\right)\in D^q$, define

$$S_{\boldsymbol{u}}(a):=\left(S_{u_q}\circ\cdots\circ S_{u_1}\right)(a)=\frac{1}{n^q}\left(a+\sum_{j=1}^{q}u_j\cdot n^{q-j}\right).$$

Hence, $C=S^q(C)=\bigcup_{\boldsymbol{u}\in D^q}S_{\boldsymbol{u}}(C)$ and $C_k=\bigcup_{\boldsymbol{u}\in D^q}S_{\boldsymbol{u}}([0,1])$.

Since $x=\frac{1}{n^q}\left(\sum_{j=1}^{q}x_j\cdot n^{q-j}\right)+\frac{1}{n^q}x$, let $\boldsymbol{w}:=\left(x_1,x_2,\cdots,x_q\right)$. Then for any $\boldsymbol{u}\in D^q$,

$$\begin{aligned}S_{\boldsymbol{u}}(C)+x&=S_{\boldsymbol{u}}(C)+\frac{1}{n^q}\left(\sum_{j=1}^{q}x_j\cdot n^{q-j}\right)+\frac{x}{n^q}\\&=\frac{1}{n^q}\left(C+x+\sum_{j=1}^{q}\left(u_j+x_j\right)\cdot n^{q-j}\right)\\&=S_{\boldsymbol{u}+\boldsymbol{w}}(C+x).\end{aligned}$$

Therefore, $S_{\boldsymbol{u}}(C)+x=S_{\boldsymbol{u}+\boldsymbol{w}}(C)+\frac{x}{n^q}=S_{\boldsymbol{u}+\boldsymbol{w}}(C+x)$ and $\bigcup_{\boldsymbol{v}\in D^q+\boldsymbol{w}} S_{\boldsymbol{v}}([0,1])=C_k+\lfloor x\rfloor_k$. Since $\sigma_x(k)=1$ then $C_k\cap(C_k+\lfloor x\rfloor_k)$ is in the interval case by definition so that the sets $S_{\boldsymbol{u}}([0,1])$ and $S_{\boldsymbol{v}}([0,1])$ are either disjoint or equal for any pair $\boldsymbol{u}\in D^q$ and $\boldsymbol{v}\in D^q+\boldsymbol{w}$. Since $C\subseteq[0,1]$ and $0<x<1$ then

$$\begin{aligned} C\cap(C+x) &= \left[\bigcup_{\boldsymbol{u}\in D^q} S_{\boldsymbol{u}}(C)\right]\bigcap\left[\bigcup_{\boldsymbol{v}\in D^q+\boldsymbol{w}} (S_{\boldsymbol{v}}(C+x))\right] \\ &\subseteq \left[\bigcup_{\boldsymbol{u}\in D^q} S_{\boldsymbol{u}}([0,1])\right]\bigcap\left[\bigcup_{\boldsymbol{v}\in D^q+\boldsymbol{w}} (S_{\boldsymbol{v}}([0,1]+x))\right] \\ &\subseteq \left[\bigcup_{\boldsymbol{u}\in D^q} S_{\boldsymbol{u}}([0,1])\right]\bigcap\left[\bigcup_{\boldsymbol{v}\in D^q+\boldsymbol{w}} S_{\boldsymbol{v}}([0,1])\right]. \end{aligned}$$

Hence, $S_{\boldsymbol{u}}(C)\cap(S_{\boldsymbol{v}}(C+x))\neq\varnothing$ only if $\boldsymbol{v}\in D^q$. Thus,

$$\begin{aligned} C\cap(C+x) &= \left[\bigcup_{\boldsymbol{u}\in D^q} S_{\boldsymbol{u}}(C)\right]\bigcap\left[\bigcup_{\boldsymbol{v}\in D^q+\boldsymbol{w}} (S_{\boldsymbol{v}}(C+x))\right] \\ &= \bigcup_{\boldsymbol{u}\in D^q\cap(D^q+\boldsymbol{w})} (S_{\boldsymbol{u}}(C)\cap S_{\boldsymbol{u}}(C+x)) \\ &= \bigcup_{\boldsymbol{u}\in D^q\cap(D^q+\boldsymbol{w})} S_{\boldsymbol{u}}(C\cap(C+x)). \end{aligned}$$

Let $E:=\left\{\sum_{j=1}^q u_j\cdot n^{q-j}\mid \boldsymbol{u}\in D^q\cap(D^q+\boldsymbol{w})\right\}$. Then $B=C_{n^q,E}$ is the unique, nonempty compact set invariant under the mapping

$$\bigcup_{e\in E}\left(\frac{1}{n^q}(\cdot+e)\right)=\bigcup_{\boldsymbol{u}\in D^q\cap(D^q+\boldsymbol{w})} S_{\boldsymbol{u}}(\cdot).$$

Hence, $C\cap(C+x)=B$. Note that $\#E=\prod_{j=1}^q\#\Big(D\cap\Big(D+x_j\Big)\Big)=\mu_x(q)$.

Suppose D is sparse. It is sufficient to show that $\gamma-\gamma'\geq 2$ for any $\gamma\neq\gamma'$ in $\Gamma:=\left\{\sum_{j=1}^q u_j\cdot n^{q-j}\mid \boldsymbol{u}\in D^q-D^q\right\}$ since $E-E\subseteq\Gamma$. Let $\gamma\neq\gamma'$ be arbitrary and i be the smallest index $1\leq i\leq q$ such that $\gamma_i\neq\gamma_i'$. Without loss of generality, assume $\gamma_i>\gamma_i'$. Since D is sparse and $\gamma_j,\gamma_j'\in\Delta$, then $\left|\gamma_j-\gamma_j'\right|\geq 2$ for all $1\leq j\leq q$. Thus, if $i=q$ then $|\gamma-\gamma'|=\left|\gamma_q-\gamma_q'\right|\geq 2$. Otherwise, if $i<q$ then

$$\begin{aligned} \left|\gamma-\gamma'\right| &= \left|\sum_{j=i}^q\gamma_j\cdot n^{q-j}-\sum_{j=i}^q\gamma_j'\cdot n^{q-j}\right| = \left|\left(\gamma_i-\gamma_i'\right)n^{q-i}+\sum_{j=i+1}^q\left(\gamma_j-\gamma_j'\right)\cdot n^{q-j}\right| \\ &\geq \left|2n^{q-i}-\sum_{j=i+1}^q(n-1)\cdot n^{q-j}\right| \geq \left|2n^{q-i}-n^{q-i}\right|\geq n. \end{aligned}$$

Therefore, E is sparse. □

Theorem 3.8 shows that any sparse set C and rational $t\in F$ is the finite, disjoint union of self-similar sets and proves the first half of Theorem 1.1.

3.3 Hausdorff Measure of $C \cap (C+t)$.

The structure of the set $C \cap (C+t)$ is given by Theorem 3.8 when t is translate equivalent to a rational, and Lemma 3.1 when $\sigma_t(k) = \pm 1$ for all k. This additional structure allows us to apply various methods for calculating the Hausdorff dimension and measure of $C \cap (C+t)$. If t is an arbitrary element of F such that $\sigma_t(k) = \pm 1$ for all k, the Hausdorff dimension of $C \cap (C+t)$ can be calculated by methods in [23] and [24]. Specifically, if $t = 0._n t_1 t_2 \cdots t_k \overline{t_{k+1} \cdots t_{k+q}}$ and $\sigma_t(k) = 1$ for all k, then the Hausdorff dimension of $C(t)$ is $\frac{1}{q}\sum_{j=1}^{q} \log_n \#\left(D \cap \left(D + t_{k+j}\right)\right)$.

Remark 3.9. Assume the notation from Theorem 3.8. Furthermore, suppose D is sparse, then $s := \log_{n^q}(\#E)$ is the Hausdorff dimension of B and

$$\mathcal{H}^s(C \cap (C+t)) = (\mu_t(k))^s \cdot \mathcal{H}^s(B).$$

An algorithm for calculating the Hausdorff measure of B in a finite number of steps is known, see [1], [19], and [20]. An estimate of the $\dim_H(C \cap (C+t))$-dimensional Hausdorff measure of $C \cap (C+t)$ is given in [24] even when the set is is not a finite union of self-similar sets.

Proposition 3.10 gives a formula for the Hausdorff measure of a deleted digits Cantor set when D contains only two digits.

Proposition 3.10. *Let $n \geq 3$ and $0 \leq a < b < n$ be non-negative integers. If $D = \{a, b\}$ and $s := \log_n(2)$, then $\mathcal{H}^s(C_{n,D}) = \left(\frac{b-a}{n-1}\right)^s$.*

Proof. Let $n \geq 3$ be given. We may assume $D = \{0, d\}$ where $d := b - a \geq 1$ and $\Delta = \{-d, 0, d\}$. If $d \geq 2$ then D is sparse.

Since $\frac{n-1}{d} \cdot D = \{0, n-1\}$ then $B = C_{n, \frac{n-1}{d} \cdot D}$ is the self-similar Cantor set generated by removing the open "middle" interval of length $1 - 2 \cdot \frac{1}{n}$. This set is well known to have measure $\mathcal{H}^s(B) = 1$, see e.g., [10], [12], [7], or [19]. Hence, $\frac{d}{n-1} B = \left\{\sum_{j=1}^{\infty} \frac{t_j}{n^j} \mid t_j \in D\right\} = C$ and $\mathcal{H}^s(C) = \mathcal{H}^s\left(\frac{d}{n-1} \cdot B\right) = \left(\frac{d}{n-1}\right)^s$. □

Example 3.11. Let $C = C_{3,\{0,2\}}$ denote the middle thirds Cantor set. Let $t = 0._3\overline{20} = \frac{3}{4}$ so that $q = 2$ is a period of $\sigma_t(k)$. By Theorem 3.8, $C \cap (C+t)$ is the self-similar set $C_{9,\{6,8\}}$. If $s := \log_9(2)$, then $\mathcal{H}^s(C \cap (C+t)) = 4^{-s}$ by Proposition 3.10.

3.4 Non-sparse digit sets.

Many of the results in Section 3 only require that $t \in F$ satisfy $\sigma_t(k) = \pm 1$ for all $k \in \mathbb{N}_0$. In this section we construct specific examples to apply these results when D is not a sparse digits set. Example 3.12 constructs a family of values $t \in F$ when D is not sparse.

Example 3.12. Let $n = 10$, $D = \{0, 1, 2, 6, 8\}$, and $C = C_{n,D}$. Then D is not sparse, yet $\{2, 8\} \subset \Delta \setminus (\Delta - 1)$ where $\setminus$ denotes set subtraction. Thus, any $t \in C_{n,\{2,8\}}$ is such that $\sigma_t(k) = 1$ for all k by definition of σ. Let $t = 0._{10}\overline{2} = \frac{2}{9}$. Since $D \cap (D+2) = \{2, 8\}$ then $C \cap (C+t) = \{0._n x_1 x_2 \ldots \mid x_k \in \{2, 8\}\} = C_{n,\{2,8\}}$. If $s := \log_n(2)$ then $\mathcal{H}^s(C \cap (C+t)) = \left(\frac{2}{3}\right)^s$ by Proposition 3.10.

In specific cases, these methods can be applied to analyze values $t \in F$ when D is not sparse and $\sigma_t(k) \neq \pm 1$ for some k.

Example 3.13. Let $D = \{0,2,4,7,\dots,4+3r\}$ for some $r > 2$ and $n > 4+3(r+1)$. Choose $t = 0._n\overline{2} \in F$. Note that D is not sparse and $\sigma_t(k) = i$ for all $k \geq 1$. For each k, $C_k \cap (C_k + \lfloor t \rfloor_k)$ contains 2^k interval cases and $r \cdot 2^{k-1}$ potential interval cases, however the potential interval cases never contain points in $C_{n,D} \cap (C_{n,D} + t)$ since 2 is neither in $n - \Delta$ nor $n - \Delta - 1$.

For each k, let I_k denote the collection of 2^k interval cases of $C_k \cap (C_k + \lfloor t \rfloor_k)$ so that $C_{n,D} \cap (C_{n,D} + t) \subset \bigcup_{J \in I_k} J$ for each k. If $E := \{0,2,4\}$, then I_k consists of the same 2^k intervals chosen from the k^{th} step in the construction of $C_{n,E} \cap (C_{n,E} + t)$. Since $C_{n,E} \cap (C_{n,E} + t) = \bigcap_{k=1}^{\infty} \left(\bigcup_{J \in I_k} J \right)$ implies $C_{n,D} \cap (C_{n,D} + t) \subseteq C_{n,E} \cap (C_{n,E} + t)$, and $E \subset D$, then

$$C_{n,D} \cap (C_{n,D} + t) = C_{n,E} \cap (C_{n,E} + t).$$

Since E is sparse, then $C_{n,E} \cap (C_{n,E} + t) = C_{n,\{2,4\}}$ and $\mathcal{H}^s(C_{n,\{2,4\}}) = \left(\frac{2}{n-1}\right)^s$ when $s := \log_n(2)$ by Proposition 3.10. Thus, for a specific choice of t, D, and n, we can apply our method even though D is not sparse and $\sigma_t(k)$ does not equal ± 1 for any k.

4 Unions of Self-Similar Sets

In this section we prove the second half of Theorem 1.1.

Remark 4.1. A real number $\alpha \in [0,1]$ has a Δ *representation,* if $\alpha = \sum_{k=1}^{\infty} \frac{\alpha_k}{n^k} = 0._n\alpha_1\alpha_2\dots$ and each α_k is a digit of Δ for all k. Let $\lfloor \alpha \rfloor_k = 0._n\alpha_1\alpha_2\dots\alpha_k$. Note Δ representations allows the digits α_k to be positive for some k and negative for other k. It is easy to see that F is the self-similar set $\{0._n\alpha_1\alpha_2\cdots \mid \alpha_k \in \Delta\}$, see e.g., [23]. Throughout this paper we will denote Δ representations as α, γ and reserve t, x, and y for n-ary representations.

The discussion in Section 2 holds for $C \cap (C+\alpha)$ with Δ representations of α in F. Detailed proofs are in [24] for the case of n-ary representations. The key observations are that $\Delta = -\Delta$, $|h| \in \Delta - 1$ iff $-|h| \in \Delta + 1$, $\sigma_\alpha(k)$ only has values in $\{1, i\}$, and the geometric configurations we called potentially empty cases in Section 2 are now potential interval cases.

Remark 4.2. Suppose $\alpha \in F$ admits finite Δ representation and let $j,k \in \mathbb{Z}$ satisfy $\alpha = \lfloor \alpha \rfloor_k = \frac{j}{n^k}$. We may define an n-ary representation $t := 0._nt_1t_2\cdots t_k = \frac{|j|}{n^k}$ so that $C(t) = C(|\alpha|) = C(\alpha)$ by definition of F. Thus, we may apply Theorem 3.1 of [24] so that $C \cap (C+\alpha) = A \cup B$, where A and B are the sets defined in Remark 2.1. Hence, we will focus our analysis on values $\alpha \in F$ which do not admit finite Δ representations.

4.1 Translation Equivalence of Δ representations.

The next result continues our investigation of translation equivalence. More precisely, we describe translation equivalence in terms of the digit set D.

Theorem 4.3. *Let $n \geq 3$ and let D be a sparse digit set. Let $\alpha = \sum_{k=1}^{\infty} \alpha_k n^{-k}, \beta = \sum_{k=1}^{\infty} \beta_k n^{-k}$, and $\delta = \sum_{k=1}^{\infty} \delta_k n^{-k}$. If $D \cap (D+\alpha_k) = D \cap (D+\beta_k) + \delta_k$ for all $k \geq 1$, then $C \cap (C+\alpha) = (C \cap (C+\beta)) + \delta$.*

Proof. Recall, if A is a set of real numbers and t is a real number then $tA = \{ta \mid a \in A\}$ and if A and B are two sets of real numbers then $A + B = \{a + b \mid a \in A, b \in B\}$.

Let $C_0 = [0,1]$. The refinement is $C_1 = \frac{1}{n}(D + C_0) = \frac{1}{n}D + [0,1/n]$. The refinement of C_1 is $C_2 = \frac{1}{n}(D + C_1) = \frac{1}{n}D + \frac{1}{n^2}D + [0,1/n^2]$. Continuing in this manner we see that

$$C_k = \frac{1}{n}D + \frac{1}{n^2}D + \cdots + \frac{1}{n^k}D + \left[0,1/n^k\right] = \sum_{i=1}^{k} \frac{1}{n^i}D + \left[0,1/n^k\right].$$

By sparsity of D the distance between any two of these intervals is ℓ/n^k for some integer $1 \le \ell < n^k$. Therefore,

$$C_k + \sum_{i=1}^{k} \alpha_i n^{-i} = \sum_{i=1}^{k} \frac{1}{n^i}(D + \alpha_i) + \left[0,1/n^k\right]$$

and consequently,

$$C_k \cap \left(C_k + \sum_{i=1}^{k} \alpha_i n^{-i}\right) = \sum_{i=1}^{k} \frac{1}{n^i}(D \cap (D + \alpha_i)) + \left[0,1/n^k\right].$$

Using $D \cap (D + \alpha_i) = (D \cap (D + \beta_i)) + \delta_i$ for $1 \le i \le k$, it follows that

$$C_k \cap \left(C_k + \sum_{i=1}^{k} \alpha_i n^{-i}\right) = \left(C_k \cap \left(C_k + \sum_{i=1}^{k} \beta_i n^{-i}\right)\right) + \sum_{i=1}^{k} \delta_i n^{-i}$$

and that this is a collection of intervals each of length $1/n^k$. Let $\beta^{(0)} = \beta$, and

$$\beta^{(k)} = \sum_{i=1}^{k} \alpha_i n^{-i} + \sum_{i=k+1}^{\infty} \beta_i n^{-i}.$$

Using $\beta^{(k)} - \sum_{i=1}^{k} \alpha_i n^{-i} = \beta^{(0)} - \sum_{i=1}^{k} \beta_i n^{-i}$ we conclude

$$C_k \cap \left(C_k + \beta^{(k)}\right) = \left(C_k \cap \left(C_k + \beta^{(0)}\right)\right) + \sum_{i=1}^{k} \delta_i n^{-i}$$

is a collection of intervals each of length $\frac{1}{n^k}\left(1 - \left|\sum_{i=k+1}^{\infty} \beta_i n^{-i}\right|\right)$. Repeatedly refining the intervals in C_k we get

$$C_j \cap \left(C_j + \beta^{(k)}\right) = \left(C_j \cap \left(C_j + \beta^{(0)}\right)\right) + \sum_{i=1}^{k} \delta_i n^{-i}$$

for $j \ge k$. Consequently,

$$C \cap \left(C + \beta^{(k)}\right) = \left(C \cap \left(C + \beta^{(0)}\right)\right) + \sum_{i=1}^{k} \delta_i n^{-i}.$$

Since $\beta^{(k)} \to \alpha$ as $k \to \infty$ the result follows from Lemma 3.3. $\square$

Example 4.4 applies Theorem 4.3 to construct an uncountable set of values $x \in F^+$ which are not only translation equivalent, but all generate the same set $C \cap (C+t)$.

Example 4.4. Let $D = \{0,5,7\}$ and $n = 8$ so that $\Delta = \{-7,-5,-2,0,2,5,7\}$ and $C = C_{n,D}$ is sparse. Choose $t := 0._8\overline{07}$. Then $C \cap (C+t) = C_{64,\{7,47,63\}}$ has dimension $s := \log_{64}(3)$ and measure $0 < \mathcal{H}^s(C \cap (C+t)) < \infty$.

Note that $D \cap (D+2) = \{7\}$, $D \cap (D+5) = \{5\}$, and $D \cap (D+7) = \{7\}$. Let $x \in [0,1]$ with ternary representation $x := 0._3 x_1 x_2 \ldots$. Let $f : [0,1] \to F$ such that $f(x) = 0._8 y_1 y_2 \ldots$ consists of digits

$$y_{2k-1} := 0$$

$$y_{2k} := \begin{cases} 2 & \text{if } x_k = 0 \\ 5 & \text{if } x_k = 1 \\ 7 & \text{if } x_k = 2. \end{cases}$$

Then, $f(1) = t$, $\sigma_{f(x)}(k) = 1$ for all k, and $f(x)$ is irrational whenever x is irrational. It follows from Theorem 4.3 that $(C \cap (C+f(x))) + c = C \cap (C+t)$ is self-similar, in particular, $f(x)$ is translation equivalent to t for all $x \in [0,1]$. It is, perhaps, interesting to note that since $D \cap (D+2) = D \cap (D+7)$ then $C \cap (C+f(x)) = C \cap (C+t)$ for any representation of x chosen from the middle thirds Cantor set $C_{3,\{0,2\}}$.

Define $\Delta^+ := \Delta \cap [0,\infty)$. We say that $\alpha \in F$ has a Δ^+ *representation* if $\alpha = 0._n\alpha_1\alpha_2\ldots$ such that each $\alpha_k \in \Delta^+$ for all k. According to Corollary 4.5, when D is sparse, we can restrict our analysis to Δ^+ representations in F without loss of generality.

Corollary 4.5. *Suppose D is sparse. If $\alpha \in F^+$ has Δ representation $\alpha = \sum_{k=1}^{\infty} \alpha_k n^{-k}$, then α is translation equivalent to $\widetilde{\alpha} := \sum_{k=1}^{\infty} |\alpha_k| n^{-k}$.*

Proof. Let $\alpha \in F$ be given with Δ representation $\alpha := \sum_{k=1}^{\infty} \alpha_k n^{-k}$. If $\alpha_k \le 0$ for some k, then $\alpha_k, \pm|\alpha_k|$ are integers such that $D \cap (D+\alpha_k) = D \cap (D+|\alpha_k|) - |\alpha_k|$. Hence α is translation equivalent to $\widetilde{\alpha} = \sum_{k=1}^{\infty} |\alpha_k| n^{-k}$ by Theorem 4.3. □

Remark 4.6. For the middle thirds Cantor set this shows that any intersection $C \cap (C+t)$ is a translate of an intersection $C \cap (C+s)$ with s in C.

4.2 Proof of Theorem 1.1.

In this section, we will prove the second part of Theorem 1.1. We begin by showing that Δ^+ representations are unique.

Lemma 4.7. *Let D be sparse. If $\alpha \in F^+$ has a representation $\alpha = \sum_{k=1}^{\infty} \frac{\alpha_k}{n^k}$ with digits $\alpha_k \in \Delta^+$ for all k, then this representation is unique.*

Proof. Suppose the Δ^+ representation $\alpha = 0._n\alpha_1\alpha_2\ldots$ is not unique. Any sequence $\{\alpha_k\} \subseteq \Delta^+$ is also a sequence of $\{0,1,\ldots,n-1\}$ so that $\sum_{k=1}^{\infty} \alpha_k n^{-k}$ is an n-ary representation of α. Thus α has two n-ary representations with digits in Δ^+, namely, $0._n\alpha_1\ldots\alpha_k b\overline{0}$ and $0._n\alpha_1\ldots\alpha_k(b-1)\overline{(n-1)}$ for some k and $1 \le b \le n-1$. Hence, b and $b-1$ are both elements of Δ, which contradicts that D is sparse by assumption. Therefore, the n-ary representation with digits in Δ^+ is unique. □

Since the Δ^+ representation is unique whenever D is sparse, we can classify the set $C \cap (C+\alpha)$ in terms of the digits $\{\alpha_k\}$.

Lemma 4.8. *Let D be sparse. If $\alpha \in F$ has Δ^+ representation $\alpha = 0._n\alpha_1\alpha_2\ldots$, then*

$$C \cap (C+\alpha) = \{0._n x_1 x_2 \ldots \mid x_k \in D \cap (D+\alpha_k)\}.$$

Proof. Let $x \in C$ be arbitrary and choose $y \in C$ such that $x = y+\alpha$. Denote $x := 0._n x_1 x_2 \ldots$ and $y := 0._n y_1 y_2 \ldots$ such that $x_k, y_k \in D$. Suppose $x_k \neq y_k + \alpha_k$ for some k. Without loss of generality, suppose $k-1 = \min\left\{j \mid x_j \neq y_j + \alpha_j\right\}$.

If $y_k + \alpha_k < n$ then x has two n-ary representations with digits strictly contained in $D \subseteq \Delta^+$, which is a contradiction by Lemma 4.7.

If $y_k + \alpha_k \geq n$ then $0 \leq y_k, \alpha_k < n$ implies $0 \leq y_k + \alpha_k - n < n$. Thus, $x = 0._n x_1 \ldots x_{k-1} x_k \ldots = 0._n x_1 \ldots (x_{k-1}+1)(y_k + \alpha_k - n)\ldots$ has two different n-ary representations. Therefore $x_j = n-1 \in D$ and $y_j + \alpha_j - n = 0$ for all $j \geq k$. However, $\alpha_k \in \Delta$ by definition and $\alpha_k - 1 = (n-1) - y_k$ is some element of $D - D = \Delta$, which contradicts that D is sparse.

Hence, $x_k = y_k + \alpha_k$ for each k and $C \cap (C+\alpha) \subseteq \{0._n x_1 x_2 \cdots \mid x_k \in D \cap (D+\alpha_k)\}$ since x is arbitrary. The reverse inclusion follows immediately since $C = \{0._n x_1 x_2 \cdots \mid x_k \in D\}$. □

When α has Δ^+ representation, then $\inf(C \cap (C+\alpha)) = \sum_{k=1}^{\infty} n^{-k} \cdot \min(D \cap (D+\alpha_k))$ according to Lemma 4.8. For each $\delta \in \Delta^+$, define $D_\delta := D \cap (D+\delta) - \min(D \cap (D+\delta))$ so that $0 \in D_\delta \subseteq \Delta^+$ and

$$C(\alpha) = C \cap (C+\alpha) - \sum_{k=1}^{\infty} \frac{\min(D \cap (D+\alpha_k))}{n^k} = \left\{\sum_{k=1}^{\infty} \frac{x_k}{n^k} \mid x_k \in D_{\alpha_k}\right\}. \tag{4.1}$$

This leads to the following Corollary to Theorem 4.3:

Corollary 4.9. *Let D be sparse and suppose $0._n\alpha_1\alpha_2\ldots$ is a Δ^+ representation for α. Then $C(\alpha) = C(\gamma)$ if and only if $D_{\alpha_k} = D_{\gamma_k}$ for all k.*

We now prove the second part of Theorem 1.1 when $C(\alpha)$ is a finite set.

Lemma 4.10. *Let D be sparse and $\alpha \in F$ given. If $C(\alpha)$ is a finite set then α is equivalent to a rational. In this case, $C \cap (C+\alpha)$ is the finite, disjoint union of trivial self-similar sets.*

Proof. Suppose $C(\alpha)$ is finite and let $K := \{k \mid \{0\} \subsetneq D_{\alpha_k}\}$. Suppose $K = \{k_1 < k_2 < \cdots\}$ is an infinite subset of $\mathbb{N}$. For any $x = 0._2 x_1 x_2 \ldots$ in $[0,1]$ with $x_k \in \{0,1\}$, define

$$f(x) := \sum_{j=1}^{\infty}\left(\frac{x_j}{n^{k_j}} \cdot \min\left\{a > 0 \mid a \in D_{\alpha_{k_j}}\right\}\right).$$

Thus, $f(x) \in C(\alpha)$ for all $x \in [0,1]$ and $f(x) \neq f(y)$ for any $x \neq y$ so that $f([0,1])$ is an uncountably infinite subset of $C(\alpha)$. This contradicts the assumption that $C(\alpha)$ is finite, hence K is either a finite subset of $\mathbb{N}$ or empty. If K is empty then $D_{\alpha_k} = \{0\}$ for all k and $C(\alpha) = \{0\} = C\left(\frac{d_m}{n-1}\right)$.

Suppose K is finite. Let $k = \max(K)$ and define $\gamma := 0._n\alpha_1\alpha_2\cdots\alpha_k\overline{d_m}$ so that $D \cap \left(D+\alpha_j\right) = D \cap \left(D+\gamma_j\right)$ for each $j \leq k$. Since $D \cap \left(D+\alpha_j\right) = \left\{d_{i_j}\right\}$ and $D \cap \left(D+\gamma_j\right) = \{d_m\}$ for $j > k$ then $D_{\alpha_j} = \{0\} = D_{\gamma_j}$. Hence, $C(\alpha) = C(\gamma)$ so that α is translation equivalent to γ. □

The proof of Lemma 4.10 shows that $C\cap(C+\alpha)$ is either finite or uncountably infinite. Note that α need not admit finite n-ary representation in the proof of Lemma 4.10. Example 4.11 exhibits an irrational number α such that $C(\alpha)$ is finite.

Example 4.11. Let $D=\{0,5,7\}$, $n=8$, and $\alpha=0._n0\overline{7}$ so that $C\cap(C+\alpha)=\left\{\frac{1}{8},\frac{3}{4},1\right\}$. By defining γ such that $\gamma_k=\alpha_k$ except $\gamma_{2k}=2$ on a sparse set of k's (larger than 1) then $C\cap(C+\gamma)=C\cap(C+\alpha)$ by Theorem 4.3 yet γ does not admit finite n-ary representation.

Lemma 4.12. *Let D be sparse and $\alpha\in F$ given. There exist non-negative integers k,q such that $D_{\alpha_j}\subseteq D_{\alpha_{j+q}}$ for all $j>k$ if and only if α is translation equivalent to a rational number.*

Proof. Suppose $D_{\alpha_j}\subseteq D_{\alpha_{j+q}}$ for all $j>k$. Since $D_{\alpha_j}\subset\{0,1,\dots,d_m\}$ for all j, then for each $1\le i\le q$ there exists a chain

$$D_{\alpha_{k+i}}\subseteq D_{\alpha_{k+i+q}}\subseteq\cdots\subseteq D_{\alpha_{k+i+jq}}\subseteq\{0,1,\dots,d_m\}.$$

Therefore equality holds for all $D_{\alpha_{k+i+jq}}$ after a certain point. For each i, let h_i be a value such that $D_{\alpha_{k+i+h_iq}}=D_{\alpha_{k+i+(h_i+j)q}}$ for all $j\ge0$. If $h:=\max_i(h_i)$ then $D_{\alpha_{k+i+hq}}=D_{\alpha_{k+i+(h+j)q}}$ for all $1\le i\le q$ and $j\ge0$. Let $\gamma:=0._n\alpha_1\cdots\alpha_{k+hq}\overline{\alpha_{k+hq+1}\cdots\alpha_{k+(h+1)q}}$. Then $D_{\alpha_j}=D_{\gamma_j}$ for all $j\in\mathbb{N}$ so that $C(\alpha)=C(\gamma)$.

Conversely, suppose $\gamma=0._n\gamma_1\cdots\gamma_k\overline{\gamma_{k+1}\cdots\gamma_{k+q}}$ is translation equivalent to α. Then $C(\alpha)=C(\gamma)$ and $D_{\alpha_j}=D_{\gamma_j}$for all $j\in\mathbb{N}$ by equation (4.1). Thus, $D_{\alpha_j}=D_{\gamma_j}=D_{\gamma_{j+q}}=D_{\alpha_{j+q}}$ for all $j>k$. □

We now have the tools required to prove the second half of Theorem 1.1.

Theorem 4.13. *Let D be sparse and $\alpha\in F$ be given. Suppose there exists $\varepsilon>0$ such that $C(\alpha)\cap[0,\varepsilon]$ is a self-similar set generated by similarities $f_j(x)=r_jx+b_j$ where $r_i=n^{-q_i}$ for some $q_i\in\mathbb{Z}$. Then α is translation equivalent to a rational number.*

Proof. According to Corollary 4.5 we may assume α has a Δ^+ representation. Let $\varepsilon>0$ be a value such that $C(\alpha)\cap[0,\varepsilon]=T$ is a self-similar set. We may assume that $b_1<b_2<\cdots<b_\ell$ so that $b_1=0$ and $f_1(x)=x\cdot n^{-q_1}$. Choose $k\in\mathbb{N}$ such that $\varepsilon\ge n^{-k}>0$ and let $j>k$ be arbitrary. Let $d\in D_{\alpha_j}$ be arbitrary so that $d\cdot n^{-j}\in T\subset C(\alpha)$ by equation (4.1). We note that the representation $d\cdot n^{-j}$ is unique by Lemma 4.7. Thus, $f_1\left(d\cdot n^{-j}\right)=d\cdot r_1\cdot n^{-j}=d\cdot n^{-(j+q_1)}\in C(\alpha)$ and $d\in D_{\alpha_{j+q_1}}$. Since j and d are arbitrary, then $D_{\alpha_j}\subseteq D_{\alpha_{j+q_1}}$ for all $j>k$ and α is translation equivalent to a rational number by Lemma 4.12. □

This completes the proof of Theorem 1.1. Theorem 4.13 shows that if $C(\alpha)\cap[0,\varepsilon]$ is constructed by specific similarity mappings, then α is translation equivalent to a rational number and, by Theorem 3.8, can be expressed as

$$C\cap(C+\alpha)=\bigcup_{j=1}^{N}\left(C_{n^{2p},E}+\eta_j\right)$$

for some $\eta_1<\eta_2<\cdots<\eta_N$.

4.3 When is $C(\alpha)$ self-similar?

Essentially, half of the answer to this question is provided by a calculation on page 307 of [18] and the other half by an elaboration on the proof of Theorem 4.13.

Note that, if $C(\alpha)$ is self-similar, then we may choose $k = 0$ in the proof of Theorem 4.13. Hence, this proof shows that for some $q > 0$ we have

$$D_{\alpha_j} \subseteq D_{\alpha_{j+q}} \text{ for all } j > 0. \tag{4.2}$$

But this condition is not sufficient for $C(\alpha)$ to be self-similar. For this we need the stronger condition that α is *strongly periodic* in the sense that there exists $\widetilde{D}_{\alpha_j}$ such that

$$D_{\alpha_j} + \widetilde{D}_{\alpha_j} = D_{\alpha_{j+q}} \text{ for all } j > 0. \tag{4.3}$$

Clearly, whether of not a given α satisfies (4.2) or (4.3) depends on the set D.

The following is a restatement of Theorem 1.2.

Theorem 4.14. *If D is sparse, then $C \cap (C+\alpha)$ is self-similar generated by a finite set of similarities $f_j(x) = n^{-q}x + b_j$ with $q \in \mathbb{N}$ if and only if α is strongly periodic.*

Proof. Suppose (4.3) holds. Since (4.3) implies (4.2) it follows that $D_{\alpha_j} = D_{\alpha_{j+q}}$ for all sufficiently large j. Hence, for some $p > 0$, we have

$$D_{\alpha_j} + \widetilde{D}_{\alpha_j} = D_{\alpha_{j+pq}} \text{ when } j \le pq \text{ and } D_{\alpha_j} = D_{\alpha_{j+pq}} \text{ when } j > pq.$$

Consequently,

$$D_{\alpha_{j+pqr}} = D_{\alpha_j} + \widetilde{D}_{\alpha_j}, \text{ when } j \le pq \text{ and } r \ge 1. \tag{4.4}$$

It follows now from the calculation on the top half of page 307 of [18] that, $C(\alpha)$ is a self-similar set. For the convenience of the reader we sketch the details. Let $x \in C(\alpha)$. Use (4.1) to write $x = \sum_k x_k n^{-k}$, with $x_k \in D_{\alpha_k}$. By (4.4) we can write

$$x_{k+pqr} = y_{r,k} + z_{r,k}, y_{r,k} \in D_{\alpha_k}, z_{r,k} \in \widetilde{D}_{\alpha_k} \text{ when 1≤k≤pq, 1≤r}$$

and $y_{0,k} = x_k$ when $1 \le k \le pq$. Then

$$\begin{aligned}
\sum_k x_k n^{-k} &= \sum_{r=0}^{\infty} \sum_{k=1}^{pq} x_{k+pqr} n^{-(k+pqr)} \\
&= \sum_{k=1}^{pq} y_{0,k} n^{-k} + \sum_{r=1}^{\infty} n^{-pqr} \sum_{k=1}^{pq} (y_{r,k} + z_{r,k}) n^{-k} \\
&= \sum_{r=0}^{\infty} \left(\sum_{k=1}^{pq} (y_{r,k} + z_{r+1} n^{-pq}) n^{-k} \right) n^{-pqr}.
\end{aligned}$$

It follows that $C(\alpha)$ is generated by the similarities

$$f_b(x) = n^{-pq}x + b, b \in B,$$

where $B = \left\{ \sum_{k=1}^{pq} (y_k + z_k n^{-pq}) n^{-k} \middle| y_k \in D_{\alpha_k}, z_k \in \widetilde{D}_{\alpha_k} \right\}$.

On the other hand, suppose $C(\alpha)$ is generated by the similarities $f_j(x) = n^{-q}x + b_j$, $j = 1,2,\ldots,L$. Since 0 is in $C(\alpha)$ it follows that b_j is in $C(\alpha)$ for all j. Write

$$b_j = \sum_k b_{j,k} n^{-k}, \text{ with } b_{j,k} \in D_{\alpha_k}.$$

Let $\widetilde{D}_{\alpha_k} = \left\{ b_{j,k+q} \mid j = 1,2,\ldots,L \right\}$. For any $x = \sum_k x_k n^{-k}$, $x_k \in D_{\alpha_k}$ we have

$$f_j(x) = \sum_{k=1}^{q} b_{j,k} n^{-k} + \sum_{k=1}^{\infty} \left(b_{j,k+q} + x_k \right) n^{-(k+q)}.$$

Since D is sparse it follows that $b_{j,k+q} + x_k$ is in $D_{\alpha_{k+q}}$. Consequently, $\widetilde{D}_{\alpha_k} + D_{\alpha_k} \subseteq D_{\alpha_{k+q}}$. If one of these inclusions is strict, then $\bigcup_j f_j(C(\alpha))$ would be a strict subset of $C(\alpha)$, by (4.1). Hence, (4.3) holds. □

Example 4.15. Let $D = \{0,2,4,6\}$, $n = 7$, and $\alpha = 0.2\overline{0}$. Then it follows from Theorem 4.14 that $C(\alpha)$ is self-similar. However, the self-similarities constructed in the proof of Theorem 4.14 do not satisfy the open set condition. Hence, $C(\alpha)$ is perhaps better understood in terms of Theorem 1.3 where $C(\alpha)$ is described as a finite union of disjoint translates of a deleted digits Cantor set.

Remark 4.16. After we circulated the first version of this paper, containing a version of Theorem 4.14 valid for uniform sets, Derong Kong asked us to provide a set of similarities for the set $C(\alpha)$, when $D = \{0,2,4,8\}$, $n = 9$, and $\alpha = 0.2\overline{0}$. This is not possible, since α satisfies (4.2), but does not satisfy (4.3). We replied to Derong Kong query that we had a proof of Theorem 4.14 as stated above. Subsequently Derong Kong sent us a preliminary version of the manuscript [14] containing a similar result. Our Theorem 4.14 is similar to [14, Theorem 2.3], however [14, Theorem 2.3] shows that $C(\alpha)$ is generated by similarities $f_b(x) = n^{-q}x + b$ from the assumption that $C(\alpha)$ is generated by similarities $f_b(x) = rx + b$, $0 < |r| < 1$.

5 A Construction of Numbers not Translation Equivalent to a Rational

The structure of $C \cap (C + \alpha)$ is determined by the previous sections whenever α is translation equivalent to a rational number. However, there exist many elements α in F such that $C(\alpha)$ is not a finite union of self-similar sets in the sense of Theorem 1.3. Lemma 4.12 allows us to construct a family of values $\gamma \in F^+$ which are not translation equivalent to a rational number. In fact, the proof below associates such an uncountable family of such γ to any rational α for which $C \cap (C + \alpha)$ is infinite.

Proposition 5.1. *Let D be sparse. There exists an uncountably infinite family of values $\gamma \in F^+$ which are not translate equivalent to any rational number.*

Proof. Let α be a rational such that $C \cap (C + \alpha)$ is not finite. We may assume $\alpha := 0._n\overline{\alpha_1 \ldots \alpha_p}$ by Lemma 3.1. Fix $i \in \mathbb{N}$ according to the proof of 4.10 such that $1 \le i \le p$ and $\{0\} \subsetneq D_{\alpha_i}$ and let $\delta \in \{\delta \in \Delta^+ \mid D_{\alpha_i} \not\subseteq D_\delta\}$ be arbitrary (this set is nonempty since $D_{d_m} = \{0\}$ for any digits set). Suppose $x \in [0,1]$ has binary representation $x := 0._2 x_1 x_2 \dots$ and define $f : [0,1] \to F$ such that $f(x) = 0._n \gamma_1 \gamma_2 \dots$ consists of digits

$$\gamma_j := \begin{cases} x_{h+1} \cdot \alpha_j + (1 - x_{h+1}) \cdot \delta & \text{if } j = i + hp \text{ for some } h \in \mathbb{N}_0 \\ \alpha_j & \text{otherwise.} \end{cases}$$

Thus, $\{0\} \subsetneq D_{\gamma_{i+jp}} = D_{\alpha_{i+jp}}$ if $x_{j+1} = 1$ and $D_{\gamma_{i+jp}} = D_\delta$ if $x_{j+1} = 0$ so that $f(x)$ is irrational whenever x is irrational. Since $C \cap (C + \alpha)$ is infinite then

$$\left\{ j \mid x_{j+1} = 1 \right\} = \left\{ j \mid D_{\gamma_{i+jp}} = D_{\alpha_{i+jp}} \right\} \tag{5.1}$$

is an infinite subset of $\mathbb{N}$.

Suppose $\tau := 0._n \tau_1 \tau_2 \cdots \tau_h \overline{\tau_{h+1} \cdots \tau_{h+q}}$ is translate equivalent to $f(x)$ for some $h \in \mathbb{N}$ and period q. Then $D_{\gamma_{h+j}} = D_{\gamma_{h+j+q}}$ for all $j > 0$ according to Corollary 4.9. If a and b are positive integers satisfying $h + a = bp$, then for each integer $j > b$,

$$D_{\gamma_{i+jp}} = D_{\gamma_{h+a+i+(j-b)p}} = D_{\gamma_{h+a+i+(j-b)p+pq}} = D_{\gamma_{i+(j+q)p}}.$$

Equivalently, $x_{j+1} = x_{j+q+1}$ for all $j > b$ by equation (5.1) so that x is rational with period q. □

If $K \subseteq \mathbb{R}^n$ is an arbitrary compact set with $\dim_H(K)$-dimensional Hausdorff measure 0 or ∞, then K is not a self-similar set, see e.g., [7] and [12]. In particular, such a set K cannot be expressed as the finite union of self-similar sets. In [24], a method was given for constructing values $y \in F$ which satisfy $0 < s := \dim_H(C(y)) < \log_n(m)$ and $\mathcal{H}^s(C(y)) = 0$ so that such elements y are not translation equivalent to any rational. Example 5.2 constructs $\gamma \in F$ which is not translation equivalent to a rational, yet $0 < \mathcal{H}^s(C(\gamma)) < \infty$.

Example 5.2. Let $D = \{0, 3, 6, 12\}$ and $n = 17$. Choose $\alpha := 0._n\overline{3}$ so that $C \cap (C + \alpha) = C_{n,\{3,6\}}$ is self-similar with Hausdorff dimension $s := \log_n(2)$ and Hausdorff measure

$$\mathcal{H}^s(C \cap (C + \alpha)) = \left(\frac{3}{16} \right)^s.$$

Since $D_3 = \{0, 3\}$ and $D_6 = \{0, 6\}$, define $\gamma := 0._n \gamma_1 \gamma_2 \dots$ such that $\gamma_j = 3 = a_j$ except $\gamma_k = 6$ on a sufficiently sparse set of k's. Thus, γ is irrational and not translate equivalent to any rational by Proposition 5.1 so that $C \cap (C + \gamma) \cap [0, \varepsilon]$ is not self-similar for any $\varepsilon > 0$. Note, however, that $\mu_\alpha(k) = \mu_\gamma(k)$ for all k. According to [24], $L_t = 1$ and $\frac{1}{4} \le \mathcal{H}^s(C \cap (C + \gamma)) \le 1$.

6 Uniform Sets

In this section we consider uniform digits sets and prove Theorem 1.4. This allows us to prove Theorem 4.13 with fewer restrictions on the similitudes and to establish connections to some of the results in the papers mentioned in the introduction.

The next lemma is a step in that direction. The lemma also allows us to consider certain β-expansions with non-uniform digit sets.

Lemma 6.1. *Let D be sparse, $N := d_m + 1$, and $C = C_{N,D}$. Fix $\beta \in \left(0, \frac{1}{N}\right]$ and suppose $\alpha := 0._N\alpha_1\alpha_2\ldots$ is a Δ^+ representation. If there exists $\varepsilon > 0$ such that $(C \cap (C+\alpha)) \cap [0,\varepsilon] = \left\{\sum_{k=1}^{\infty} x_k \left(\frac{1}{N}\right)^k \mid x_k \in D \cap (D+\alpha_k)\right\} \cap [0,\varepsilon]$ is a self-similar set generated by similarities $\left\{f_j\right\}_{j=1}^{\ell}$, then there exists $\delta > 0$ such that*

$$\left\{\sum_{k=1}^{\infty} x_k\beta^k \mid x_k \in D \cap (D+\alpha_k)\right\} \cap [0,\delta]$$

is also a self-similar set.

Proof. Let D be sparse, $N := d_m + 1$, and $\alpha \in F^+$ be fixed. The result is trivial if $\beta = N^{-1}$, so suppose $\beta \in \left(0, \frac{1}{N}\right)$ and $(C \cap (C+\alpha)) \cap [0,\varepsilon] = T$ is self-similar for some $\varepsilon > 0$. Since $C \cap (C+\alpha)$ is compact, we may assume $\varepsilon = \sup(T) \in C$ without loss of generality.

Each $\gamma \in C$ has a unique representation $0._N\gamma_1\gamma_2\cdots$ where each $\gamma_k \in D$ by equation (1.1) and Lemma 4.7. Define $g_\beta : C \to \mathbb{R}$ such that

$$g_\beta\left(\sum_{k=1}^{\infty} \frac{\gamma_k}{N^k}\right) = \sum_{k=1}^{\infty} \gamma_k\beta^k.$$

The function g_β is both continuous and increasing on C, and $g_\beta(X) \cup g_\beta(Y) = g_\beta(X \cup Y)$ for any $X, Y \subseteq C$. By equation (4.1), $g_\beta(C \cap (C+\alpha)) = \left\{\sum_{k=1}^{\infty} x_k \cdot \beta^k \mid x_k \in D \cap (D+\alpha_k)\right\}$. Since any $\varepsilon < \gamma \in C$ implies $g_\beta(\varepsilon) < g_\beta(\gamma)$ then $g_\beta(T) = g_\beta(C \cap (C+\alpha)) \cap [0,\delta]$ where $\delta := g_\beta(\varepsilon)$.

For arbitrary elements $\gamma \neq \xi$ in C, there exists $k \in \mathbb{N}$ such that $\gamma_k \neq \xi_k$ and $g_\beta(\gamma) \neq g_\beta(\xi)$. Hence, g_β has unique inverse for any element of $g_\beta(C) = \left\{\sum_{k=1}^{\infty} x_k \cdot \beta^k \mid x_k \in D\right\}$ and

$$\begin{aligned}\bigcup_{j=1}^{\ell}\left(g_\beta \circ f_j \circ g_\beta^{-1}\right)\left(g_\beta(T)\right) &= \bigcup_{j=1}^{\ell} g_\beta\left(f_j(T)\right) \\ &= g_\beta\left(\bigcup_{j=1}^{\ell} f_j(T)\right) \\ &= g_\beta(T).\end{aligned}$$

Therefore, $g_\beta(T)$ is a self-similar set generated by similarities $\left\{g_\beta \circ f_j \circ g_\beta^{-1}\right\}_{j=1}^{\ell}$. □

Corollary 6.2. *Let D be sparse, $N := d_m + 1$, $\beta \in \left(0, \frac{1}{N}\right]$, and α have Δ representation. The β-expansion Cantor set $g_\beta(C \cap (C+\alpha))$ can be expressed as the disjoint union*

$$\bigcup_{j=1}^{\ell}\left(g_\beta\left(C_{n^{2p},E}\right) + \eta_j\right)$$

for some $\eta_1 < \eta_2 < \cdots < \eta_\ell$ if and only if there exist integers k, q such that $D_{|\alpha_j|} \subseteq D_{|\alpha_{j+q}|}$ for all $j > k$.

The additional structure of uniform sets allows us to prove Theorem 1.1 with fewer restrictions on the similitudes. Theorem 4.13 requires the contraction ratios to be of the form $r_j = n^{-q_j}$ for some integers q_j, however, when D is uniform we require only that the contraction ratios r_j are positive.

Theorem 6.3. *Let D be uniform and α have Δ^+ representation. Suppose there exists $\varepsilon > 0$ such that $C(\alpha) \cap [0, \varepsilon]$ is a self-similar set generated by similarities $f_j(x) = r_j x + b_j$ where $r_j > 0$. Then α is translation equivalent to a rational number.*

Proof. Since D is uniform, then there exists $d \geq 2$ such that $d_j = (j-1)d \in D$ for each $1 \leq j \leq m < n$. Furthermore, $D_{\alpha_i} = \{a - \alpha_i \mid a \in D \text{ and } a \geq \alpha_i\} \subseteq D$ by Remark 6.4 so that $C(\alpha) \subseteq C$ and if $(j-1)d \in D_{\alpha_i}$ for some i and j then $(k-1)d \in D_{\alpha_i}$ for all $1 \leq k \leq j$. We may assume that $b_1 < b_2 < \cdots < b_\ell$ so that $b_1 = 0$ and $f_1(x) = r_1 \cdot x$.

We only need show that there exist integers k, q such that $D_{\alpha_j} \subseteq D_{\alpha_{j+q}}$ for all $j > k$ by Lemma 4.12. According to Lemma 6.1, the result holds if there exists an $n > d_m$ such that $D_{\alpha_j} \subseteq D_{\alpha_{j+q}}$ for all sufficiently large j. Suppose $n > d_m \cdot (m-1)$.

Let $\varepsilon > 0$ be a value such that $C(\alpha) \cap [0, \varepsilon] = T$ is a self-similar set. We may assume by Lemma 4.10 that $C(\alpha)$ is not a finite set and $K := \{k \mid \{0\} \subsetneq D_{\alpha_k}\}$ is an infinite subset of $\mathbb{N}$.

Choose $k \in \mathbb{N}$ such that $\varepsilon \geq n^{-k}$ and let $j > k$ be an arbitrary element of K. Then $d \in D_{\alpha_j}$ so that $d \cdot n^{-j} \in T \subseteq C(\alpha)$. Thus, $d \cdot r_1 \cdot n^{-j} \in T \subset C$ and $d \cdot r_1 \cdot n^{-j} = \sum_{i=1}^{\infty} x_i \cdot n^{-i}$ has a unique expression with each $x_i \in D$ by definition of C and Lemma 4.7. Now, $0 < r_1 < 1$ so that $d \cdot r_1 = \sum_{i=1}^{\infty} x_i \cdot n^{j-i} < d$ and $x_i = 0$ for all $1 \leq i \leq j$. Since each $x_i \in D$ we can write the n-ary representation $r_1 = 0._n r_{1,1} r_{1,2} \ldots$ where $r_{1,i} = \frac{x_{i-j}}{d} \in \{0, 1, \ldots, m-1\}$ for each $i \in \mathbb{N}$.

Suppose there exists $q \in \mathbb{N}$ such that $r_{1,q} \geq 2$. Let $a_1 := \max\left(D_{\alpha_j}\right) > 0$ so that $a_1 \cdot n^{-j} \in T$. We will inductively define a sequence $\{a_h\} \subseteq D$ so suppose $a_h \in D_{\alpha_{j+q(h-1)}}$ for some h. Then $0 < a_h \leq d_m$ and $1 < r_{1,i} \leq m-1$ so that $a_h \cdot r_{1,i} \leq d_m \cdot (m-1)$ for all i. Hence, $\frac{a_h}{n^{j+q(h-1)}} \cdot r_1 = \sum_{i=1}^{\infty} a_h \cdot r_{1,i} \cdot n^{-i-j-q(h-1)}$ and $a_h < a_h \cdot r_{1,q} \in D_{\alpha_{j+qh}}$. Define $a_{h+1} := \max\left(D_{\alpha_{j+q \cdot h}}\right) > a_h$.

Thus, we have defined $a_1 < a_2 < \ldots < a_m$ such that $a_h = \max\left(D_{\alpha_{j+q \cdot h}}\right) \in D$ for all $1 \leq h \leq m$. Since D is a uniform set containing m elements then $D = \{a_1, a_2, \ldots, a_m\}$. This leads to a contradiction since $a_1 > 0$ yet $0 \in D$.

Therefore, $r_{1,i} \in \{0, 1\}$ for all $i \in \mathbb{N}$ and there exists q such that $r_{1,q} = 1$ since $r_1 > 0$. If $d_i \in D_{\alpha_j}$ for some $1 \leq i \leq m$, then $r_1 \cdot d_i \cdot n^{-j} \in C_\alpha$ and $d_i \in D_{\alpha_{j+q}}$. Hence, $D_{\alpha_j} \subseteq D_{\alpha_{j+q}}$ for all $j > k$. $\square$

It is a simple consequence of the proof of Theorem 6.3 that each element $q \in \{i \mid r_{1,i} = 1\}$ is a period of the rational number that is translation equivalent to α. It is also interesting to note that the proof of Theorem 3.8 constructs a collection $\left\{f_j(x) = r_j x + b_j\right\}_{j=1}^{\ell}$ where each $r_1 = r_2 = \cdots = r_\ell = n^{-q}$ for some $q \in \mathbb{N}$, however the set $\{i \mid r_{1,i} = 1\}$ could be countably infinite in the proof of Theorem 6.3.

Remark 6.4. Suppose D is a uniform digit set. Then $D_{\alpha_j} = \left\{a - \alpha_j \mid a \in D \text{ and } a \geq \alpha_j\right\}$ for each $\alpha_j \in \Delta^+$ and $D_{\alpha_j} \subseteq D$ for all j by definition of uniform sets so that $C(\alpha) \subseteq C$. This also implies that $D_{\alpha_j} = \left\{0, d, \ldots, d\left(d_m - \alpha_j\right)\right\} = \max\left\{D_{\alpha_j}\right\} - D_{\alpha_j}$ for all j so that $C(\alpha) = z - C(\alpha)$ is centrally symmetric when $z := \sum_{j=1}^{\infty}\left(\max\left\{D_{\alpha_j}\right\} \cdot n^{-j}\right)$.

When D is regular, but not uniform, then $C(\alpha)$ need not be a subset of C. For example, we can choose $D = \{0,4,6,8\}$, $n \geq 9$, and $\alpha := \frac{2}{n}$. Thus, $D_{\alpha_1} = \{0,2\}$ so that $\alpha \in C(\alpha)$ yet $\alpha \notin C$.

According to Remark 6.4, if D is uniform and $C(\alpha)$ is a self-similar set then we may choose $\varepsilon = 1$ to obtain the following Corollary to Theorem 6.3:

Corollary 6.5. *Let D be uniform and α have Δ^+ representation. If $C(\alpha)$ is a self-similar set generated by similarities $f_j(x) = r_j x + b_j$ then α is translation equivalent to a rational number.*

Proof. It is sufficient to show that $T := C(\alpha)$ can be generated by a collection of similarity mappings with positive contraction ratios. Since T is centrally symmetric, then $T = z - T$ for some $z \in [-1,1]$. For each $1 \leq j \leq \ell$, let $h_j(x) := -r_j x + \left(b_j + z \cdot r_j\right)$ if $r_j < 0$, otherwise let $h_j(x) := f_j(x)$. Then

$$\bigcup_{j=1}^{\ell} h_j(T) = \left(\bigcup_{r_j<0} f_j(z-T)\right) \cup \left(\bigcup_{r_j>0} f_j(T)\right) = \bigcup_{j=1}^{\ell} f_j(T) = T.$$

□

This completes the proof of Theorem 1.4. A special case of Corollary 6.5 is proven in [18] when $d_m = n-1$.

6.1 Uniform sets and strongly periodic rationals

In this section we assume that D is a uniform digits set. Note that if $\alpha_j \in \Delta^+$ then $D_{\alpha_j} = \left\{0, d, \ldots, \left(d_m - \alpha_j\right)\right\}$ so that (4.2) and (4.3) are equivalent when D is assumed to be a uniform set. Thus, when D is uniform, a sequence $\{\alpha_k\} \subseteq \Delta^+$ is strongly periodic if and only if there exists an integer $q > 0$ such that $D_{\alpha_j} \subseteq D_{\alpha_{j+q}}$ for all $j > 0$. We show that this is consistent with the definition given in [4] and [18].

Proposition 6.6. *Let D be uniform, $n = d_m + 1$, and $\alpha = 0._n\alpha_1\alpha_2\ldots$ have Δ representation. Define $\hat{\alpha} := 0._n\hat{\alpha}_1\hat{\alpha}_2\cdots$ such that $\hat{\alpha}_k := d_m - |\alpha_k|$ for each k. There exists an integer $q > 0$ such that $D_{|\alpha_j|} \subseteq D_{|\alpha_{j+q}|}$ for all $j > 0$ if and only if there exist $\boldsymbol{u}, \boldsymbol{v} \in D^p$ for some integer $p > 0$ such that $u_j \leq v_j$ for all $1 \leq j \leq p$ and $\hat{\alpha} = 0._n u_1 \cdots u_p \overline{v_1 \cdots v_p}$.*

Proof. Suppose $\boldsymbol{u}, \boldsymbol{v} \in D^p$ such that $u_j \leq v_j$ for all $1 \leq j \leq p$ and $\hat{\alpha} = 0._n u_1 \cdots u_p \overline{v_1 \cdots v_p}$. Since $u_j, v_j \in D$ then the Δ^+ representation of $\hat{\alpha}$ is unique, so that $u_j \leq v_j$ for all $1 \leq j \leq p$ is equivalent to $d_m - \left|\alpha_j\right| \leq d_m - \left|\alpha_{j+p}\right|$ for all $j > 0$ by definition of $\hat{\alpha}_j$. Furthermore, $d_m - \left|\alpha_j\right| \leq d_m - \left|\alpha_{j+p}\right|$ if and only if $D_{|\alpha_j|} \subseteq D_{|\alpha_{j+p}|}$ since $D_{|\alpha_j|} = \left\{0, d, \ldots, \left(d_m - \left|\alpha_j\right|\right)\right\}$ for all $\alpha_j \in \Delta$. □

Thus, when D is uniform, we extend the definition of strongly periodic to mean there exists $q > 0$ such that $D_{|\alpha_j|} \subseteq D_{|\alpha_{j+q}|}$ for all $j > 0$. We note that α need not be rational to satisfy this equation, but any such α is translation equivalent to a rational.

Remark 6.7. Suppose D is uniform and α has a Δ^+ representation. Since $\Delta^+ = D$, then $\alpha \in C_{n,D} \subset F$ and $D_{\alpha_j} = D_{\alpha_i}$ if and only if $\alpha_j = \alpha_i$. Hence, an irrational value $\alpha \in C_{n,D}$ is not translation equivalent to any rational by Corollary 4.9 and Lemma 4.12.

We point out that if D is uniform, then F contains three disjoint partitions:

1. According to Theorem 1.4, if $\alpha \in F$ is translation equivalent to a strongly periodic rational then $C(\alpha) \cap [0,\varepsilon]$ is a self-similar set for $\varepsilon = 1$.
2. If $\alpha \in F$ is translation equivalent to a rational γ, but not to any strongly periodic rational, then $C(\alpha) \cap [0,\varepsilon]$ is a self-similar set for some $0 < \varepsilon < 1$.
3. Otherwise, if $\alpha \in F$ is not translation equivalent to any rational, then $C(\alpha) \cap [0,\varepsilon]$ is not a self-similar set for any $\varepsilon > 0$

Example 6.8 illustrates a case when α is a strongly periodic rational.

Example 6.8. Let $C = C_{3,\{0,2\}}$ denote the middle thirds Cantor set and $\alpha := 0.{}_3 02\overline{0}$. Then $C \cap (C+\alpha)$ consists of $\mu_\alpha(2) = 2$ disjoint copies of $\frac{1}{9}C$ by Theorem 3.8. Let $q = 2$ so that $D_{\alpha_j} \subseteq D_{\alpha_{j+q}}$ for all $j > 0$ and α is strongly periodic. Thus, $C \cap (C+\alpha)$ is a self-similar set composed of two "smaller" copies of C. Furthermore, the Hausdorff dimension of $C \cap (C+\alpha)$ is $s := \log_3(2)$ and the Hausdorff measure is $\mathcal{H}^s(C \cap (C+\alpha)) = \frac{1}{2}$.

Example 6.9 demonstrates a rational in F that is not strongly periodic.

Example 6.9. Let $C = C_{3,\{0,2\}}$ denote the middle thirds Cantor set and $\alpha := 0.{}_3 02\overline{20}$. Then $C \cap (C+\alpha)$ consists of $\mu_\alpha(2) = 2$ disjoint copies of $\frac{1}{9}C_{9,\{6,8\}}$ by Theorem 3.8. If $s := \log_9(2)$ then $\mathcal{H}^s(C \cap (C+\alpha)) = 4^{-s}$ according to Proposition 3.10. However, if $q = 2k$ is even then $D_{\alpha_1} = \{0,2\}$ and $D_{\alpha_{1+2k}} = \{0\}$. Similarly, if $q = 2k+1$ then $D_{\alpha_4} = \{0,2\}$ and $D_{\alpha_{4+2k+1}} = \{0\}$. Hence, α is not translation equivalent to any strongly periodic rational and $C \cap (C+\alpha)$ is not a self-similar set.

6.2 β-expansion Cantor Sets

Let $N \geq 2$, $\Omega \subseteq \{0,1,\ldots,N-1\}$ be an arbitrary set containing at least two elements, and $\beta \in \left(0, \frac{1}{N}\right)$. If $\phi_d(x) := \beta x + d(1-\beta)/(N-1)$, then the set generated by $\{\phi_d \mid d \in \Omega\}$ is the *β-expansion Cantor set*

$$\Gamma_{\beta,\Omega} := \left\{ \sum_{k=1}^{\infty} \frac{x_k \beta^{k-1}(1-\beta)}{(N-1)} \mid x_k \in \Omega \right\}.$$

According to Lemma 6.1, if there exists an integer $d \geq 1$ such that $D = d \cdot \Omega$ is a sparse digits set and $d_m \leq N-1$, then $\Gamma_{\beta,D}$ can be expressed as

$$\Gamma_{\beta,D} = \frac{(1-\beta)}{\beta(N-1)} \cdot g_\beta(C_{N,D})$$

for some sparse deleted digits Cantor set $C_{N,D}$. Therefore, when β is small it is sufficient to consider the structure of deleted digits Cantor sets. We point out that g_β only preserves the structure of these sets; the Hausdorff dimension and measure are not necessarily preserved since $g_\beta(C_{3,\{0,2\}})$ has dimension $\log_{\frac{1}{\beta}}(2)$ for any $\beta \in \left(0, \frac{1}{3}\right)$. Our results do not necessarily hold for larger values of β since $\Gamma_{\beta,\Omega} - \Gamma_{\beta,\Omega}$ may not satisfy the open set condition. We refer to [29] and [13] for analysis of uniform β-expansion Cantor sets when $\beta > \frac{1}{d_m+1}$.

Many of our results support the idea that self-similarity structure is determined by the sequence $\{\alpha_k\} \subseteq \Omega$, sometimes called the Ω-*code*. If D is sparse and $d_m < n$, then F satisfies the open set condition and any Δ^+ representations are unique by Lemma 4.7. We avoid (direct) discussion of Ω-codes to focus on the geometry of n-ary intervals $J^{(h)} \subset C_k$. Lemma 6.1 directly supports the idea that self-similarity is independent of the chosen base when β is small.

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