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Decay Property for Solutions in the Three-Phase-Lag Heat Conduction

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Abstract

In this paper, we consider the Cauchy problem of two models of the theory of heat conduction with three−phase−lag. Under appropriate assumptions on the material parameters, we show the optimal decay rate of the L^2-norm of solutions. More precisely, we prove that in each model the L^2-norm of the solution is decaying with the rate $(1+t)^{-1/4}$ for initial data in $L^1(\mathbb{R})$. This decay rate is similar to the one of the heat kernel. Some faster decay rates have been also given for some weighted initial data in $L^1(\mathbb{R})$.

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1 Introduction

In 1807, the French mathematical physicist Joseph Fourier proposed a constitutive relation of the heat flux of the form

$$q(x,t) = -\kappa\nabla\theta(x,t), \tag{1.1}$$

where x stands for the material point, t is the time, q is the heat flux, θ is the temperature, ∇ is the gradient operator and κ is the thermal conductivity of the material, which is a thermodynamic state property. Using the Fourier law (1.1), the behavior of an elastic heat

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body can be described by a coupled system of hyperbolic−parabolic type. This hyperbolic−parabolic system is interesting due to its large applications in mechanics, physics and engineering problems.

Over the past two decades, there has been a lot of work on local existence, global existence, well-posedeness, and asymptotic behavior of solutions to some initial-boundary value problems as well as to Cauchy problems in both one-dimensional and multi-dimensional thermoelasticity. See for instance [15, 16, 22, 24, 29] and references therein.

The Cauchy problem, in which a thermoelastic body occupies the entire real line, was investigated by Kawashima & Okada [12], Zheng & Shen [30], and Hrusa & Tarabek [9] proving the global existence in time. In particular, Hrusa & Tarabek [9] combined certain estimates of Slemrod [24] that remain valid on unbounded intervals with some additional ones which exploited some relations associated with the second law of thermodynamics, to obtain the energy estimate for lowest order terms, an estimate that cannot be obtained using Poincaré's inequality as in Slemrod's paper.

The Fourier law of heat conduction is an early empirical law. It assumes that q and $\nabla\theta$ appear at the same time instant t and consequently implies that thermal signals propagate with an infinite speed. That is, any thermal disturbance at a single point has an instantaneous effect everywhere in the medium. More precisely, the thermoelastic theory based on the Fourier law has some disadvantages such as:

- infinite velocity of thermoelastic disturbances;
- unsatisfactory thermoelastic response of a solid to short laser pulses;
- poor description of thermoelastic behavior at low temperature.

With the development of science and technology such as the application of ultra-fast pulse-laser heating on metal films, heat conduction appears in the range of high heat flux and high unsteadiness. The drawback of infinite heat propagation speed in the Fourier law becomes unacceptable. This has inspired the work of searching for new constitutive relations. Consequently, a number of modifications of the basic assumption on the relation between the heat flux and the temperature have been made, such as: Cattaneo's law, Gurtin & Pipkin's theory, Jeffreys' law, Green & Naghdi's theory and others. The common feature of these theories is that all lead to hyperbolic differential equation and permit transmission of heat flow as thermal waves at finite speed. See [2, 11] for more details.

The Cattaneo law (it is also known as Maxwell−Cattaneo or Maxwell−Cattaneo−Vernotte or Lord−Shulman model)

$$\tau_q q_t + q + \kappa\nabla\theta = 0, \qquad (\tau_q > 0, \text{ relatively small}) \tag{1.2}$$

was proposed by Cattaneo in [1]. It is perhaps the most obvious, the most widely accepted and simplest generalization of Fourier's law that gives rise to a finite speed of propagation of heat.

When the Fourier law (1.1) is replaced by the Cattaneo law (1.2) for the heat conduction, the equations of thermoelasticity become purely hyperbolic. Indeed, from the energy balance law

$$\rho\theta_t + \varrho \operatorname{div} q = 0 \tag{1.3}$$

and (1.2), we obtain the telegraph equation

$$\rho\theta_{tt} - \frac{\varrho\kappa}{\tau_q}\Delta\theta + \frac{\rho}{\tau_q}\theta_t = 0, \tag{1.4}$$

which is a hyperbolic equation and predicts a finite signal speed limited by $\left(\varrho\kappa/(\rho\tau_q)\right)^{1/2}$.

Concerning the thermoelasticity of second sound, the interested reader is refereed to Tarabek [25], Hrusa & Tarabek [9], Hrusa & Messaoudi [8], Racke [19, 20, 21], Messaoudi & Said-Houari [13, 14] and references therein.

Note that the Cattaneo constitutive relation (1.2) can be seen as a first-order approximation of a more general constitutive relation (single−phase−lagging model; Tzou [26]),

$$q(x,t+\tau_q) = -\kappa\nabla\theta(x,t). \tag{1.5}$$

The relation (1.5) states that the temperature gradient established at a point x at time t gives rise to a heat flux vector at x at a later time $t+\tau_q$. The delay time τ_q is interpreted as the relaxation time due to the fast-transient effects of thermal inertia (or small-scale effects of heat transport in time) and is called the phase−lag of the heat flux. It has been confirmed by many experiments that the Cattaneo law generates a more accurate prediction than the classical Fourier law. However, some studies show that the Cattaneo constitutive relation has only taken account of the fast-transient effects, but not the micro-structural interactions. See Tzou [27] for more details.

In [27], Tzou proposed a new theory of heat conduction which describes the interactions between phonons and electrons on the microscopic level as retarding sources causing a delayed response on the macroscopic scale. The physical meanings and the applicability of the dual−phase−lag model have been supported by the experimental results [28]. In this theory the Fourier law is replaced by an approximation of the equation

$$q(x,t+\tau_q) = -\kappa\nabla\theta(x,t+\tau_\theta), \qquad \tau_q > 0, \quad \tau_\theta > 0, \tag{1.6}$$

where τ_q is the phase lag of the heat flux and τ_θ is the phase lag of the gradient of the temperature. According to the relation (1.6), the temperature gradient at a point x of the material at time $t+\tau_\theta$ corresponds to the heat flux density vector at x at time $t+\tau_q$. The delay time τ_θ is interpreted as being caused by the micro-structural interactions such as phonon-electron interaction or phonon scattering, and is called the phase−lag of the temperature gradient (Tzou [27]).

If the two phase lags are equal, that is $\tau_q = \tau_\theta$, then, the relation (1.6) is identical with the classical Fourier law (1.1). While in the absence of the phase lag of the temperature gradient, $\tau_\theta = 0$ and by taking the first-order approximation for q, then equation (1.6) reduces to the Cattaneo law (1.2).

A combination of the constitutive equation (1.6) with the classical energy equation, leads to an ill-posed problem (see [4]). However, if we replace the delay expressions in (1.6) by their Taylor expansions at different orders, we obtain several heat conduction theories. Indeed, in the case that we only consider the development until the first order in τ_θ and a second order in τ_q, we obtain a hyperbolic theory, which has been studied and analysed by Horgan & Quintanilla [7] and Quintanilla & Racke [17]. In particular, the authors in [17] analyzed the dual−phase−lag thermoelasticity, where the heat condition is given by (1.6) with second order approximation for q and first order approximation for θ were used. They showed that under the condition

$$\tau_\theta > \tau_q/2, \tag{1.7}$$

then solutions of the problem are generated by a semigroup of quasi-contractions. In addition, they showed that solutions of the one-dimensional problem are exponentially stable.

Choudhuri [3] proposed a three−phase− lag heat conduction model

$$q(x,t+\tau_q) = -[k\nabla\theta(x,t+\tau_\theta)+k^*\nabla\nu(x,t+\tau_\nu)], \tag{1.8}$$

where $\nu_t = \theta$ and ν is the thermal displacement gradient, k and k^* are two positive constants.

Equation (1.8) states that the temperature gradient and the thermal displacement gradient established across a material volume located at a position x at time $t+\tau_\theta$ and $t+\tau_\nu$ result in heat flux to flow at a different instant of time $t+\tau_q$. The third delay time $t+\tau_\nu$ may be interpreted, as the phase−lag of the thermal displacement gradient.

The case $\tau_\theta = \tau_q = \tau_\nu$ corresponds to the Green & Naghdi type III model. See [5, 6] for more details.

In this paper we consider two models that can be obtained by taking the Taylor series expansion of (1.8) up to the first or second-order terms in τ_θ, τ_q and τ_ν. These models have been recently investigated in [18] where an exponential stability has been shown in bounded domains.

Here we consider the Cauchy problem of these models and by using the energy method in the Fourier space, we build the appropriate Lyapunov functional for each model and show the optimal (compared to the heat kernel) decay rate of solutions provided that the coefficients satisfy appropriate assumptions. This paper is organized as follows: In Section 2, we study the stability of the solutions of the first model, while Section 3 is devoted to the analysis of the second model.

Before going on, let us introduce some notations used throughout this paper. Throughout this paper, $\|.\|_{L^q}$ and $\|.\|_{H^l}$ stand for the $L^q(\mathbb{R})$-norm $(1 \le q \le \infty)$ and the $H^l(\mathbb{R})$-norm. Also, for $\gamma \in [0,+\infty)$, we define the weighted function space $L^{1,\gamma}(\mathbb{R})$, as follows: $u \in L^{1,\gamma}(\mathbb{R})$ iff $u \in L^1(\mathbb{R})$ and

$$\|u\|_{L^{1,\gamma}} = \int_{\mathbb{R}} (1+|x|)^\gamma |u(x)| dx < +\infty.$$

Let us also denote $\hat{f} = \mathcal{F}(f)$ be the Fourier transform of f:

$$\hat{f}(\xi) = \mathcal{F}(f)(\xi) = \int_{\mathbb{R}} f(x) e^{-i\xi x} dx,$$

and let $\mathcal{F}^{-1}(f)$ be the inverse Fourier transform of f defined by the formula

$$\mathcal{F}^{-1}(f)(x) = \frac{1}{2\pi} \int_{\mathbb{R}} f(\xi) e^{i\xi x} d\xi.$$

2 The third order model

In this section, we consider the Taylor series expansion of (1.8) up to the first-order terms in τ_θ, τ_q and τ_ν. That is

$$\begin{aligned} q+\tau_q q_t &= -\left[k\left(\theta_x+\tau_\theta\theta_{xt}\right)+k^*\left(\nu_x+\tau_\nu\nu_{xt}\right)\right] \\ &= -\left[\tau_\nu^*\theta_x+k\tau_\theta\theta_{xt}+k^*\nu_x\right], \end{aligned} \tag{2.1}$$

where we have used the relation $\nu_t=\theta$ with $\tau_\nu^*=k+k^*\tau_\nu$.

Equation (2.1) together with the heat conservation law:

$$\rho c_\upsilon\theta_t(x,t)+q_x(x,t)=0, \tag{2.2}$$

leads to the following equation

$$\begin{cases} \tau_q\rho c_\upsilon\theta_{ttt}+\rho c_\upsilon\theta_{tt}-k^*\theta_{xx}-\tau_\nu^*\theta_{txx}-k\tau_\theta\theta_{ttxx}=0, \\ \theta(x,0)=\theta_0(x), \quad \theta_t(x,0)=\theta_1(x), \quad \theta_{tt}(x,0)=\theta_2(x), \end{cases} \tag{2.3}$$

where $x\in\mathbb{R}$ and $t\geq 0$. All the coefficients are positive constants. Our goal now is to show the optimal decay rate for the solution of (2.3).

2.1 The energy method in the Fourier space

This subection is devoted to the proof of the pointwise estimates of the Fourier image of the solution of (2.3). For simplicity, we write our system (2.3) as a first-order (in time) system. Indeed, we introduce the following variables:

$$y=\theta_t+\tau_q\theta_{tt}, \qquad w=\theta_x+\tau_q\theta_{tx}, \qquad z=\theta_{tx}, \tag{2.4}$$

then, the resulting system takes the form

$$\begin{cases} w_t-y_x=0, \\ \rho c_\upsilon y_t+\left(\dfrac{k\tau_\theta}{\tau_q}+k^*\tau_q-\tau_\nu^*\right)z_x-k^*w_x-\dfrac{k\tau_\theta}{\tau_q}y_{xx}=0, \qquad x\in\mathbb{R},\ t>0, \\ \tau_q z_t-y_x+z=0, \end{cases} \tag{2.5}$$

with the initial data

$$(w,y,z)(x,0)=(y_0,w_0,z_0)(x). \tag{2.6}$$

Taking the Fourier transform of system (2.5), we obtain

$$\begin{cases} \hat{w}_t-i\xi\hat{y}=0, \\ \rho c_\upsilon\hat{y}_t+i\xi\left(\dfrac{k\tau_\theta}{\tau_q}+k^*\tau_q-\tau_\nu^*\right)\hat{z}-i\xi k^*\hat{w}+\xi^2\dfrac{k\tau_\theta}{\tau_q}\hat{y}=0, \qquad \xi\in\mathbb{R},\ t>0. \\ \tau_q\hat{z}_t-i\xi\hat{y}+\hat{z}=0, \end{cases} \tag{2.7}$$

The initial data (2.6) takes the form

$$(\hat{w},\hat{y},\hat{z})(\xi,0) = (\hat{w}_0,\hat{y}_0,\hat{z}_0)(\xi). \tag{2.8}$$

Let us define the energy functional of system (2.7)-(2.8) as follows:

$$\hat{E}(\xi,t) = k^* |\hat{w}|^2 + \rho c_\upsilon |\hat{y}|^2 + \Big(\tau_q(\tau_\nu^* - k^*\tau_q) - k\tau_\theta\Big)|\hat{z}|^2. \tag{2.9}$$

Lemma 2.1. *Let $\hat{U}(\xi,t) = (\hat{w},\hat{y},\hat{z})^T(\xi,t)$ be the solution of (2.7)-(2.8), then for any $t \geq 0$, the identity*

$$\frac{d}{dt}\hat{E}(\xi,t) = \Big(\frac{k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_\nu^*\Big)|\hat{z}|^2 - \xi^2\frac{k\tau_\theta}{\tau_q}|\hat{y}|^2, \tag{2.10}$$

holds.

Proof. Multiplying the first equation in (2.7) by $k^*\bar{\hat{w}}$, the second equation by $\bar{\hat{y}}$, the third equation by $(\frac{k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_\nu^*)\bar{\hat{z}}$, adding the resulting equalities and taking the real part, then (2.10) follows. This finishes the proof of Lemma 2.1. □

To ensure that the energy is positive and non-increasing function, it is necessary to impose the assumption

$$\tau_\nu^* > \frac{k\tau_\theta}{\tau_q} + k^*\tau_q. \tag{2.11}$$

Lemma 2.2. *Assume that (2.11) is satisfied. Let $\hat{U}(\xi,t) = (\hat{w},\hat{y},\hat{z})^T(\xi,t)$ be the solution of system (2.7)-(2.8). Then for any $t \geq 0$ and $\xi \in \mathbb{R}$, we have the following pointwise estimate*

$$\left|\hat{U}(\xi,t)\right|^2 \leq Ce^{-c\rho_1(\xi)t}\left|\hat{U}(\xi,0)\right|^2, \tag{2.12}$$

where

$$\rho_1(\xi) = \frac{\xi^2}{1+\xi^2}. \tag{2.13}$$

Here C and c are two positive constants.

Proof. Multiplying the first equation in (2.7) by $-i\rho c_\upsilon\xi\bar{\hat{y}}$ and the second equation by $i\xi\bar{\hat{w}}$, adding the resulting equalities and taking the real part, we have

$$\frac{d}{dt}\mathrm{Re}(i\rho c_\upsilon\xi\bar{\hat{w}}\hat{y}) + \xi^2(k^*|\hat{w}|^2 - \rho c_\upsilon|\hat{y}|^2) = \mathrm{Re}\left(\xi^2\Big(\frac{k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_\nu^*\Big)\hat{z}\bar{\hat{w}}\right) - \mathrm{Re}\left(i\xi^3\frac{k\tau_\theta}{\tau_q}\hat{y}\bar{\hat{w}}\right)$$

Applying Young's inequality, we obtain, for any $\varepsilon > 0$

$$\frac{d}{dt}F(\xi,t) + (k^* - \varepsilon)\xi^2|\hat{w}|^2 \leq C(\varepsilon)(\xi^2 + \xi^4)|\hat{y}|^2 + C(\varepsilon)\xi^2|\hat{z}|^2, \tag{2.14}$$

where

$$F(\xi,t) = \mathrm{Re}(i\rho c_\upsilon\xi\bar{\hat{w}}\hat{y}). \tag{2.15}$$

We define the Lyapunov functional $L(\xi,t)$ as:

$$L(\xi,t) = \frac{1}{1+\xi^2}F(\xi,t) + N\hat{E}(\xi,t), \tag{2.16}$$

where N is a large positive constant that will be chosen later. Taking the derivative of $L(\xi,t)$ with respect to t, we obtain

$$\frac{d}{dt}L(\xi,t)+\frac{\xi^2}{1+\xi^2}(k^*-\varepsilon)|\hat{w}|^2+\left\{N\left(-\frac{k\tau_\theta}{\tau_q}-k^*\tau_q+\tau_\nu^*\right)-C(\varepsilon)\right\}|\hat{z}|^2$$
$$+\xi^2\left(N\frac{k\tau_\theta}{\tau_q}-C(\varepsilon)\right)|\hat{y}|^2\leq 0,\qquad \forall t\geq 0. \tag{2.17}$$

Keeping in mind (2.11), choosing ε small enough such that $\varepsilon<k^*$ and N large enough such that

$$N>\max\left\{\frac{C(\epsilon)}{(\tau_\nu^*-\frac{k\tau_\theta}{\tau_q}-k^*\tau_q)},\frac{C(\epsilon)\tau_q}{k\tau_\theta}\right\}.$$

With these choices, (2.17) takes the from

$$\frac{d}{dt}L(\xi,t)+\eta\Phi(\xi,t)\leq 0,\qquad \forall t\geq 0, \tag{2.18}$$

where

$$\Phi(\xi,t)=\frac{\xi^2}{1+\xi^2}|\hat{w}|^2+\xi^2|\hat{y}|^2+|\hat{z}|^2 \tag{2.19}$$

and η is a positive constant . On the other hand, it is not hard to see that for N large enough there exist two positive constants $\delta_1$and δ_2 such that for all $t\geq 0$, we have

$$\delta_1\hat{E}(\xi,t)\leq L(\xi,t)\leq\delta_2\hat{E}(\xi,t). \tag{2.20}$$

Also, from (2.19) and (2.9), we deduce that

$$\Phi(\xi,t)\geq c\frac{\xi^2}{1+\xi^2}\hat{E}(\xi,t),\qquad \forall t\geq 0. \tag{2.21}$$

Now, combining (2.18), (2.20) and (2.21), we have

$$\frac{dL(\xi,t)}{dt}\leq-\frac{\eta c}{\delta_2}\frac{\xi^2}{(1+\xi^2)}L(\xi,t),\qquad \forall t\geq 0. \tag{2.22}$$

Integrating (2.22) with respect to t and exploiting once again (2.20), we get (2.12). This completes the proof of Lemma 2.2. □

2.2 Decay estimates

In this subsection, we prove the decay estimates of the L^2-norm for the solution of (2.5)-(2.6). Thus, we have

Theorem 2.3. *(L^1-initial data) Let s be a nonnegative integer and assume that $U_0=(w_0,y_0,z_0)^T\in H^s(\mathbb{R})\cap L^1(\mathbb{R})$. Assume that (2.11) holds. Then the solution $U=(w,y,z)^T$ of system (2.5)-(2.6) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_{L^2}\leq C(1+t)^{-1/4-k/2}\|U_0\|_{L^1}+Ce^{-ct}\left\|\partial_x^k U_0\right\|_{L^2}, \tag{2.23}$$

for $k\leq s$ be a nonnegative integer and C and c are two positive constants.

Proof. The proof is essentially based on the pointwise estimate in Lemma 2.2. Indeed, applying the Plancherel theorem, we may write

$$\begin{aligned}
\left\|\partial_x^k U(t)\right\|_{L^2}^2 &= \int_{\mathbb{R}} |\xi|^{2k} |U(\xi,t)|^2 \, d\xi \\
&\leq C \int_{\mathbb{R}} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left|\hat{U}(\xi,0)\right|^2 d\xi \\
&\leq C \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left|\hat{U}(\xi,0)\right|^2 d\xi + C \int_{|\xi|\geq 1} |\xi|^{2k} e^{-c\rho_1(\xi)t} \left|\hat{U}(\xi,0)\right|^2 d\xi \\
&= L_1 + L_2. \qquad (2.24)
\end{aligned}$$

The integral here is divided into two parts: the low-frequency part ($|\xi| \leq 1$) and the high-frequency part ($|\xi| \geq 1$). As $\rho_1(\xi) \geq \frac{1}{2}\xi^2$ for $|\xi| \leq 1$, then we have for the low-frequency part that

$$L_1 \leq C \left\|\hat{U}_0\right\|_{L^\infty}^2 \int_{|\xi|\leq 1} |\xi|^{2k} e^{-c\xi^2 t} d\xi. \qquad (2.25)$$

Using the inequality

$$\int_0^1 |\xi|^\sigma e^{-c\xi^2 t} d\xi \leq C (1+t)^{-\frac{(\sigma+1)}{2}}, \qquad (2.26)$$

we deduce from (2.25) that

$$L_1 \leq C (1+t)^{-\frac{1}{2}-k} \|U_0\|_{L^1}^2. \qquad (2.27)$$

For the high-frequency part, L_2 we have $\rho(\xi) \geq \frac{1}{2}$, and therefore

$$\begin{aligned}
L_2 &\leq Ce^{-ct} \int_{|\xi|\geq 1} |\xi|^{2k} \left|\hat{U}(\xi,0)\right|^2 d\xi \\
&\leq Ce^{-ct} \left\|\partial_x^k U_0\right\|_2^2. \qquad (2.28)
\end{aligned}$$

Inserting (2.27) and (2.28) into (2.24), we obtain the estimate (2.23). □

In the next theorem, we show that the decay rate given in Theorem 2.3 can be improved for initial data in some weighted spaces of L^1 and with zero total mass.

Theorem 2.4. *($L^{1,\gamma}$-initial data) Let $\gamma \in [0,1]$. Let s be a nonnegative integer and assume $U_0 = (w_0, y_0, z_0)^T \in H^s(\mathbb{R}) \cap L^{1,\gamma}(\mathbb{R})$ such that $\int_{\mathbb{R}} U_0(x)dx = 0$. Assume that (2.11) holds. Then the solution $U = (w,y,z)^T$ of system (2.5)-(2.6) satisfies the following decay estimates:*

$$\left\|\partial_x^k U(t)\right\|_{L^2} \leq C(1+t)^{-1/4-(k+\gamma)/2} \|U_0\|_{L^{1,\gamma}} + Ce^{-ct} \left\|\partial_x^k U_0\right\|_{L^2} \qquad (2.29)$$

for $k \leq s$ and C and c are two positive constants.

Proof. It is clear that from [10] and for $0 \leq \gamma \leq 1$, and if $\int_{\mathbb{R}} U_0(x)dx = 0$, then we deduce

$$\left|\hat{U}(\xi,0)\right| \leq C_\gamma |\xi|^\gamma \|U_0\|_{L^{1,\gamma}} \tag{2.30}$$

with C_γ is a positive constant depending on γ.

Therefore, the integral L_1 in (2.24) can be estimated as follows:

$$L_1 \leq C\|U_0\|^2_{L^{1,\gamma}} \int_{|\xi|\leq 1} |\xi|^{2(k+\gamma)} e^{-c\xi^2 t} d\xi. \tag{2.31}$$

Thus, applying (2.26), we find

$$L_1 \leq C(1+t)^{-1/2-(k+\gamma)} \|U_0\|^2_{L^{1,\gamma}}. \tag{2.32}$$

Consequently, (2.32) together with (2.28) lead to the desired result. □

3 The Fourth order model

Retaining terms of the order of τ_q^2 in the Taylor expansion of (1.8), we obtain

$$q + \tau_q q_t + \tau_q^2 q_{tt} = -\left[\tau_\nu^* \theta_x + k\tau_\theta \theta_{xt} + k^* \nu_x\right]. \tag{3.1}$$

Taking the divergence of both sides in (3.1) and combining the result with (2.2), we get

$$\begin{cases} \tau_q \rho c_\upsilon \theta_{ttt} + \rho c_\upsilon \theta_{tt} + \dfrac{\tau_q^2}{2} \rho c_\upsilon \theta_{tttt} - k^* \theta_{xx} - \tau_\nu^* \theta_{txx} - k\tau_\theta \theta_{ttxx} = 0, \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \quad \theta_{tt}(x,0) = \theta_2(x), \quad \theta_{ttt}(x,0) = \theta_3(x), \end{cases} \tag{3.2}$$

where $x \in \mathbb{R}$ and $t \geq 0$. All the coefficients are positive constants. We introduce the following variables:

$$y = \theta_t + \tau_q \theta_{tt} + \frac{\tau_q^2}{2}\theta_{ttt}, \quad w = \theta_x + \tau_q \theta_{xt} + \frac{\tau_q^2}{2}\theta_{xtt}, \quad z = \theta_{xt} + \frac{\tau_q}{2}\theta_{xtt}, \quad h = \theta_{xt}, \quad \varphi = \theta_{xtt}.$$

Then, the resulting system takes the from

$$\begin{cases} w_t - y_x = 0, \\ \tau_q z_t - y_x + h = 0, \\ \rho c_\upsilon y_t - k^* w_x - \dfrac{2k\tau_\theta}{\tau_q} z_x + \left(\dfrac{2k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_\nu^*\right) h_x + \dfrac{k^*\tau_q^2}{2}\varphi_x = 0, \quad x \in \mathbb{R}, \quad t > 0, \\ h_t - \varphi = 0, \\ \dfrac{\tau_q^2}{2}\varphi_t + \tau_q \varphi + h - y_x = 0, \end{cases} \tag{3.3}$$

with the initial data

$$(w,z,y,h,\varphi)(x,0) = (w_0,z_0,y_0,h_0,\varphi_0)(x). \tag{3.4}$$

Taking the Fourier transform of system (3.3), we obtain

$$\begin{cases} \hat{w}_t - i\xi\hat{y} = 0, \\ \tau_q\hat{z}_t - i\xi\hat{y} + \hat{h} = 0, \\ \rho c_v\hat{y}_t - i\xi k^*\hat{w} - i\xi\dfrac{2k\tau_\theta}{\tau_q}\hat{z} + i\xi\left(\dfrac{2k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_v^*\right)\hat{h} + i\xi\dfrac{k^*\tau_q^2}{2}\hat{\varphi} = 0, \quad \xi \in \mathbb{R}, t > 0 \\ \hat{h}_t - \hat{\varphi} = 0, \\ \dfrac{\tau_q^2}{2}\hat{\varphi}_t + \tau_q\hat{\varphi} + \hat{h} - i\xi\hat{y} = 0, \end{cases} \tag{3.5}$$

The initial data (3.4) takes from

$$\left(\hat{w},\hat{z},\hat{y},\hat{h},\hat{\varphi}\right)(\xi,0) = \left(\hat{w}_0,\hat{z}_0,\hat{y}_0,\hat{h}_0,\hat{\varphi}_0\right)(\xi). \tag{3.6}$$

Let us define the energy functional of system (3.5)-(3.6) as follows:

$$\hat{E}(\xi,t) = \frac{1}{2}\left\{k^*|\hat{w}|^2 + \tau_q\left(\tau_v^* - k^*\tau_q\right)|\hat{z}|^2 + \rho c_v|\hat{y}|^2 + \left(k\tau_\theta - k^*\frac{\tau_q^2}{2}\right)\left|\hat{h}\right|^2 + \frac{\tau_q^2}{2}\left(k\tau_\theta - \frac{\tau_q\tau_v^*}{2}\right)|\hat{\varphi}|^2\right\}. \tag{3.7}$$

Lemma 3.1. *Let $\hat{V}(\xi,t) = \left(\hat{w},\hat{z},\hat{y},\hat{h},\hat{\varphi}\right)^T(\xi,t)$ be the solution of (3.5)-(3.6), then for any $t \geq 0$, the identity*

$$\frac{d}{dt}\hat{E}(\xi,t) = -\tau_q\left(k\tau_\theta - \frac{\tau_q\tau_v^*}{2}\right)|\hat{\varphi}|^2 - \left(\tau_v^* - k^*\tau_q\right)\left|\hat{h}\right|^2, \tag{3.8}$$

holds.

Proof. Multiplying the first equation in (3.5) by $k^*\bar{\hat{w}}$, the second equation by $\left(\tau_v^* - k^*\tau_q\right)\bar{\hat{z}}$, the third equation by $\bar{\hat{y}}$, the fourth equation by $\left(k\tau_\theta - k^*\frac{\tau_q^2}{2}\right)\bar{\hat{h}}$ and the last equation by $\left(k\tau_\theta - \frac{\tau_q\tau_v^*}{2}\right)\bar{\hat{\varphi}}$ adding the results and taking the real part to get

$$\begin{aligned} \frac{d}{dt}\hat{E}(\xi,t) = {} & \mathrm{Re}\left\{\left(\frac{2k\tau_\theta}{\tau_q} - \tau_v^* + k^*\tau_q\right)i\xi\bar{\hat{y}}\hat{z} - \left(\frac{2k\tau_\theta}{\tau_q} + k^*\tau_q - \tau_v^*\right)i\xi\bar{\hat{y}}\hat{h}\right\} \\ & -\mathrm{Re}\left(\left(\tau_v^* - k^*\tau_q\right)\bar{\hat{z}}\hat{h}\right) \\ & +\mathrm{Re}\left\{\left(\frac{\tau_q\tau_v^*}{2} - k^*\frac{\tau_q^2}{2}\right)\hat{\varphi}\bar{\hat{h}}\right\} - \tau_q\left(k\tau_\theta - \frac{\tau_q\tau_v^*}{2}\right)|\hat{\varphi}|^2 \\ & +\mathrm{Re}\left\{\left(k\tau_\theta + \frac{k^*\tau_q}{2} - \frac{\tau_q\tau_v^*}{2}\right)i\xi\hat{y}\bar{\hat{\varphi}}\right\}. \end{aligned} \tag{3.9}$$

Now, we have

$$\begin{aligned} \mathrm{Re}\left(i\xi\bar{\hat{y}}\hat{z}\right) &= \mathrm{Re}\left\{i\xi\left(\bar{\hat{u}}_t + \tau_q\bar{\hat{u}}_{tt} + \frac{\tau_q^2}{2}\bar{\hat{u}}_{ttt}\right)i\xi\left(\hat{u}_t + \frac{\tau_q}{2}\hat{u}_{tt}\right)\right\} \\ &= -\xi^2\left\{|\hat{u}_t|^2 + \tau_q\mathrm{Re}\left(\bar{\hat{u}}_{tt}\hat{u}_t\right) + \frac{\tau_q^2}{2}\mathrm{Re}\left(\bar{\hat{u}}_{ttt}\hat{u}_t\right) + \frac{\tau_q}{2}\mathrm{Re}\left(\bar{\hat{u}}_t\hat{u}_{tt}\right) + \frac{\tau_q^2}{2}|\hat{u}_{tt}| + \frac{\tau_q^3}{4}\mathrm{Re}\left(\bar{\hat{u}}_{ttt}\hat{u}_{tt}\right)\right\}, \end{aligned}$$

and

$$\begin{aligned}\mathrm{Re}\left(i\xi\bar{\hat{y}}\hat{h}\right) &= -\xi^2\mathrm{Re}\left\{\left(\bar{\hat{u}}_t+\tau_q\bar{\hat{u}}_{tt}+\frac{\tau_q^2}{2}\bar{\hat{u}}_{ttt}\right)\hat{u}_t\right\}\\ &= -\xi^2\left\{|\hat{u}_t|^2+\tau_q\mathrm{Re}\left(\bar{\hat{u}}_{tt}\hat{u}_t\right)+\frac{\tau_q^2}{2}\mathrm{Re}\left(\bar{\hat{u}}_{ttt}\hat{u}_t\right)\right\}.\end{aligned}$$

Also,

$$\begin{aligned}\mathrm{Re}\left(\bar{\hat{z}}\hat{h}\right) &= \xi^2\mathrm{Re}\left\{\left(\bar{\hat{u}}_t+\frac{\tau_q}{2}\bar{\hat{u}}_{tt}\right)\hat{u}_t\right\}\\ &= \xi^2\left\{|\hat{u}_t|^2+\frac{\tau_q}{2}\mathrm{Re}\left(\bar{\hat{u}}_{tt}\hat{u}_t\right)\right\}.\end{aligned}$$

Furthermore,

$$\mathrm{Re}\left(\hat{\varphi}\bar{\hat{h}}\right)=\xi^2\mathrm{Re}\left(\bar{\hat{u}}_t\hat{u}_{tt}\right).$$

Finally,

$$\begin{aligned}\mathrm{Re}\left(i\xi\hat{y}\bar{\hat{\varphi}}\right) &= \xi^2\mathrm{Re}\left\{\left(\hat{u}_t+\tau_q\hat{u}_{tt}+\frac{\tau_q^2}{2}\hat{u}_{ttt}\right)\bar{\hat{u}}_{tt}\right\}\\ &= \xi^2\left\{\mathrm{Re}\left(\hat{u}_t\bar{\hat{u}}_{tt}\right)+\tau_q|\hat{u}_{tt}|^2+\frac{\tau_q^2}{2}\mathrm{Re}\left(\hat{u}_{ttt}\bar{\hat{u}}_{tt}\right)\right\}.\end{aligned}$$

Inserting the above identities into (3.9), then (3.8) is fulfilled. □

To ensure that the energy is positive and non-increasing function, it is necessary to impose the assumption

$$\tau_\nu^*>k^*\tau_q \qquad \text{and} \qquad k\tau_\theta>\frac{\tau_q\tau_\nu^*}{2}\ . \tag{3.10}$$

Lemma 3.2. *Let $\hat{V}(\xi,t)=\left(\hat{w},\hat{z},\hat{y},\hat{h},\hat{\varphi}\right)^T(\xi,t)$ be the solution of system (3.5)-(3.6). Assume that (3.10) holds. Then for any $t\geq 0$ and $\xi\in\mathbb{R}$, we have the following pointwise estimate*

$$\left|\hat{V}(\xi,t)\right|^2\leq Ce^{-\tilde{c}\rho_2(\xi)t}\left|\hat{V}(\xi,0)\right|^2, \tag{3.11}$$

where

$$\rho_2(\xi)=\frac{\xi^2}{1+\xi^2}. \tag{3.12}$$

Here C and $\tilde{c}$ are two positive constants.

Proof. Multiplying the third equation in (3.5) by $i\xi\bar{\hat{w}}$ and the first equation by $-i\xi\rho c_\upsilon\bar{\hat{y}}$, adding the resulting equalities and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\left(i\xi\rho c_\upsilon\hat{y}\bar{\hat{w}}\right)-\rho c_\upsilon\xi^2|\hat{y}|^2+\xi^2k^*|\hat{w}|^2\\ =\ &-\mathrm{Re}\left(\frac{2k\tau_\theta}{\tau_q}\xi^2\hat{z}\bar{\hat{w}}\right)+\mathrm{Re}\left(\frac{2k\tau_\theta}{\tau_q}\xi^2\hat{h}\bar{\hat{w}}\right)+\mathrm{Re}\left(k^*\tau_q\xi^2\hat{h}\bar{\hat{w}}\right)\\ &-\mathrm{Re}\left(\tau_\nu^*\xi^2\hat{h}\bar{\hat{w}}\right)+\mathrm{Re}\left(\frac{k^*\tau_q^2}{2}\xi^2\hat{\varphi}\bar{\hat{w}}\right).\end{aligned} \tag{3.13}$$

Multiplying the fifth equation in (3.5) by $i\xi\rho c_\upsilon\bar{\hat{y}}$ and the third equation by $-i\xi\frac{\tau_q^2}{2}\bar{\hat{\varphi}}$, adding these equalities and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\left(i\xi\frac{\tau_q^2}{2}\rho c_\upsilon\bar{\hat{y}}\hat{\varphi}\right)+\mathrm{Re}\left(i\xi\rho\tau_q c_\upsilon\bar{\hat{y}}\hat{\varphi}\right)+\mathrm{Re}\left(i\xi\rho c_\upsilon\bar{\hat{y}}\hat{h}\right)+\rho c_\upsilon\xi^2|\hat{y}|^2\\&-\mathrm{Re}\left(k^*\frac{\tau_q^2}{2}\xi^2\bar{\hat{\varphi}}\hat{w}\right)-\mathrm{Re}\left(k\tau_\theta\tau_q\xi^2\bar{\hat{\varphi}}\hat{z}\right)+\left(k\tau_\theta\tau_q+\frac{\tau_q^3}{2}k^*-\frac{\tau_q^2}{2}\tau_\nu^*\right)\xi^2\bar{\hat{\varphi}}\hat{h}\\&+\frac{k^*\tau_q^4}{4}\xi^2|\hat{\varphi}|^2=0.\end{aligned}\tag{3.14}$$

Computing $\tau_\nu^*(3.14)+k^*\tau_q(3.13)$, we have

$$\begin{aligned}&\frac{d}{dt}\mathscr{F}(\xi,t)+\rho c_\upsilon(\tau_\nu^*-k^*\tau_q)\xi^2|\hat{y}|^2+\xi^2k^{*2}\tau_q|\hat{w}|^2+\tau_\nu^*\frac{k^*\tau_q^4}{4}\xi^2|\hat{\varphi}|^2\\&+\mathrm{Re}\left(2\xi^2k\tau_\theta k^*\hat{z}\bar{\hat{w}}\right)-\mathrm{Re}\left(2\xi^2k\tau_\theta k^*\hat{h}\bar{\hat{w}}\right)-\mathrm{Re}\left(\xi^2k^{*2}\tau_q^2\hat{h}\bar{\hat{w}}\right)\\&+\mathrm{Re}\left(\xi^2k^*\tau_q\tau_\nu^*\hat{h}\bar{\hat{w}}\right)-\mathrm{Re}\left(\xi^2\frac{k^{*2}\tau_q^3}{2}\hat{\varphi}\bar{\hat{w}}\right)\\&+\mathrm{Re}\left(i\xi\tau_\nu^*\rho\tau_q c_\upsilon\bar{\hat{y}}\hat{\varphi}\right)+\mathrm{Re}\left(i\xi\tau_\nu^*\rho c_\upsilon\bar{\hat{y}}\hat{h}\right)-\mathrm{Re}\left(\xi^2\tau_\nu^*k^*\frac{\tau_q^2}{2}\bar{\hat{\varphi}}\hat{w}\right)\\&-\mathrm{Re}\left(\xi^2\tau_\nu^*k\tau_\theta\tau_q\bar{\hat{\varphi}}\hat{z}\right)+\left(k\tau_\theta\tau_\nu^*\tau_q+\frac{\tau_q^3}{2}\tau_\nu^*k^*-\frac{\tau_q^2}{2}\tau_\nu^{*2}\right)\mathrm{Re}(\xi^2\bar{\hat{\varphi}}\hat{h})\\=\;&0,\end{aligned}\tag{3.15}$$

where

$$\mathscr{F}(\xi,t)=\mathrm{Re}\left(i\xi\frac{\tau_q^2}{2}\rho\tau_\nu^*c_\upsilon\bar{\hat{y}}\hat{\varphi}\right)+\mathrm{Re}\left(i\xi\rho k^*\tau_q c_\upsilon\hat{y}\bar{\hat{w}}\right).\tag{3.16}$$

Using the fact that $\hat{z}=\hat{h}+\frac{\tau_q}{2}\hat{\varphi}$, we have

$$\begin{aligned}&\frac{d}{dt}\mathscr{F}(\xi,t)+\rho c_\upsilon(\tau_\nu^*-k^*\tau_q)\xi^2|\hat{y}|^2+\xi^2k^{*2}\tau_q|\hat{w}|^2+\left(\tau_\nu^*\frac{k^*\tau_q^4}{4}\xi^2-\tau_\nu^*k\tau_\theta\frac{\tau_q^2}{2}\right)|\hat{\varphi}|^2\\&+\mathrm{Re}\left(\xi^2k\tau_q\tau_\theta k^*\hat{\varphi}\bar{\hat{w}}\right)-\mathrm{Re}\left(\xi^2k^{*2}\tau_q^2\hat{h}\bar{\hat{w}}\right)\\&+\mathrm{Re}\left(\xi^2k^*\tau_q\tau_\nu^*\hat{h}\bar{\hat{w}}\right)-\mathrm{Re}\left(\xi^2\frac{k^{*2}\tau_q^3}{2}\hat{\varphi}\bar{\hat{w}}\right)\\&+\mathrm{Re}\left(i\xi\tau_\nu^*\rho\tau_q c_\upsilon\bar{\hat{y}}\hat{\varphi}\right)+\mathrm{Re}\left(i\xi\tau_\nu^*\rho c_\upsilon\bar{\hat{y}}\hat{h}\right)-\mathrm{Re}\left(\xi^2\tau_\nu^*k^*\frac{\tau_q^2}{2}\bar{\hat{\varphi}}\hat{w}\right)\\&+\mathrm{Re}\left(\frac{\tau_q^3}{2}\tau_\nu^*k^*-\frac{\tau_q^2}{2}\tau_\nu^{*2}\right)\xi^2\bar{\hat{\varphi}}\hat{h}=0.\end{aligned}\tag{3.17}$$

Applying Young's inequality, we get, for any $\varepsilon_1>0$,

$$\begin{aligned}&\frac{d}{dt}\mathscr{F}(\xi,t)+\left(\rho c_\upsilon(\tau_\nu^*-k^*\tau_q)-\varepsilon_1\right)\xi^2|\hat{y}|^2+(k^{*2}\tau_q-\varepsilon_1)\xi^2|\hat{w}|^2\\\le\;&c(\varepsilon_1)(1+\xi^2)(|\hat{h}|^2+|\hat{\varphi}|^2).\end{aligned}\tag{3.18}$$

Next, multiplying the third equation in (3.5) by $i\xi\tau_q\bar{\hat{z}}$ and the second equation by $-i\xi\rho c_\upsilon\bar{\hat{y}}$, summing up the results and taking the real part, we have

$$\begin{aligned}&\frac{d}{dt}\mathrm{Re}\left(i\xi\tau_q\rho c_\upsilon\hat{y}\bar{\hat{z}}\right)+2k\tau_\theta\xi^2|\hat{z}|^2-\rho c_\upsilon\xi^2|\hat{y}|^2+\mathrm{Re}\left(\xi^2\tau_q k^*\bar{\hat{z}}\hat{w}\right)\\&-\mathrm{Re}(\xi^2\frac{k^*\tau_q^3}{2}\hat{\varphi}\bar{\hat{z}})-\mathrm{Re}\left(i\xi\rho c_\upsilon\bar{\hat{y}}\hat{h}\right)-\mathrm{Re}(2\xi^2k\tau_\theta\bar{\hat{z}}\hat{h})\\&-\mathrm{Re}(\xi^2k^*\tau_q^2\bar{\hat{z}}\hat{h})+\mathrm{Re}(\xi^2\tau_\nu^*\tau_q\bar{\hat{z}}\hat{h})=0.\end{aligned}\tag{3.19}$$

Applying Young's inequality, we find for any $\varepsilon_2>0$,

$$\begin{aligned}\frac{d}{dt}\mathscr{G}(\xi,t)+(2k\tau_\theta-\varepsilon_2)\xi^2|\hat{z}|^2\quad\leq\quad & c(\varepsilon_2)(1+\xi^2)\left|\hat{h}\right|^2+c(\varepsilon_2)\xi^2|\hat{y}|^2\\&+c(\varepsilon_2)\xi^2|\hat{w}|^2+c(\varepsilon_2)\xi^2|\hat{\varphi}|^2,\end{aligned}\tag{3.20}$$

where

$$\mathscr{G}(\xi,t)=\mathrm{Re}\left(i\xi\tau_q\rho c_\upsilon\hat{y}\bar{\hat{z}}\right).\tag{3.21}$$

Now, we define the Lyapunov functional $\Phi(\xi,t)$ as:

$$\Psi(\xi,t)=N_1(1+\xi^2)\hat{\mathscr{E}}(\xi,t)+\lambda\mathscr{F}(\xi,t)+\mathscr{G}(\xi,t).\tag{3.22}$$

where N_1 and λ are two positive constants that have to be chosen later.

Taking the derivate of $\Phi(\xi,t)$ with respect to t, we obtain

$$\begin{aligned}&\frac{d}{dt}\Psi(\xi,t)+\left(\lambda(\rho c_\upsilon(\tau_\nu^*-k^*\tau_q)-\varepsilon_1))-c(\varepsilon_2)\right)\xi^2|\hat{y}|^2+\left(\lambda(k^{*2}\tau_q-\varepsilon_1)-c(\varepsilon_2)\right)\xi^2|\hat{w}|^2\\&+\left((2k\tau_\theta-\varepsilon_2)\right)\xi^2|\hat{z}|^2+\left(N_1\left(\tau_\nu^*-k^*\tau_q\right)-c(\lambda,\varepsilon_1,\varepsilon_2)\right)(1+\xi^2)\left|\hat{h}\right|^2\\&+\left(N_1\tau_q\left(k\tau_\theta-\frac{\tau_q\tau_\nu^*}{2}\right)-c(\lambda,\varepsilon_1,\varepsilon_2)\right)(1+\xi^2)|\hat{\varphi}|^2\\\leq\quad&0.\end{aligned}\tag{3.23}$$

Our goal now is to choose the constants $\varepsilon_1,\varepsilon_2,\lambda$ and N_1 so that all the coefficients in (3.23) are positive. Indeed, keeping in mind the assumption (3.10) and fixing ε_1 and ε_2 small enough such that

$$\varepsilon_1<\min\left(\tau_q k^{*2},(\tau_\nu^*-k^*\tau_q)\rho c_\upsilon\right),$$

and $\varepsilon_2<2k\tau_\theta$. Once ε_1 and ε_2 are fixed, then we choose λ large enough such that

$$\begin{cases}\lambda(\rho c_\upsilon(\tau_\nu^*-k^*\tau_q)-\varepsilon_1)-c(\varepsilon_2)>0,\\\lambda(k^{*2}\tau_q-\varepsilon_1)-c(\varepsilon_2)>0.\end{cases}$$

Finally, we choose N_1 large enough such that

$$N_1>\max\left(c(\lambda,\varepsilon_1,\varepsilon_2)/\tau_q\left(k\tau_\theta-\frac{\tau_q\tau_\nu^*}{2}\right),c(\lambda,\varepsilon_1,\varepsilon_2)/\left(\tau_\nu^*-k^*\tau_q\right)\right).$$

Consequently, the estimate (3.22) takes the form

$$\frac{d}{dt}\Psi(\xi,t)+\eta\mathscr{W}(\xi,t)\leq0,\qquad\forall t\geq0,\tag{3.24}$$

where η is a positive constant and

$$\mathscr{W}(\xi,t)=\xi^2\left(|\hat{w}|^2+|\hat{z}|^2+|\hat{y}|^2\right)+(1+\xi^2)\left(|\hat{h}|^2+|\hat{\varphi}|^2\right). \tag{3.25}$$

On the other hand, it is not hard to see that for N large enough there exist two positive constants $\beta_1$and $\beta_2$such that for all $t \geq 0$, we have

$$\beta_1(1+\xi^2)\hat{\mathscr{E}}(\xi,t) \leq \Psi(\xi,t) \leq \beta_2(1+\xi^2)\hat{\mathscr{E}}(\xi,t), \qquad \forall t \geq 0. \tag{3.26}$$

Also, from (3.25) and (3.7), we deduce that

$$\mathscr{W}(\xi,t) \geq \eta_1\xi^2\hat{\mathscr{E}}(\xi,t), \qquad \forall t \geq 0, \tag{3.27}$$

for some positive constants η_1. Now, combining (3.24), (3.26) and (3.27), we have

$$\frac{d\Psi(\xi,t)}{dt} \leq -\eta_2\frac{\xi^2}{1+\xi^2}\Psi(\xi,t), \qquad \forall t \geq 0, \tag{3.28}$$

where η_2 is a positive constant. Integrating (3.28) with respect to t and exploiting once again (3.26), we get (3.12). This completes the proof of Lemma 3.2. ◻

3.1 Decay estimates

In this subsection, we prove the L^2 decay estimates of the solution of system (3.3)-(3.4). Thus we have:

Theorem 3.3. *(L^1-initial data) Let s be a nonnegative integer and assume $V_0=(w_0,z_0,y_0,h_0,\varphi_0)^T \in H^s(\mathbb{R})\cap L^1(\mathbb{R})$. Assume that (3.10) holds. Then the solution $V=(w,z,y,h,\varphi)^T$ of system (3.3)-(3.4) satisfies the following decay estimates:*

$$\left\|\partial_x^k V(t)\right\|_{L^2} \leq C(1+t)^{-1/4-k/2}\|V_0\|_{L^1}+Ce^{-ct}\left\|\partial_x^k V_0\right\|_2. \tag{3.29}$$

where $k \leq s$, is a nonnegative integer. Here C and c are two positive constants.

The proof of Theorem 3.3 is quite similar to the one of Theorem 2.3. We omit the details.

Similarly to Theorem 2.4, we can also further improve the above decay result as follows:

Theorem 3.4. *($L^{1,\gamma}$-initial data) Let $\gamma \in [0,1]$. Let s be a nonnegative integer and assume $V_0=(w_0,z_0,y_0,h_0,\varphi_0)^T \in H^s(\mathbb{R})\cap L^{1,\gamma}(\mathbb{R})$ such that $\int_{\mathbb{R}} V_0(x)dx=0$. Assume that (3.10) holds. Then the solution $V=(w,z,y,h,\varphi)^T$of system (3.3)-(3.4) satisfies the following decay estimates:*

$$\left\|\partial_x^k V(t)\right\|_{L^2} \leq C(1+t)^{-1/4-(k+\gamma)/2}\|V_0\|_{L^{1,\gamma}}+Ce^{-ct}\left\|\partial_x^k V_0\right\|_{L^2}. \tag{3.30}$$

for $k \leq s$. Here C and c are two positive constants.

The proof of Theorem 3.4 can be done exactly as the one of Theorem 2.4.

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