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Boundary Value Problems for a Class of Fractional Differential Equations Depending on First Derivative

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Abstract

This paper deals with the existence and uniqueness results for nonlinear and double perturbed BVPs for fractional differential equations with first-order dependence derivative. Our approach is based on fixed point theorems and monotone iterative technique. Some illustrative examples are also presented.

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1 Introduction

Fractional differential equations can be extensively applied for various disciplines such as physics, mechanics, chemistry and engineering, see [12, 16, 17, 18]. Recently, boundary value problems (BVP for short) for fractional differential equations have been addressed by several researchers. Some recent work on boundary value problems of fractional order can be found in [1, 2, 3, 4, 5, 6, 8, 11] and the references therein.

In this paper, we consider the BVP

$$\begin{cases} -D^{\alpha}_{0^+}u(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \ 2 < \alpha \leq 3, \\ u(0) = u'(0) = u'(1) = 0, \end{cases} \tag{1.1}$$

where $D^{\alpha}_{0^+}$ denote the standard Riemann-Liouville fractional derivative of order α and $f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function.

To the best of our knowledge, exists few papers concerned with the nonlinear fractional differential equations with first-order dependence derivative. In [13] Kosmatov studied the existence of at least one positive solution using the Leray-Schauder Continuation Principle of the BVP

$$\begin{cases} D^{\alpha}_{0^+}u(t) = f(t, u(t), u'(t)), \quad 0 < t < 1, \ 1 < \alpha \leq 2, \\ u(0) = u(1) = 0. \end{cases} \tag{1.2}$$

By means of fixed point theorem for the mixed monotone operator, Zhang [20] studied the existence of the following higher-order singular BVP

$$\begin{cases} D^{\alpha}_{0^+}u(t) + q(t)f(t, u(t), u'(t), \ldots, u^{(n-2)}(t)), \quad 0 < t < 1, \ n-1 < \alpha \leq n, \\ u(0) = 0, \ u'(0) = u'(1) = 0. \end{cases} \tag{1.3}$$

In [19] Stanek discussed the existence of positive solutions for the singular BVP

$$\begin{cases} D^{\alpha}_{0^+}u(t) + q(t)f(t, u(t), u'(t), D^{\mu}u(t)) = 0, \quad 0 < t < 1, \ 2 < \alpha < 3, \ 0 < \mu < 1, \\ u(0) = u'(0) = \ldots = u^{(n-2)}(0) = u^{(n-2)}(1) = 0. \end{cases} \tag{1.4}$$

In [10] El-Shahed established the existence and nonexistence of the problem

$$\begin{cases} D^{\alpha}_{0^+}u(t) + \lambda a(t)f(u(t)) = 0, \quad 0 < t < 1, \ 2 < \alpha < 3, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{1.5}$$

Very recently Moussaoui and Ntouyas [14] investigated some new existence and uniqueness results for nonlinear fractional differential equations with four-point nonlocal integral boundary conditions by applying fixed point theorems, and gave some extensions to a boundary value problem with first-order dependence derivative.

In this paper we prove some new existence and uniqueness results for a class of BVPs (1.1), by using fixed point theorems. Our results extend and supplement the results mentioned above. Thus, in Theorem 3.1 we prove an existence and uniqueness result by using a fixed point theorem of Boyd and Wong [9] for nonlinear contractions, while in Theorem 4.1 we prove existence of two positive solutions via iteration process. In Theorem 5.1 we prove the existence of solution of a double perturbed boundary value problem for fractional differential equations, by using a fixed point theorem of Krasnoselskii-Nonlinear Alternative type [15], which extends some results proved by Benchohra, Djebali and Moussaoui in [7]. Finally the paper close with some illustrative examples.

2 Preliminaries

For the reader's convenience, let us recall some basic definitions and preliminary results of fractional calculus and fixed point theory.

Definition 2.1. [16, 18] For a continuous function $g : [0,\infty) \to \mathbb{R}$, the Rimann-Liouville derivative of fractional order $\alpha > 0$ is defined as

$$D_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^t \frac{g(s)}{(t-s)^{\alpha-n-1}} ds, \quad n = [\alpha]+1,$$

where $[\alpha]$ denotes the integer part of the real number α.

Definition 2.2. [16, 18] The Riemann-Liouville fractional integral of order α is defined as

$$I_{0^+}^{\alpha} g(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{g(s)}{(t-s)^{1-\alpha}} ds, \quad \alpha > 0,$$

provided the integral exists.

Lemma 2.3. *(see [6]) Let* $\alpha > 0$, *if* $u \in C(0,1) \cap L(0,1)$, *then the general solution of the fractional differential equation* $D_{0^+}^{\alpha} x(t) = 0$ *is given by*

$$x(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{N-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, N$.

In view of Lemma 2.3, it follows that

Lemma 2.4. *(see [6]) Assume that* $u \in C(0,1) \cap L(0,1)$ *with a fractional derivative of order* $\alpha > 0$. *Then*

$$I_{0^+}^{\alpha} D_{0^+}^{\alpha} x(t) = x(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \ldots + c_N t^{N-1} \tag{2.1}$$

for some $c_i \in \mathbb{R}$, $i = 0, 1, 2, \ldots, N-1$.

The following lemma was proved in [10].

Lemma 2.5. *Let* $g : [0,1] \to \mathbb{R}$ *be a given continuous function. Then a unique solution of the boundary value problem*

$$\begin{cases} D_{0^+}^{\alpha} x(t) + g(t) = 0, \quad 0 < t < 1, \quad 2 < \alpha \leq 3, \\ x(0) = x'(0) = x'(1) = 0, \end{cases} \tag{2.2}$$

is given by

$$x(t) = \int_0^1 G(t,s) g(s) ds, \tag{2.3}$$

where

$$G(t,s) = \begin{cases} \dfrac{1}{\Gamma(\alpha)} (1-s)^{\alpha-2} t^{\alpha-1}, & \text{if } 0 \leq t \leq s \leq 1, \\ \dfrac{1}{\Gamma(\alpha)} \left[(1-s)^{\alpha-2} t^{\alpha-1} - (t-s)^{\alpha-1}\right], & \text{if } 0 \leq s \leq t \leq 1. \end{cases} \tag{2.4}$$

It is obvious that

$$G(t,s) \geq 0, \quad G(1,s) \geq G(t,s), \quad 0 \leq t, \; s \leq 1.$$

For more details on fractional calculus we refer to [12, 16, 18].

Now we present some results from fixed point theory. Firstly we recall Boyd and Wong's lemma, after we give the necessary definitions.

Definition 2.6. Let E be a real Banach space. A nonempty closed set $P \subset E$ is said to be a cone provided that

(i) $au + bv \in P$ for all $u, v \in P$ and all $a \geq 0$, $b \geq 0$, and

(ii) u, $-u \in P$ implies $u = 0$.

Definition 2.7. Let E be a Banach space and let $F : E \to E$ be a mapping. F is said to be a nonlinear contraction if there exists a continuous nondecrasing function $\Psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that $\Psi(0) = 0$ and $\Psi(\xi) < \xi$ for all $\xi > 0$ with the property:

$$\|Fx - Fy\| \leq \Psi(\|x - y\|), \quad \forall x, y \in E.$$

Lemma 2.8. *(Boyd and Wong) [9]. Let E be a Banach space and let $F : E \to E$ be a nonlinear contraction. Then F has a unique fixed point in E.*

Next we recall the following fixed point theorem of Krasnoselskii-Nonlinear Alternative type, for the sum of a contraction and a completely continuous map due to Ntouyas and Tsamatos [15].

Lemma 2.9. *Let $(X, \|\cdot\|)$ be a Banach space, B_1, B_2 be operators from X into X such that B_1 is contraction, and B_2 is completely continuous. Assume also that*

(A) There exist a sphere $B(0,r) \in X$ with center 0 and radius r such that for every $y \in B(0,r)$

$$r(1-\gamma) \geq \|B_1 0 - B_2 y\|.$$

Then either

(a) the operator equation $x = (B_1 + B_2)x$ has a solution with $\|x\| \leq r$, or

(b) there exist a point $x_0 \in \partial B(0,r)$, and $\lambda \in (0,1)$ such that $x_0 = \lambda B_1\left(\frac{x_0}{\lambda}\right) + \lambda B_2 x_0$.

3 Existence and uniqueness of solutions

In this section we give an existence and uniqueness result for the BVP (1.1) by using Boyd and Wong's fixed point theorem.

Theorem 3.1. *Assume that*

(H) $|f(t,u_1,v_1)-f(t,u_2,v_2)| \le h_1(t)\dfrac{|u_1-u_2|}{H^*+|u_1-u_2|}+h_2(t)\dfrac{|v_1-v_2|}{H^*+|v_1-v_2|}$, $t\in(0,1)$, $u_1,v_1,u_2,$ $v_2\in\mathbb{R}$, *where* $h_1,h_2:(0,1)\to\mathbb{R}^+$, $h_1,h_2\in L^1(0,1)$ *with*

$$H^* \;=\; \frac{\alpha}{\Gamma(\alpha)}(|h_1|_{L^1}+|h_2|_{L^1}).$$

Then the boundary value problem (1.1) has a unique solution.

Proof. In view of Lemma 2.5 we define the operator $F: C^1([0,1],\mathbb{R})\to C^1([0,1],\mathbb{R})$ by

$$Fu(t) \;=\; \int_0^1 G(t,s)f(s,u(s),u'(s))ds, \tag{3.1}$$

where $G(t,s)$ is give by (2.4).

Let us consider the Banach space $E=C^1([0,1],\mathbb{R})$ endowed with the norm,

$$\|u\|_1=\sup_{t\in[0,1]}|u(t)|+\sup_{t\in[0,1]}|u'(t)|=\|u\|_0+\|u'\|_0.$$

Let the continuous nondecreasing function $\Psi:\mathbb{R}^+\to\mathbb{R}^+$ satisfying $\Psi(0)=0$ and $\Psi(\xi)<\xi$ for all $\xi>0$ defined by

$$\Psi(\xi)=\frac{H^*\xi}{H^*+\xi},\quad \forall\xi\ge 0.$$

Let $u,v\in E$. Then

$$|f(s,u(s),u'(s))-f(s,v(s),v'(s))|\le\frac{h_1(s)+h_2(s)}{H^*}\Psi(\|u-v\|_1).$$

We remark that

$$\begin{aligned}|G(t,s)| &\le \frac{1}{\Gamma(\alpha)}, && \forall t,s\in[0,1],\\ \left|\frac{\partial G}{\partial t}(t,s)\right| &\le \frac{\alpha-1}{\Gamma(\alpha)}, && \forall t,s\in[0,1],\ t\neq s,\end{aligned}$$

so that

$$\begin{aligned}|Fu(t)-Fv(t)| &\le \int_0^1 G(t,s)\left[h_1(s)\frac{|u(s)-v(s)|}{H^*+|u(s)-v(s)|}+h_2(s)\frac{|u'(s)-v'(s)|}{H^*+|u'(s)-v'(s)|}\right]ds\\ &\le \frac{1}{\Gamma(\alpha)}(|h_1|_{L^1}+|h_2|_{L^1})\frac{\|u-v\|_1}{H^*+\|u-v\|_1},\end{aligned}$$

and

$$\begin{aligned}|(Fu)'(t)-(Fv)'(t)| &\le \int_0^t \frac{\partial G}{\partial t}(t,s)\left[h_1(s)\frac{|u(s)-v(s)|}{H^*+|u(s)-v(s)|}+h_2(s)\frac{|u'(s)-v'(s)|}{H^*+|u'(s)-v'(s)|}\right]ds\\ &\le \frac{\alpha-1}{\Gamma(\alpha)}(|h_1|_{L^1}+|h_2|_{L^1})\frac{\|u-v\|_1}{H^*+\|u-v\|_1}.\end{aligned}$$

Then $\|Fx-Fy\|_1\le\Psi(\|x-y\|_1)$ and F is a nonlinear contraction and it has a unique fixed point in E, by Lemma 2.8. This complete the proof. □

4 Existence and iteration of two positive solutions

Our aim in this section is to use the monotone iterative technique to study the existence of two positive solutions of the boundary value problem (1.1).

In the Banach space $E = C^1([0,1],\mathbb{R})$ we define the cone $P \subset E$ by

$$P = \{u \in E \;\; u(t) \geq 0\}.$$

We will prove the following existence result.

Theorem 4.1. *Assume that* $f : [0,1]\times\mathbb{R}_+\times\mathbb{R} \to \mathbb{R}_+$ *is a nonnegative continuous function and there exists* $a > 0$ *such that*

(S1) $f(t,u_1,v_1) \leq f(t,u_2,v_2)$ *for any* $0 \leq t \leq 1$, $0 \leq u_1 \leq u_2 \leq a$, $\;0 \leq |v_1| \leq |v_2| \leq a$;

(S2) $\max\limits_{0\leq t\leq 1} f(t,a,a) \leq \dfrac{a}{3H_1^*}$, *for some constant* $H_1^* \geq \dfrac{\alpha-1}{3\Gamma(\alpha)}$;

(S3) $f(t,0,0) \neq 0,\; 0 \leq t \leq 1.$

Then the boundary value problem (1.1) has two positive solutions.

Proof. We define the operator $T : P \to E$ by

$$Tu(t) \;\; = \;\; \int_0^1 G(t,s)f(s,u(s),u'(s))ds, \tag{4.1}$$

where $G(t,s)$ is given by (2.4).

Then, from the definition of T, it is easy to prove that $T : P \to P$. In what follows, we will prove that T is a completely continuous operator. The continuity of T is obvious from the continuity of the nonlinear function f. Now, it is easy to prove that the operator T is compact, by using the Arzelà-Ascoli theorem. Then, $T : P \to P$ is a completely continuous, and each fixed point of T in P is a solution of BVP (1.1).

We denote

$$\overline{P}_a = \{u \in P\,; \;\; \|u\|_1 \leq a\}.$$

Then, in what follows, we first prove that $T : \overline{P}_a \to \overline{P}_a$. If $u \in \overline{P}_a$, then $\|u\|_1 \leq a$, we have

$$0 \leq u(t) \leq \max_{0\leq t\leq 1} |u(t)| \leq \|u\|_1 \leq a,$$

$$|u'(t)| \leq \max_{0\leq t\leq 1} |u'(t)| \leq \|u\|_1 \leq a.$$

So by (S1), (S2) we have

$$0 \leq f(t,u(t),u'(t)) \leq f(t,a,a) \leq \max_{0\leq t\leq 1} f(t,a,a) \leq \frac{a}{3H_1^*} \qquad \text{for} \quad 0 \leq t \leq 1.$$

In fact,

$$\|Tu\|_1 = \max\{\max_{t\in[0,1]} |Tu(t)|, \max_{t\in[0,1]} |(Tu)'(t))|\}.$$

Then we have

$$\begin{aligned}|Tu(t)| &\leq \int_0^1 |G(t,s)f(s,u(s),u'(s))|ds\\ &\leq \frac{1}{\Gamma(\alpha)}\cdot\frac{a}{3H_1^*}\leq a,\end{aligned}$$

and

$$\begin{aligned}|(Tu)'(t)| &\leq \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)f(s,u(s),u'(s))\right|ds\\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\cdot\frac{a}{3H_1^*}\leq a.\end{aligned}$$

Thus, we obtain that

$$\|Tu\|_1 \leq a,$$

which proves that $T:\overline{P}_a\to\overline{P}_a$.
Let

$$w_0(t) = \left(\frac{\alpha-1}{\Gamma(\alpha)}\cdot\frac{a}{3H_1^*}\right)(t+1), \qquad 0\leq t\leq 1;$$

then $w_0(t)\in\overline{P}_a$. Let $w_1=Tw_0$, $w_2=Tw_1=T^2w_0$; then denote $w_{n+1}=Tw_n=T^nw_0$, $n=1,2,\ldots.$ Since the operator $T:\overline{P}_a\to\overline{P}_a$, we have $w_n\in T\overline{P}_a\subseteq\overline{P}_a$, $n=1,2,...$ Since

$$\begin{aligned}w_1(t) &= Tw_0\\ &= \int_0^1 G(t,s)f(s,w_0(s),w_0'(s))ds\\ &\leq \frac{1}{\Gamma(\alpha)}\cdot\frac{a}{3H_1^*}\leq w_0(t), \qquad 0\leq t\leq 1,\end{aligned}$$

and

$$\begin{aligned}|w_1'(t)| &= |(Tw_0)'(t)|\\ &\leq \int_0^1 \left|\frac{\partial G}{\partial t}(t,s)f(s,w_0(s),w_0'(s))\right|ds\\ &\leq \frac{\alpha-1}{\Gamma(\alpha)}\cdot\frac{a}{3H_1^*}=|w_0'(t)|, \qquad 0\leq t\leq 1,\end{aligned}$$

we obtain $\|w_1\|_1\leq\|w_0\|_1$. So

$$\begin{aligned}w_2(t) &= Tw_1(t)\leq Tw_0(t)=w_1(t), \qquad 0\leq t\leq 1,\\ |w_2'(t)| &= |(Tw_1)'(t)|\leq|(Tw_0)'(t)|=|w_1'(t)|, \qquad 0\leq t\leq 1.\end{aligned}$$

Hence, by induction, we have

$$w_{n+1}(t)\leq w_n(t), \qquad |w_{n+1}'(t)|\leq|w_n'(t)|, \qquad 0\leq t\leq 1,\ n=1,2\ldots.$$

Because the operator T is completely continuous, then $(w_n)_n$ is sequentially compact. Thus, there exists $(w_{n_k})_k \subset (w_n)_n$ such that $w_{n_k} \longrightarrow w^*$ with $w^* \in \overline{P}_a$. Since $(w_n)_n$ satisfies the above mentioned monotonicity, we assert that $w_n \longrightarrow w^*$. Applying the continuity of T and $w_{n+1}(t) = Tw_n(t)$, we get $Tw^* = w^*$. Thus w^* is a positive solution of the BVP (1.1).

Let $v_0(t) = 0, \quad 0 \le t \le 1$, then $v_0(t) \in \overline{P}_a$. Let $v_1 = Tv_0, \quad v_2 = Tv_1 = T^2 v_0$; then denote $v_{n+1} = Tv_n = T^n v_0, \; n = 1,2,\ldots$. Since the operator $T : \overline{P}_a \to \overline{P}_a$, we have $v_n \in T\overline{P}_a \subseteq \overline{P}_a$, $n = 1,2,\ldots$. Since T is completely continuous, $(v_n)_n$ is sequentially compact. Since $v_1(t) = Tv_0(t) = T0 \in T\overline{P}_a \subseteq \overline{P}_a$, we have

$$\begin{aligned} v_1(t) &= Tv_0(t) = T0(t) \ge 0, \quad 0 \le t \le 1 \\ |v_1'(t)| &= |(Tv_0)'(t)| = |(T0)'(t)| \ge 0, \quad 0 \le t \le 1. \end{aligned}$$

So

$$\begin{aligned} v_2(t) &= Tv_1(t) \ge T0(t) = v_1(t), \quad 0 \le t \le 1, \\ |v_2'(t)| &= |(Tv_1)'(t)| \ge |(T0)'(t)| = v_1'(t), \quad 0 \le t \le 1. \end{aligned}$$

By an induction argument similar to the above we obtain

$$v_{n+1}(t) \ge v_n(t), \quad |v_{n+1}'(t)| \ge |v_n'(t)|, \quad 0 \le t \le 1, \; n = 1,2\ldots$$

Thus, there exists $v^* \in \overline{P}_a$ such that $v_n \longrightarrow v^*$. Applying the continuity of T and $v_{n+1}(t) = Tv_n(t)$, we get $Tv^* = v^*$. Since $f(t,0,0) \neq 0, \quad 0 \le t \le 1$, by (S3), the zero function is not solution of the BVP (1.1). Thus v^* is a positive solution of BVP (1.1).

It is well known that each fixed point of T is a solution of the BVP (1.1). Hence, we assert that w^* and v^* are two positive solutions of the BVP (1.1), and the proof is completed. □

5 Existence of solutions for double perturbed BVPs

In this section we shall prove the existence of solutions of the following double perturbed fractional boundary value problem

$$\begin{cases} -D_{0^+}^{\alpha} u(t) = g(t,u(t),u'(t)) + h(t,u(t),u'(t)), \quad 0 < t < 1, \; 2 < \alpha \le 3, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{5.1}$$

This result extends a result proved in [7].

Let us introduce the following hypotheses:

(H1) The function $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is Carathéodory and there exist $p_1, p_2 \in L^1([0,1],\mathbb{R}_+)$, such that

$$|g(t,u_1,v_1) - g(t,u_2,v_2)| \le p_1(t)|u_1 - u_2| + p_2(t)|v_1 - v_2|$$

for almost each $t \in [0,1]$ and all $u_1, v_1, u_2, v_2 \in \mathbb{R}$;

(H2) the function h is continuous and there exist a function $q \in L^1([0,1],\mathbb{R}_+)$ with $q(t) > 0$ for each $t \in [0,1]$ and a continuous nondecreasing function $\psi : \mathbb{R}_+ \to (0,+\infty)$ such that

$$|h(t,u,v)| \le q(t)\psi(|u| + |v|)$$

for each $t \in [0,1]$ and for all $u, v \in \mathbb{R}$;

(H3) Assume that $\frac{\alpha}{\Gamma(\alpha)}(|p_1|_{L^1}+|p_2|_{L^1})<1$, and there exist $b>0$ such that

$$b>\frac{\frac{\alpha}{\Gamma(\alpha)}\left[|q|_{L^1}\psi(b)+g^*\right]}{1-\frac{\alpha}{\Gamma(\alpha)}(|p_1|_{L^1}+|p_2|_{L^1})}\quad\text{with } g^*=\int_0^1 g(s,0,0)ds.$$

Now, we present our main result of this section.

Theorem 5.1. *Assume that assumptions (H1)-(H3) hold true. Then the perturbed BVP (5.1) has at least one solutions in* $C^1([0,1],\mathbb{R})$.

Proof. In view of Lemma 2.5 we define the operator $T: C^1([0,1],\mathbb{R})\to C^1([0,1],\mathbb{R})$ by

$$Tu(t)=\int_0^1 G(t,s)\left[g(s,u(s),u'(s))+h(s,u(s),u'(s))\right]ds, \tag{5.2}$$

where $G(t,s)$ is given by (2.4).

Consider the Banach space $E=C^1([0,1],\mathbb{R})$ endowed with the norm $\|u\|_1$ and define two operator B_1,B_2 on E by

$$B_1u(t)=\int_0^1 G(t,s)g(s,u(s),u'(s))ds,\quad B_2u(t)=\int_0^1 G(t,s)h(s,u(s),u'(s))ds.$$

Let $u,v\in E$ and $t\in[0,1]$; then by $(H1)$ we obtain

$$\begin{aligned}|B_1u(t)-B_1v(t)| &\le \int_0^1 G(t,s)|g(s,u(s),u'(s))-g(s,v(s),v'(s))|ds\\ &\le \frac{1}{\Gamma(\alpha)}(|p_1|_{L^1}+|p_2|_{L^1})\|u-v\|_1,\end{aligned}$$

and

$$\begin{aligned}|(B_1u)'(t)-(B_1v)'(t)| &\le \int_0^1 \frac{\partial G}{\partial t}(t,s)\left|g(s,u(s),u'(s))-g(s,v(s),v'(s))\right|ds\\ &\le \frac{\alpha-1}{\Gamma(\alpha)}(|p_1|_{L^1}+|p_2|_{L^1})\|u-v\|_1.\end{aligned}$$

Then by $(H3)$ we obtain

$$\|B_1u-B_1v\|_1\le\eta\|u-v\|_1,\ \text{with}\quad \eta=\frac{\alpha}{\Gamma(\alpha)}(|p_1|_{L^1}+|p_2|_{L^1})<1.$$

i. e. B_1 is a contraction.

Now, for proving that B_2 is continuous, let $x_n,x\in E$ with $x_n\to x$, that is

$$\forall\varepsilon>0,\ \exists n_0\in\mathbb{N}^*,\quad (n\ge n_0\Longrightarrow \|x_n-x\|_1<\varepsilon).$$

For each $t\in[0,1]$, we have

$$|B_2x_n(t)-B_2x(t)| \le \frac{1}{\Gamma(\alpha)}\int_0^1 |h(s,x_n(s),(x_n)'(s))-h(s,x(s),x'(s))|ds,$$

$$|(B_2x_n)'(t)-(B_2x)'(t)| \leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 |h(s,x_n(s),(x_n)'(s))-h(s,x(s),x'(s))|ds.$$

Thus we have

$$\|B_2x_n(t)-B_2x(t)\|_1 \leq \frac{\alpha}{\Gamma(\alpha)}\int_0^1 \|h(s,x_n(s),(x_n)'(s))-h(s,x(s),x'(s))\|_0 ds.$$

Since the convergence of a sequence implies its boundedness, then there exist a number $k>0$ such that

$$\|x_n\|_1 \leq k, \quad \|x\|_1 \leq k, \quad \forall t \in [0,1]$$

and thus h is uniformly continuous on the compact set $\{(t,x)\in \mathbb{R}^+\times\mathbb{R},\ t\in[0,1],\ \|x\|_1\leq k\}$. Thus $\|B_2x_n-B_2x\|_1 \leq \varepsilon,\ \forall n \geq n_0$. Then B_2 is continuous.

For proving that B_2 is totally bounded, we consider the closed ball $C=\{x\in E; \|x\|_1\leq R\}$, and by $(H2)$ we prove that $B_2(C)$ is relatively compact in E. We have

$$\begin{aligned}
|B_2x(t)| &\leq \frac{1}{\Gamma(\alpha)}\int_0^1 |h(s,x(s),x'(s))|ds \leq \frac{1}{\Gamma(\alpha)}\psi(R)|q|_{L^1},\\
|(B_2x)'(t)| &\leq \frac{\alpha-1}{\Gamma(\alpha)}\int_0^1 |h(s,x(s),x'(s))|ds \leq \frac{\alpha-1}{\Gamma(\alpha)}\psi(R)|q|_{L^1},
\end{aligned}$$

and thus $\|B_2x\|_1 \leq \dfrac{\alpha}{\Gamma(\alpha)}\psi(R)|q|_{L^1}$. Then $B_2(C)$ is uniformly bounded. Also we have

$$\begin{aligned}
|B_2x(t_2)-B_2x(t_1)| &\leq \psi(R)\int_0^1 |q(s)|\,|G(t_2,s)-G(t_1,s)|\,ds,\\
|(B_2x)'(t_2)-(B_2x)'(t_1)| &\leq \psi(R)\int_0^1 |q(s)|\left|\frac{\partial G}{\partial t}(t_2,s)-\frac{\partial G}{\partial t}(t_1,s)\right|ds.
\end{aligned}$$

Then $B_2(C)$ is equicontinuous because the right-hand side tends to 0 as $t_1\to t_2$. By the Arzelá-Ascoli Theorem, the mapping B_2 is completely continuous on E.

Now, let b being defined in $(H3)$, and we consider the sphere $B(0,b)$. For $x\in B(0,b)$, we have

$$\|B_1 0+B_2u\|_1 = \sup_{t\in[0,1]}|B_1 0+B_2u| + \sup_{t\in[0,1]}|(B_1 0)'-(B_2u)'|.$$

Then

$$\begin{aligned}
|B_1 0+B_2u| &\leq \frac{1}{\Gamma(\alpha)}\left[\int_0^1 |g(s,0,0)|ds+|q|_{L^1}\psi(\|u(s)\|_1)\right] \leq \frac{1}{\Gamma(\alpha)}\left[g^*+|q|_{L^1}\psi(b)\right],\\
|(B_1 0)'-(B_2u)'| &\leq \frac{\alpha-1}{\Gamma(\alpha)}\left[\int_0^1 |g(s,0,0)|ds+|q|_{L^1}\psi(\|u(s)\|_1)\right] \leq \frac{\alpha-1}{\Gamma(\alpha)}\left[g^*+|q|_{L^1}\psi(b)\right]
\end{aligned}$$

and consequently

$$\|B_1 0+B_2u\|_1 \leq \frac{\alpha}{\Gamma(\alpha)}\left[g^*+|q|_{L^1}\psi(b)\right] < b\left[1-\frac{\alpha}{\Gamma(\alpha)}\left(|p_1|_{L^1}+|p_2|_{L^1}\right)\right].$$

Therefore the assumptions of Lemma 2.9 are satisfied and a direct application of it yields that either the conclusion (a) or the conclusion (b) holds. We show that the conclusion (b) is not possible.

By contradiction, we assume that there exist $\lambda \in (0,1)$ and $u \in \partial B(0,b)$, with

$$u = \lambda B_1\left(\frac{u}{\lambda}\right) + \lambda B_2 u.$$

Then we have

$$\begin{aligned}
|u(t)| &= \lambda \int_0^1 G(t,s) g\left(s, \frac{u(s)}{\lambda}, \frac{u'(s)}{\lambda}\right) ds + \lambda \int_0^1 G(t,s) h(s,u(s),u'(s)) ds \\
&\leq \lambda \int_0^1 G(t,s)\left(|p_1(s)| \cdot \left|\frac{u(s)}{\lambda}\right| + |p_2(s)| \cdot \left|\frac{u'(s)}{\lambda}\right|\right) ds + \lambda \int_0^1 G(t,s) g(s,0,0) ds \\
&\quad + \lambda \int_0^1 G(t,s) q(s) \psi(\|u\|_1) ds \\
&\leq \frac{1}{\Gamma(\alpha)} \left[(|p_1|_{L^1} + |p_2|_{L^1}) \|u\|_1 + |q|_{L^1} \psi(b) + g^* \right]
\end{aligned}$$

and

$$\begin{aligned}
|u'(t)| &= \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s) g(s, \frac{u(s)}{\lambda}, \frac{u'(s)}{\lambda}) ds + \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s) h(s,u(s),u'(s)) ds \\
&\leq \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s)\left(|p_1(s)| \cdot \left|\frac{u(s)}{\lambda}\right| + |p_2(s)| \cdot \left|\frac{u'(s)}{\lambda}\right|\right) ds + \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s) g(s,0,0) ds \\
&\quad + \lambda \int_0^1 \frac{\partial G}{\partial t}(t,s) q(s) \psi(\|u\|_1) ds \\
&\leq \frac{(\alpha-1)}{\Gamma(\alpha)} \left[(|p_1|_{L^1} + |p_2|_{L^1}) \|u\|_1 + |q|_{L^1} \psi(b) + g^* \right].
\end{aligned}$$

Thus $\|u\|_1 \leq \frac{\alpha}{\Gamma(\alpha)} \left[(|p_1|_{L^1} + |p_2|_{L^1}) \|u\|_1 + |q|_{L^1} \psi(b) + g^* \right]$. Hence

$$b = \|u\|_1 \leq \frac{\frac{\alpha}{\Gamma(\alpha)} \left[|q|_{L^1} \psi(b) + g^* \right]}{1 - \frac{\alpha}{\Gamma(\alpha)} (|p_1|_{L^1} + |p_2|_{L^1})}.$$

This contradicting the condition in $(H3)$. We conclude that the second alternative of Lemma 2.9 is not valid. Then we conclude that assertion (a) is satisfied. Then the double perturbed BVP (5.1) has at least one solutions in $C^1([0,1],\mathbb{R})$. □

6 Examples

Example 6.1. Consider the following fractional BVP

$$\begin{cases} D_{0^+}^{5/2} u(t) = \dfrac{1}{(1+t)} \cdot \dfrac{|u(t)|}{H^* + |u(t)|} + \dfrac{1}{(1+t)} \cdot \dfrac{|u'(t)|}{H^* + |u'(t)|} + 1, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{6.1}$$

Here $\alpha = \frac{5}{2}$, $f(t,u,u') = \frac{1}{(1+t)} \cdot \frac{|u(t)|}{H^* + |u(t)|} + \frac{1}{(1+t)} \cdot \frac{|u'(t)|}{H^* + |u'(t)|} + 1.$
Clearly $H^* = \frac{20 \ln 2}{3\sqrt{\pi}}$. With $h_1(t) = h_2(t) = \frac{1}{(1+t)}$ we have:

$$\begin{aligned}
|f(t,u,u') - f(t,v,v')| &\leq h_1(t)\frac{H^*||u|-|v||}{(H^*+|u|)(H^*+|v|)} + h_2(t)\frac{H^*||u'|-|v'||}{(H^*+|u'|)(H^*+|v'|)} \\
&\leq h_1(t)\frac{H^*|u-v|}{(H^*)^2+H^*(|u|+|v|)} + h_2(t)\frac{H^*|u'-v'|}{(H^*)^2+H^*(|u'|+|v'|)} \\
&\leq h_1(t)\frac{|u-v|}{H^*+|u-v|} + h_2(t)\frac{|u'-v'|}{H^*+|u'-v'|}
\end{aligned}$$

Thus, by Theorem 3.1, the boundary value problem (6.1) has a unique solution.

Example 6.2. We consider the following BVP

$$\begin{cases} D_{0^+}^{5/2} u(t) = t^3 + \frac{u(t)}{4} + \frac{u'(t)}{6}, \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{6.2}$$

Here $\alpha = \frac{5}{2}$ and $f(t,u,v) = t^3 + \frac{u(t)}{4} + \frac{u'(t)}{6}$.
Choose $a = 2$, and $H_1^* = 1$. Then $H_1^* = 1 \geq \frac{\alpha - 1}{\Gamma(\alpha)} = \frac{2}{\sqrt{\pi}}$. Also, $f(t,u,v)$ satisfies:

(1) $f(t,u_1,v_1) \leq f(t,u_2,v_2)$ for any $0 \leq t \leq 1$, $0 \leq u_1 \leq u_2 \leq 2$, $0 \leq |v_1| \leq |v_2| \leq 2$;

(2) $\max_{0 \leq t \leq 1} f(t,2,2) \leq 2$;

(3) $f(t,0,0) \neq 0, \quad 0 \leq t \leq 1.$

So, by Theorem 4.1, the BVP (6.2) has two positive solutions.

Example 6.3. We consider the following double perturbed BVP for fractional differential equations with first-order dependence derivative

$$\begin{cases} D_{0^+}^{5/2} u(t) = t^2 (|u(t)| + |u'(t)|) + t^3 \left(\frac{|u(t)+u'(t)|}{1 + |u(t)+u'(t)|} \right), \quad 0 < t < 1, \\ u(0) = u'(0) = u'(1) = 0. \end{cases} \tag{6.3}$$

Here

$$g(t,u,v) = t^2(|u(t)| + |v(t)|), \qquad h(t,u,v) = t^3 \left(\frac{|u(t)+v(t)|}{1 + |u(t)+v(t)|} \right).$$

It is clear that $(H1), (H2)$ are satisfied with $p_1(t) = p_2(t) = t^2$, $q(t) = t^3$ and $\psi(\xi) = \frac{\xi}{1+\xi}$, $\xi = |u(t)| + |u'(t)|$. Finally there exists $b = 2 > 0$ that $(H3)$ is satisfied. Then by Theorem 5.1 the double perturbed BVP (6.3) has at least one solutions in $C^1([0,1],\mathbb{R})$.

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