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Nonlinear Stability of Periodic Traveling Waves of the BBM System

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Abstract

This paper is concerned with the nonlinear stability of periodic traveling wave solutions for the coupled Benjamin-Bona-Mahony system. We show the existence of a family of dnoidal type traveling waves. We find conditions on parameters of the waves which imply the nonlinear stability of periodic traveling waves.

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1 Introduction

In [10], Bona-Chen-Saut derived the following system

$$\left| \begin{array}{l} \eta_t + u_x + (\eta u)_x + a u_{xxx} - b\eta_{xxt} = 0 \\ u_t + \eta_x + u u_x + c\eta_{xxx} - d u_{xxt} = 0. \end{array} \right. \tag{1.1}$$

which is the motion of small-amplitude long waves on the surface of an ideal fluid under the force of gravity. Here, η represents the vertical deviation of the free surface from its rest position, while u is the horizontal velocity at time t. The existence of traveling wave solutions for equation (1.1) in various cases is considered in [12, 13].

In [15], Chen-Chen-Nguyen consider another relevant case, namely the BBM system, in which ($a = c = 0, b = d = \frac{1}{6}$). They construct periodic traveling wave solutions for the BBM case, as well as in more general situations. The stability of solitary wave solutions for the equation (1.1) was considered in [16, 20].

The orbital stability for periodic traveling waves goes back to the work of Benjamin [8]. Recently, the orbital stability of periodic traveling waves for many dispersive equations has

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been obtained in [4, 5, 6, 18, 19, 21, 24]. The nonlinear stability of periodic traveling waves of cnoidal type for the Korteweg-de Vries equation was considered in [5]. The stability of dnoidal type periodic traveling waves for the modified Korteweg-de Vries and nonlinear Schrodinger equation is obtained in [4]. The approach is based on the theory developed in [1, 2, 3, 7, 9, 17, 26, 27] for the stability of solitary waves. The stability of periodic traveling waves of cnoidal and dnoidal type for the generalized BBM equation is obtained in [18].

In this paper, we consider the following system

$$\begin{cases} u_t - u_{txx} + u_x + (uv^2)_x = 0 \\ \\ v_t - v_{txx} + v_x + (vu^2)_x = 0 \end{cases} \tag{1.2}$$

which is the generalized BBM system. We investigate the orbital stability of dnoidal waves for the system (1.2) with respect to the perturbation of the same period. We extend the theory developed in [22, 23] for the stability of solitary waves, to the periodic case. Our approach is to verify that a periodic traveling wave of dnoidal type is a minimized of a properly chosen functional $M(\vec{u})$, which is conservative with respect to time over the solutions of (1.2). In order to get the required spectral conditions for the operator of linearization we use the well-known theory of Hill operators [25].

The paper is organized as follows. In Section 2, we consider the existence of periodic traveling wave solutions for the equation (1.2). In Section 3, we show that under some conditions on the parameters, the periodic wave solutions of dnoidal type are stable.

2 Periodic traveling waves

We are looking for a solution of equation (1.2) in the form $u(t,x) = \varphi(x-ct)$, $v(t,x) = \psi(x-ct)$, where φ and ψ are smooth periodic functions with the fundamental period T and $c \in \mathbb{R}$. Substituting, in (1.2), for φ and ψ one obtains the system

$$\begin{cases} -c\varphi' + \varphi' + (\varphi\psi^2)' + c\varphi''' = 0 \\ \\ -c\psi' + \psi' + (\psi\varphi^2)' + \chi''' = 0 \end{cases} \tag{2.1}$$

To solve the system, we consider $\varphi = \psi$. Thus

$$-c\varphi' + \varphi' + (\varphi^3)' + c\varphi''' = 0. \tag{2.2}$$

Integrating (2.2) twice, one obtain

$$c\varphi'' - (c-1)\varphi + \varphi^3 = b \tag{2.3}$$

$$c\varphi'^2 - (c-1)\varphi^2 + \frac{1}{2}\varphi^4 = b\varphi + a. \tag{2.4}$$

with some constants a, b. Below we will consider the symmetric case $b = 0$. Thus, for φ we get the equation

$$\varphi'^2 = \frac{1}{2c}[4a + 2(c-1) - \varphi^4]. \tag{2.5}$$

Consider in the plane $(X,Y) = (\varphi,\varphi')$ the Hamiltonian system

$$\begin{aligned}\dot{Y} &= cY = H_Y\\ \dot{X} &= (c-1)X - X^3 = -H_X.\end{aligned}$$

with the Hamiltonian function

$$H(X,Y) = \frac{c}{2}Y^2 + \frac{1}{4}X^4 - \frac{c-1}{2}X^2$$

Then (2.5) becomes $H(\varphi,\varphi') = a$ and the curve $s \to (\varphi(s),\varphi'(s)$ determined (2.5) lines on the energy level $H = a$. It is well-known that there are two cases related to the above Hamiltonian

(i) global center $c < 1$

(ii) Duffing oscillator $c > 1$

There is one continuous family of periodic orbits in case (i) and three families (left, right, outer) in case (ii). We will consider here only the case (ii) right family, ($c > 1$) of periodic traveling waves.

Let us denote

$$U(s) = \frac{4a + 2(c-1)s^2 - s^4}{2c}$$

and $\varphi_1 > \varphi_0 > 0$ are the positive roots of the polynomial $4a + 2(c-1)s^2 - s^4$. Then for $\varphi_0 < \varphi < \varphi_1$ we can write (2.5) as

$$\varphi'(\sigma) = \sqrt{U(\sigma)} = \sqrt{\frac{(\sigma^2 - \varphi_0^2)(\varphi_1^2 - \sigma^2)}{2c}}. \tag{2.6}$$

Then, up to translation, we obtain the explicit formula [see [18]]

$$\varphi_1^2 + \varphi_0^2 = 2(c-1), \quad \varphi_1^2\varphi_0^2 = -4a \tag{2.7}$$

$$\varphi_c(x) = \varphi_1 dn(\alpha x;\kappa) \quad \kappa^2 = \frac{\varphi_1^2 - \varphi_0^2}{\varphi_1^2} = \frac{2\varphi_1^2 - 2(c-1)}{\varphi_1^2}, \quad \alpha = \frac{\varphi_1}{\sqrt{2c}}. \tag{2.8}$$

Since $dn(x)$ has fundamental period $2K(\kappa)$, we get the following relations

$$T = \frac{2K(\kappa)}{\alpha}, \quad T \in I = \left(2\pi\sqrt{\frac{c}{2(c-1)}}, \infty\right). \tag{2.9}$$

The above formula for the period $T = T(\kappa)$ implies that the period is increasing function. Indeed, result follows from

$$\frac{d}{d\kappa}[\sqrt{(2-\kappa^2)}K(\kappa)] = \frac{(2-\kappa^2)K'(\kappa) - \kappa K(\kappa)}{\sqrt{2-\kappa^2}} = \frac{E'(\kappa) + K'(\kappa)}{\sqrt{2-\kappa^2}}.$$

On the other hand, from (2.7), we have

$$\frac{d\kappa}{d\varphi_0} = \frac{1}{2\kappa}\frac{d\kappa^2}{d\varphi_0} = -2\frac{2(c-1)+\varphi_0^2}{[2(c-1)-\varphi_0^2]^2}.$$

Thus $\frac{d}{d\varphi_0}[\sqrt{(2-\kappa^2)}K(\kappa)] \neq 0$. Therefore, for the given $T \in I$, the condition holds in order to determine φ_0 by the implicit function theorem so that φ has the fundamental period T.

We end this section with comments on the initial value problem

$$\begin{cases} u_t - u_{txx} + u_x + (uv^2)_x = 0 \\ v_t - v_{txx} + v_x + (vu^2)_x = 0 \\ \vec{u}(0,x) = \vec{u_0}. \end{cases} \tag{2.10}$$

First, rewrite (1.2) in the form

$$\begin{cases} u_t = G(u + uv^2) \\ v_t = G(v + u^2 v), \end{cases} \tag{2.11}$$

where G is given by the Fourier transform $\widehat{Gu}(n) = \frac{-in}{1+n^2}\widehat{u}(n)$. Integrating (2.10), we get

$$\vec{u}(t,x) = \vec{u}_0 + \int_0^t \vec{G}\vec{f}d\tau, \tag{2.12}$$

where $\vec{G} = \begin{pmatrix} G & 0 \\ 0 & G \end{pmatrix}$ and $\vec{f} = \begin{pmatrix} u + uv^2 \\ v + u^2 v \end{pmatrix}$.

Now, by standard contraction-mapping argument we have the following well-posedness result for the equation (1.2)(for more details see [10]).

Theorem 2.1. *If $\vec{u}_0 \in X = H^1 \times H^1$, then there is unique global solution of (2.10) in $C([0,\infty), X)$.*

3 Stability

In this section we prove our main stability result which concerns the right Duffing oscillator cases. Take $a < 0$, $T > 2\pi/\sqrt{2(c-1)/c}$ and determine $a = a(c)$ so that the orbit given by $H(\varphi, \varphi') = a$ has period T. Then (φ, φ) is a solution of (2.1) having a period T with respect to x.

Take a solution (u, v) of (1.2) of period T in x and introduce the pseudometric

$$d(\overrightarrow{u}(t), \overrightarrow{\varphi}) = \inf_{\zeta\in\mathbb{R}} ||\overrightarrow{u}(t,x) - \overrightarrow{\varphi}(x - \zeta - ct)||_X. \tag{3.1}$$

Clearly, the infimum in (3.1) is attained at some point ζ in the interval $[0, T]$.

The equation (1.2) possesses the following conservation laws

$$E(u,v) = -\frac{1}{2}\int_0^T [u^2 + v^2 + u^2v^2]dx$$

$$Q(u,v) = \frac{1}{2}\int_0^T [u^2 + v^2 + u_x^2 + v_x^2]dx.$$

Let

$$M(u) = E(u,v) + cQ(u,v).$$

Now, we can formulate our main result in the paper.

Theorem 3.1. *Let φ be given by* (2.8). *For each $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u(t,x), v(t,x))$ is a solution of* (1.2) *and $d((u,v),(\varphi,\varphi))_{|t=0} < \delta$, then $d((u,v),(\varphi,\varphi)) < \varepsilon$ $\forall t \in [0,\infty)$ in any of the cases:*

(i) $3c^2 - 8 \geq 0$

(ii) $3c^2 - 8 < 0$, $2c^2 - 2c - 1 > 0$ *and the period T of φ is sufficiently large.*

The crucial step in the proof will be to verify the following statement.

Proposition 3.2. *Let satisfies the conditions (i) and (ii) of Theorem 3.1. There exist positive constants m, δ_0 such that if (u,v) is a solution of* (1.2) *such that $Q(u,v)) = Q(\varphi,\varphi)$ and $d((u,v),(\varphi,\varphi)) < \delta_0$, then*

$$M(u,v) - M(\varphi,\varphi) \geq m d^2((u,v),(\varphi,\varphi)). \tag{3.2}$$

The proof consists of several steps. The first one concerns the metric d introduced above.

Lemma 3.3. *The metric $d(u,\varphi)$ is a continuous function of $t \in [0,\infty)$.*

Proof. The proof of the lemma is similar to the proof of Lemmas 1, 2 in [9] □.

We fix $t \in [0,\infty)$ and assume that the minimum in (3.1) is attained at the point $\zeta = \zeta(t)$.

Consider the perturbation of periodic traveling wave (φ,φ),

$$\begin{cases} u(t, x + \zeta(t)) = \varphi(x) + p(t,x) \\ \\ v(t, x + \zeta(t)) = \varphi(x) + q(t,x). \end{cases}$$

Let $M = cQ + E$, then

$$\begin{aligned} \Delta M &= M(\varphi + p, \varphi + q) - M(\varphi,\varphi) \\ &= \int_0^T (-c\varphi'' + (c-1)\varphi - \varphi^3)(p+q)dx \\ &+ \tfrac{1}{2}\int_0^T [cp_x^2 + cq_x^2 + (c-1)p^2 + (c-1)q^2 - \varphi^2(p^2+q^2) - 4\varphi^2 pq]dx \\ &- \tfrac{1}{2}\int_0^T [2\varphi(pq^2 + p^2 q) + p^2 q^2]dx. \end{aligned}$$

Using that $\varphi(x)$ satisfies the equation (2.3), we get that the first term in the above equality is zero. Thus,

$$\Delta M = \left\langle \mathcal{L}\begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} p \\ q \end{pmatrix} \right\rangle - \frac{1}{2}\int_0^T [2\varphi(pq^2 + p^2 q) + p^2 q^2]dx,$$

where

$$\mathcal{L} = \begin{pmatrix} -c\partial_x^2 + (c-1) - \varphi^2 & -2\varphi^2 \\ \\ -2\varphi^2 & -c\partial_x^2 + (c-1) - \varphi^2 \end{pmatrix}. \tag{3.3}$$

Let consider in $L^2[0,T]$ the self-adjoint operators L_1 and L_2 defined by

$$L_1 = -c\partial_x^2 + (c-1) - 3\varphi^2$$

$$L_2 = -c\partial_x^2 + (c-1) + \varphi^2.$$

Using (2.8) and that $dn^2 = 1 - \kappa^2 sn^2$, we get

$$\begin{aligned} L_1 &= -c\partial_x^2 + (c-1) - 3\varphi^2 = c\alpha^2[-\partial_y^2 + \tfrac{c-1}{c\alpha^2} - 3\tfrac{\varphi_1^2}{c\alpha^2} dn^2(y;\kappa^2)] \\ &= c\alpha^2[-\partial_y^2 + 6\kappa^2 sn^2(y;\kappa) + r], \end{aligned}$$

where $r = -4 - \kappa^2$.

The first five eigenvalues of the operator Λ defined by the differential expression $\Lambda = -\partial_y^2 + 6k^2 sn^2(y;k)$, with periodic boundary conditions on $[0, 4K(k)]$, are simple. These eigenvalues and their respective eigenfunctions are:

$$\begin{array}{ll} \mu_0 = 2 + 2\kappa^2 - 2\sqrt{1-\kappa^2+\kappa^4}, & \psi_0(y) = 1 - (1 + \kappa^2 - \sqrt{1-\kappa^2+\kappa^4}) sn^2(y;\kappa), \\ \mu_1 = 1 + \kappa^2, & \psi_1(y) = cn(y;\kappa) dn(y;\kappa) = sn'(y;\kappa), \\ \mu_2 = 1 + 4\kappa^2, & \psi_2(y) = sn(y;\kappa) dn(y;\kappa) = -cn'(y;\kappa), \\ \mu_3 = 4 + \kappa^2, & \psi_3(y) = sn(y;\kappa) cn(y;\kappa) = -k^{-2} dn'(y;\kappa), \\ \mu_4 = 2 + 2\kappa^2 + 2\sqrt{1-\kappa^2+\kappa^4}, & \psi_4(y) = 1 - (1 + \kappa^2 + \sqrt{1-\kappa^2+\kappa^4}) sn^2(y;\kappa). \end{array}$$

From the relation between operator L_1 and Λ, we get: The first three eigenvalues of the operator L_1, equipped with periodic boundary conditions on $[0, 2K(k)]$, are simple and equal to $c\alpha^2(\mu_0 - 4 - \kappa^2) < 0$, $c\alpha^2(\mu_3 - 4 - \kappa^2) = 0$, $c\alpha^2(\mu_4 - 4 - \kappa^2) > 0$.

Since $1 - \kappa^2 < dn^2(y;\kappa) < 1$, we get $\langle L_2\varrho, \varrho\rangle = c\|\varrho\|_1^2 + (c-1)\|\varrho\|^2 + \langle \varphi^2\varrho, \varrho\rangle > \gamma\|\varrho\|^2$, where $\gamma = \gamma(c,\kappa) > 0$. Therefore, $KerL_2 = \{0\}$.

Now, with obtained spectral information for operators L_1 and L_2, we are ready to get spectral properties for the operator of linearization $\mathcal{L}$ with periodic boundary conditions.

Lemma 3.4. *The operator $\mathcal{L}$, has the following spectral properties:*
(i) has a negative simple eigenvalue
(ii) zero is the simple eigenvalue
(iii) the rest of the spectrum is positive and bounded away from zero.

Proof. From (2.2) and (3.3), we have that $\mathcal{L}\begin{pmatrix} \varphi' \\ \varphi' \end{pmatrix} = 0$. Suppose that $\mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix} = 0$. From (3.3), we obtain

$$\begin{cases} -cf'' + (c-1)f - \varphi^2 f - 2\varphi^2 g = 0 \\ -cg'' + (c-1)g - \varphi^2 g - 2\varphi^2 f = 0. \end{cases}$$

Then

$$\begin{cases} -c(f-g)'' + (c-1)(f-g) + \varphi^2(f-g) = 0 \\ -c(f+g)'' + (c-1)(f+g) - 3\varphi^2(f+g) = 0. \end{cases}$$

or $L_1(f+g) = 0$, $L_2(f-g) = 0$. Since $KerL_2 = \{0\}$, then from the first relation in the above system we get $f = g$. Moreover $dimKerL_1 = 1$ and $L_1\varphi' = 0$. This yields that $Ker\mathcal{L} = 1$.

If $\mathcal{L}\begin{pmatrix} f \\ g \end{pmatrix} = -\lambda^2 \begin{pmatrix} f \\ g \end{pmatrix}$, similarly as in the above, we get $L_1(f+g) = -\lambda^2(f+g)$ and $L_2(f-g) = -\lambda^2(f-g)$. Since the negative eigenvalue of L_1 is simple and L_2 is positive operator, then the negative eigenvalue of $\mathcal{L}$ is simple. □

The following lemma is proved in [18] and we sketch the proof for reader's convenience.

Lemma 3.5. *The function $d(c) = \frac{d}{dc}\int_0^T [\varphi^2 + \varphi'^2]$ is positive provided conditions (i) or (ii) of Theorem 3.1.*

Proof. One has

$$y^2 = U(x,h) = \frac{2h}{c} - \frac{1-c}{c}x^2 - \frac{1}{2c}x^4, \quad h = \frac{1-c}{2}\varphi_0^2 + \frac{1}{4}\varphi_0^4.$$

Given a nonnegative even integer n, denote $I_n(h) = \oint_{H=h} x^n y dx$.

The function $d(c)$ can be expressed by the integrals I_n. Below, we will denote for short the derivatives with respect to h by I_n', I_n'' etc. We have

$$I_n'(h) = \oint_{H=h} \frac{x^n dx}{vy}.$$

On the other hand,

$$\oint_{H=h} y^3 dx = \oint_{H=h} U(x,h) y dx = \frac{2h}{c} I_0 - \frac{1-c}{c} I_2 - \frac{1}{2c} I_4.$$

By using these expressions and performing some direct calculations, the following formula is derived [18]

$$d(c) = \frac{(1-c)^2 I_0'^2(h)}{72c\beta^2 I_0(h)} w(h,R)$$

where

$$w(h,R) = (4c^2 - 2c - \frac{1}{2})R^2 + (1+4c-5c^2)hR + (1-c)^2h^2 + 3c^2(R-h) - \frac{6}{4}.$$

Since $I_0(h) > 0$, then the sign of $d(c)$ determine by $w(h,R)$. The curve $\gamma : w(h,R) = 0$ divide the (,) plane in two parts according to the sign of $w(h,R)$. It is proved in [18] that the trajectory corresponding to Duffing oscillator Γ lies on positive part, provided (i) or (ii) of Theorem 3.1.□

3.1 Proof of Proposition 3.2.

Differentiating (2.3) with respect to c, we get $L_1 \frac{d\varphi}{dc} = \varphi'' - \varphi$. Since $\mathcal{L}\begin{pmatrix} f \\ f \end{pmatrix} = \begin{pmatrix} L_1 f \\ L_1 f \end{pmatrix}$, then $\mathcal{L}\begin{pmatrix} \frac{d\varphi}{dc} \\ \frac{d\varphi}{dc} \end{pmatrix} = \begin{pmatrix} \varphi'' - \varphi \\ \varphi'' - \varphi \end{pmatrix}$, and by Lemma 3.5

$$\langle \mathcal{L}\frac{d\vec{\varphi}}{dc}, \frac{d\vec{\varphi}}{dc} \rangle = \left\langle \begin{pmatrix} \varphi'' - \varphi \\ \varphi'' - \varphi \end{pmatrix}, \frac{d\vec{\varphi}}{dc} \right\rangle = -\frac{d}{dc}\int_0^T [\varphi^2 + \varphi'^2] dx < 0. \tag{3.4}$$

From the condition $Q(u,v) = Q(\varphi,\varphi)$, we get

$$2\left\langle \begin{pmatrix} p \\ q \end{pmatrix}, \begin{pmatrix} \varphi-\varphi'' \\ \varphi-\varphi'' \end{pmatrix} \right\rangle = -(\|p\|^2_{H^1} + \|q\|^2_{H^1}). \tag{3.5}$$

Let denote $\overrightarrow{p} = \begin{pmatrix} p \\ q \end{pmatrix}$, $\frac{d\overrightarrow{\varphi}}{dc} = \begin{pmatrix} \frac{d\varphi}{dc} \\ \frac{d\varphi}{dc} \end{pmatrix}$, and $\overrightarrow{\varphi} = \begin{pmatrix} \varphi \\ \varphi \end{pmatrix}$. From the spectral properties of the operator $\mathcal{L}$, we write

$$\overrightarrow{p} = \overrightarrow{p}_- + \overrightarrow{p}_0 + \overrightarrow{p}_+, \quad \overrightarrow{p}_- = \langle \overrightarrow{p}, \overrightarrow{\varphi}_-\rangle \overrightarrow{\varphi}_- \quad \overrightarrow{p}_0 = \frac{\langle \overrightarrow{p}, \overrightarrow{\varphi'}\rangle}{\|\overrightarrow{\varphi'}\|^2}\overrightarrow{\varphi'} \tag{3.6}$$

and

$$\frac{d\overrightarrow{\varphi}}{dc} = \overrightarrow{f}_- + \overrightarrow{f}_0 + \overrightarrow{f}_+, \quad \overrightarrow{f}_- = \langle \frac{d\overrightarrow{\varphi}}{dc}, \overrightarrow{\varphi}_-\rangle \overrightarrow{\varphi}_- \quad \overrightarrow{f}_0 = \frac{\langle \frac{d\overrightarrow{\varphi}}{dc}, \overrightarrow{\varphi'}\rangle}{\|\overrightarrow{\varphi'}\|^2}\overrightarrow{\varphi'} \tag{3.7}$$

We have

$$\begin{cases} \langle \mathcal{L}\frac{d\overrightarrow{\varphi}}{dc}, \frac{d\overrightarrow{\varphi}}{dc}\rangle = -\lambda_0^2\|\overrightarrow{f}_-\|^2 + \langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{f}_+\rangle \\ \langle \mathcal{L}\frac{d\overrightarrow{\varphi}}{dc}, \overrightarrow{p}\rangle = -\lambda_0^2\langle \overrightarrow{f}_-, \overrightarrow{p}_-\rangle + \langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{p}_+\rangle \\ \langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle = -\lambda_0^2\|\overrightarrow{p}_-\|^2 + \langle \mathcal{L}\overrightarrow{p}_+, \overrightarrow{p}_+\rangle \end{cases} \tag{3.8}$$

From (3.4) and the first relation in (3.8), we have

$$\langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{f}_+\rangle < \lambda_0^2\|\overrightarrow{f}_-\|^2 \tag{3.9}$$

and since the both sides in the above inequality independent of t, then there exists m_1, $0 < m_1 < 1$ such that

$$\langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{f}_+\rangle \le m_1\lambda_0^2\|\overrightarrow{f}_-\|^2. \tag{3.10}$$

From (3.5) and (3.8), we have

$$\langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{p}_+\rangle = \lambda_0^2\langle \overrightarrow{f}_-, \overrightarrow{p}_-\rangle + \frac{1}{2}\|\overrightarrow{p}\|_1^2. \tag{3.11}$$

From the spectral properties of the operator $\mathcal{L}$, we have

$$\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle \ge \lambda_2\|\overrightarrow{p}_+\|^2, \tag{3.12}$$

and from the Cauchy inequality

$$\langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{p}_+\rangle^2 \le \langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{f}_+\rangle\langle \mathcal{L}\overrightarrow{p}_+, \overrightarrow{p}_+\rangle. \tag{3.13}$$

On the other hand, the inequality (3.11) leads to

$$\left|\langle \mathcal{L}\overrightarrow{f}_+, \overrightarrow{p}_+\rangle\right| = \left|\frac{1}{2}\|\overrightarrow{p}\|_1^2 - \lambda_0^2\|\overrightarrow{f}_-\|\|\overrightarrow{p}_-\|\right|. \tag{3.14}$$

Combining inequalities (3.10), (3.13) and (3.14), leads to the inequality

$$\langle \mathcal{L}\overrightarrow{p}_+, \overrightarrow{p}_+\rangle \geq \frac{\lambda_0^2}{m_1}\|\overrightarrow{p}_-\|^2 - \frac{\|\overrightarrow{p}_-\|\cdot\|\overrightarrow{p}\|_1^2}{m_1\|\overrightarrow{f}_-\|} + \frac{\|\overrightarrow{p}\|_1^4}{4m_1\lambda_0^2\|\overrightarrow{f}_-\|^2}. \tag{3.15}$$

Finally, from the last inequality, (3.12) and (3.13) we get

$$\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle \geq m_2\|\overrightarrow{p}_- + \overrightarrow{p}_+\|^2 - m_3\|\overrightarrow{p}\|_1^3 + m_4\|\overrightarrow{p}\|_1^4, \tag{3.16}$$

where m_i, $i = 2,3,4$ are positive constants may depends on c, but not on t.

Now, we will estimate the norm of $\overrightarrow{p}_0$ by the norm of $\overrightarrow{p}_- + \overrightarrow{p}_+$. From Lemma 3.3, let the infimum in (3.1) attained in $\zeta = \zeta(t)$. Thus $\frac{d}{d\zeta}d^2(\overrightarrow{u}, \overrightarrow{\varphi}) = 0$, which is equivalent to the equalities $\left\langle \overrightarrow{p}, \begin{pmatrix} \varphi''' - \varphi' \\ \varphi''' - \varphi' \end{pmatrix} \right\rangle = 0$,

$$\left\langle \overrightarrow{p}_0, \begin{pmatrix} \varphi' - \varphi''' \\ \varphi' - \varphi''' \end{pmatrix} \right\rangle = \left\langle \overrightarrow{p}_- + \overrightarrow{p}_+, \begin{pmatrix} \varphi''' - \varphi' \\ \varphi''' - \varphi' \end{pmatrix} \right\rangle.$$

Since $\overrightarrow{p}_0 = \beta\overrightarrow{\varphi'}$, then $|\beta| = \|\overrightarrow{p}_0\|\cdot\|\overrightarrow{\varphi'}\|^{-1}$ and

$$|\beta|(\|\overrightarrow{\varphi'}\|^2 + \|\overrightarrow{\varphi''}\|^2) \leq \|\overrightarrow{\varphi'''} - \overrightarrow{\varphi'}\|\cdot\|\overrightarrow{p}_- + \overrightarrow{p}_+\|.$$

Hence

$$\|\overrightarrow{p}_0\| \leq m_0\|\overrightarrow{p}_- + \overrightarrow{p}_+\|,$$

where $m_0 = \|\overrightarrow{\varphi'''} - \overrightarrow{\varphi'}\|\cdot\|\overrightarrow{\varphi'}\|\cdot(\|\overrightarrow{\varphi'}\|^2 + \|\overrightarrow{\varphi''}\|^2)^{-1}$, depends only on c. From the above inequality and (3.16), we get

$$\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle \geq m_5\|\overrightarrow{p}\|^2 - m_3\|\overrightarrow{p}\|_1^3 + m_4\|\overrightarrow{p}\|_1^4, \tag{3.17}$$

On the other hand, estimating directly $\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle$ from below (for this purpose we use the special form of the operator $\mathcal{L}$), for sufficiently small $\|\overrightarrow{p}\|$ ($\|\overrightarrow{p}\|_1 < \delta_0$) we have

$$\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle = c\|\overrightarrow{p}_x\|^2 + (c-1)\|\overrightarrow{p}\|^2 - \langle\varphi^2, p^2 + q^2\rangle - 4\langle\varphi^2, pq\rangle \geq m_6\|\overrightarrow{p}_x\|_1^2 - m_7\|\overrightarrow{p}\|^3. \tag{3.18}$$

Similarly, $\left|\int_0^T [2\varphi(pq^2 + p^2q) + p^2q^2]dx\right| \leq m_8\|\overrightarrow{p}\|^3$. Now

$$\langle \mathcal{L}\overrightarrow{p}, \overrightarrow{p}\rangle \geq D_1\|\overrightarrow{p}\|_1^2 - D_2\|\overrightarrow{p}\|^3 - D_4\|\overrightarrow{p}\|^4,$$

where the constants D_i, $i = 1,2,3$ are positives and independent of t.

Finally, we obtain that if $d(\overrightarrow{u}, \overrightarrow{\varphi}) = \|\overrightarrow{p}\|_1 < \delta_0$, then $\Delta M \geq md^2(\overrightarrow{u}, \overrightarrow{\varphi})$. Proposition 3.2 is completely proved. □

3.2 Proof of Theorem 3.1.

We split the proof of the theorem into two steps. First we consider the special case $Q(u,v) = Q(\varphi,\varphi)$. Assume that m, δ_0 have been selected according to Proposition 3.2. Since ΔM does not depend on $t, t \in [0,\infty)$, there exists a constant l such that $\Delta M \leq ld^2(u,\varphi)|_{t=0}$. Let

$$\varepsilon > 0, \ \ \delta = \frac{m}{l}\min\left(\frac{\delta_0}{2}, \varepsilon\right)$$

and $d(\overrightarrow{u},\overrightarrow{\varphi})|_{t=0} < \delta$. Then

$$d(\overrightarrow{u},\overrightarrow{\varphi}) \le 1/2d(u,\varphi)|_{t=0} < \frac{\delta_0}{2}$$

and Lemma 3.3 yields that there exists a $t_0 > 0$ such that $d(u,\varphi) < \delta_0$ if $t \in [0,t_0)$. Then, by virtue of Proposition 3.2 we have

$$\Delta M \ge md^2(\overrightarrow{u},\overrightarrow{\varphi}),\ \ t \in [0,t_0).$$

Let t_{max} be the largest value such that

$$\Delta M \ge md^2(\overrightarrow{u},\overrightarrow{\varphi}),\ \ t \in [0,t_{max}).$$

We assume that $t_{max} < \infty$. Then, for $t \in [0,t_{max}]$ we have

$$d^2(\overrightarrow{u},\overrightarrow{\varphi}) \le \frac{\Delta M}{m} \le \frac{l}{m}d^2(\overrightarrow{u},\overrightarrow{\varphi})|_{t=0} < \frac{l}{m}\delta^2 \le \frac{\delta_0^2}{4}.$$

Applying once again Lemma 3.3, we obtain that there exists $t_1 > t_{max}$ such that

$$d(\overrightarrow{u},\overrightarrow{\varphi}) < \delta_0,\ \ t \in [0,t_1).$$

By virtue of the proposition, this contradicts the assumption $t_{max} < \infty$. Consequently, $t_{max} = \infty$,

$$\Delta M \ge md_q^2(\overrightarrow{u},\overrightarrow{\varphi}) \ge md^2(\overrightarrow{u},\overrightarrow{\varphi}),\ \ t \in [0,\infty).$$

Therefore,

$$d^2(\overrightarrow{u},\overrightarrow{\varphi}) \le \frac{\Delta M}{m} \le \frac{l}{m}\delta^2 < \varepsilon^2,\ \ t \in [0,\infty),$$

which proves the theorem in the special case.

Now we proceed to get rid the restriction. $Q(\vec{u}) = ||\vec{u}||^2 = ||\vec{\varphi}||^2 = Q(\vec{\varphi})$. We have $||\vec{\varphi}|| = (8c\alpha E(\kappa))^{1/2}$.

We claim there are respective parameter values a^*, c^*, and corresponding φ^*, α^*, κ^*, see (2.7), (2.8) and (2.9), such that φ^* has a period T in x and moreover, $8c^*\alpha^*E(k^*) = ||\vec{u}||^2$. By (2.9), we obtain the equations

$$\begin{aligned} &\frac{2K(k^*)}{\alpha^*} - T = 0,\\ &8c^*\alpha^*E(\kappa^*) - ||\vec{u}||^2 = 0. \end{aligned} \tag{3.19}$$

If (3.19) has a solution $\kappa^* = \kappa^*(T, ||\vec{u}||)$, $\alpha^* = \alpha^*(T, ||\vec{u}||)$, then the parameter values we need are given by

$$a^* = \frac{4\alpha^{*2}(\kappa^{*2}-1)}{[1-\alpha^{*2}(2-\kappa^{*2})]^2}, \quad c^* = \frac{1}{1-\alpha^{*2}(2-\kappa^{*2})}.$$

Moreover, one has $||\vec{\varphi^*}|| = ||\vec{u}||$ and we could use the restricted result we established above. As $\kappa^* = \kappa^*(T, ||\varphi||)$, $\alpha^* = \alpha^*(T, ||\vec{\varphi}||)$, it remains to apply the implicit function theorem to (3.19). Since the corresponding functional determinant reads

$$\begin{vmatrix} \frac{2K'(k^*)}{\alpha^*} & -\frac{2K(k^*)}{\alpha^{*2}} \\ 8c^*\alpha^*E'(k^*) & 8c^*E(k^*) \end{vmatrix} = \frac{16c^*}{\alpha^*}(KE)' > 0$$

(by Legendre's identity), the existence of a^* and c^* with the needed properties is established. By (3.19) and our assumption, we have

$$\frac{K(\kappa^*)}{\alpha^*} = \frac{K(\kappa)}{\alpha} = \frac{T}{2}. \tag{3.20}$$

We have

$$d^2(\vec{\varphi^*},\vec{\varphi}) \le 2(||\varphi^* - \varphi||^2 + ||\varphi^{*'} - \varphi^{'}||^2).$$

Denote $\Phi(\rho) = \sqrt{\frac{2}{1-\rho^2(2-\kappa(\rho)^2)}}\rho dn(\rho z,\kappa(\rho))$, where $K = K(\rho)$ is determined from $K(\kappa) = \frac{1}{2}\rho T$. Since

$$\begin{aligned}\Phi'(\rho) \quad &= \tfrac{\rho(2-\kappa^2)+\rho\kappa\frac{T}{2K'}}{[1-\rho^2(2-\kappa(\rho)^2)]^{3/2}}\rho dn(\rho z,\kappa(\rho)) \\ &+ \sqrt{\tfrac{2}{1-\rho^2(2-\kappa(\rho)^2)}}\left[dn(y,\kappa)+\rho\left(z\tfrac{\partial dn}{\partial y}+\tfrac{T}{2K'(\kappa)}\tfrac{\partial dn}{\partial \kappa}\right)\right],\end{aligned}$$

satisfies the inequality $|\Phi'(\rho)| \le C_0$ with a constant C_0 independent of the values with $*$ accent, then $|\varphi^* - \varphi| \le C_0|\alpha^* - \alpha|$. Similarly $|\varphi'^* - \varphi'| \le C_1|\alpha^* - \alpha|$.

All this, together with (3.20) yields

$$d(\vec{\varphi^*},\vec{\varphi}) \le C|\alpha^* - \alpha| = \frac{2C}{T}|K(\kappa^*) - K(\kappa)| = \frac{2C}{T}|K'(\kappa)||\kappa^* - \kappa|. \tag{3.21}$$

Let $\varepsilon > 0$. From the inequalities

$$\left|||\vec{\varphi^*}|| - ||\vec{\varphi}||\right| = \left|||\vec{u}|| - ||\vec{\varphi}||\right| \le d(\vec{u},\vec{\varphi})|_{t=0} < \delta$$

follows that

$$-(8c\alpha E)^{-1/2}\delta < (||\vec{\varphi}||)^{-1}||\vec{\varphi^*}|| - 1 < (8c\alpha E)^{-1/2}\delta$$

and, consequently, $1 - \delta_1 < \frac{c^*\alpha^* E(\kappa^*)|}{c\alpha E(\kappa)} < 1 + \delta_1$, i.e. $|c^*\alpha^* E(\kappa^*) - c\alpha E(\kappa)| < c\alpha E(\kappa)\delta_1$, where $\delta_1 = (1 + (2c\alpha E(\kappa))^{-1/2}\delta)^2 - 1$. On the other hand, using (2.9) and (3.20), we get

$$c\alpha = \frac{\alpha}{1-\alpha^2(2-\kappa^2)} = \frac{2TK(\kappa)}{T^2 - 4(2-\kappa^2)K^2(\kappa)}.$$

and

$$\begin{aligned}|c^*\alpha^* E(\kappa^*) - c\alpha E(\kappa)| \quad &= 2T\left|\tfrac{K(\kappa^*)E(\kappa^*)}{T^2-4(2-\kappa^{*2})K^2(\kappa*)} - \tfrac{K(\kappa)E(\kappa)}{T^2-4(2-\kappa^2)K^2(\kappa)}\right| \\ &= 2T\left|\left(\tfrac{K(\kappa)E(\kappa)}{T^2-4(2-\kappa^2)K^2(\kappa)}\right)'(\kappa)\right|\cdot|\kappa^* - \kappa| \ge C_2|\kappa^* - \kappa|,\end{aligned} \tag{3.22}$$

with appropriate C_2 independent of the values with $*$ accent.

Thus combining (3.21) and (3.22), we get

$$d(\vec{u},\vec{\varphi^*})|_{t=0} \le d(\vec{u},\vec{\varphi})|_{t=0} + d(\vec{\varphi},\vec{\varphi^*})|_{t=0} < \delta + c\alpha E(\kappa)\widetilde{C}\delta_1 = \delta_0.$$

We select δ, δ_0, and δ_1 sufficiently small and apply the part of the theorem which has been already proved,

$$d(\vec{u},\vec{\varphi^*})|_{t=0} < \delta_0 \Rightarrow d(\vec{u},\vec{\varphi^*}) < \frac{\varepsilon}{2},\ \ t \in [0,\infty).$$

Choosing an appropriate $\delta > 0$, we obtain that

$$d(u,\varphi) \le d(u,\varphi^*) + d(\varphi,\varphi^*) < \frac{\varepsilon}{2} + c\alpha E(\kappa)\widetilde{C}\delta_1 < \varepsilon,$$

for all $t \in [0,\infty)$. The theorem is completely proved. □

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