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Asymptotic Behavior of Higher-Order Neutral Advanced Difference Equations

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Abstract

In this paper, we study the asymptotic behavior of neutral advanced difference equations of the form

$$\Delta^m [x(n) + cx(n+a)] + p(n)x(\sigma(n)) = 0, \quad m \in \mathbb{N}, \quad n \geq 0,$$

where $c \in \mathbb{R}$, $\mathbb{N} \ni a \geq 2$, $(\sigma(n))$ is a sequence of positive integers such that $(\sigma(n)) \geq n+2$ for all $n \geq 0$, $(p(n))_{n\geq 0}$ is a sequence of real numbers, Δ denotes the forward difference operator $\Delta x(n) = x(n+1) - x(n)$, and Δ^j denotes the j^{th} forward difference operator $\Delta^j(x(n)) = \Delta\left(\Delta^{j-1}(x(n))\right)$ for $j = 2, 3, ..., m$. Examples illustrating the results are also given.

AMS Subject Classification: 39A10; 39A21.

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1 Introduction

In the present paper, we study the asymptotic behavior of a first-order neutral advanced difference equation (1^{st}-order NADE) of the form

$$\Delta[x(n) + cx(n+a)] + p(n)x(\sigma(n)) = 0, \quad n \geq 0, \tag{E$_1$}$$

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where $c \in \mathbb{R}$, $\mathbb{N} \ni a \geq 2$, $(\sigma(n))$ is a sequence of positive integers such that $(\sigma(n)) \geq n+2$ for all $n \geq 0$, and $(p(n))_{n\geq 0}$ is a sequence of real numbers.

Next, we study the asymptotic behavior of a higher-order neutral advanced difference equation (m^{th}-order NADE) of the form

$$\Delta^m [x(n) + cx(n+a)] + p(n)x(\sigma(n)) = 0, \quad \mathbb{N} \ni m \geq 2, \quad n \geq 0. \tag{E_m}$$

For the general theory of difference equations the reader is referred to the monographs [1, 15, 18].

By a *solution* of (E_1) or (E_m), we mean a sequence of real numbers $(x(n))_{n\geq 0}$ which satisfies (E_1) or (E_m) for all $n \geq 0$.

A solution $(x(n))_{n\geq 0}$ of (E_1) or (E_m) is called *oscillatory,* if the terms $x(n)$ of the sequence are neither eventually positive nor eventually negative. Otherwise, the solution is said to be *nonoscillatory*.

In the last few decades, several papers have been written on the asymptotic and oscillatory behavior of neutral difference equations, significantly contributing to the research and advancing our knowledge and insight in this subject. See, for example [2]-[14], [16]-[17], [19]-[28] and the references cited therein. However, far less work has been carried out on neutral advanced difference equations. That is our aim in this paper. We study the general form of neutral advanced difference equations that are formally described by Eq. (E_1) or (E_m) and postulate theorems and corollaries on the asymptotic behavior of the solutions of these equations.

We examine two cases, according to whether the coefficients $p(n)$ are all non-negative (Case 1) or are all non-positive (Case 2). Examples illustrating the results are also given.

2 Some preliminaries

The following lemmas provide us with some useful tools, for establishing the main results.

Lemma 2.1. *Assume that* $(x(n))_{n\geq 0}$ *is a positive solution of* (E_1) *or* (E_m). *Set*

$$z(n) := x(n) + cx(n+a). \tag{2.1}$$

The following statements hold:

(i) *If* $\lim_{n\to\infty} z(n) = -\infty$ *and:*

(ia) $c < 0$*, then* $(x(n))$ *tends to infinity.*

(ib) $c \geq 0$*, then* (E_1) *or* (E_m) *has no positive solution.*

(ii) *If* $\lim_{n\to\infty} z(n) = A < 0$ *and:*

(iia) $c < -1$*, then* $(x(n))$ *tends to* $\frac{A}{1+c}$.

(iib) $-1 \leq c < 0$*, then* $(x(n))$ *tends to infinity.*

(iic) $c \geq 0$*, then* (E_1) *or* (E_m) *has no positive solution.*

(iii) *If* $\lim_{n\to\infty} z(n) = 0$ *and:*

(iiia) $c < -1$*, then* $(x(n))$ *tends to zero.*

(iiib) $c = -1$*, then* $(x(n))$ *tends to zero or to infinity or* $(x(n))$*,* $(x(n+a))$ *have the same set of accumulation points. In the case where* $z(n) > 0$*, then* $(x(n))$ *is bounded, and if* $z(n) < 0$ *then* $\liminf x(n) > 0$.

(iiic) $-1 < c < 0$, *then* $(x(n))$ *tends to zero or to infinity or* $(x(n))$ *has infinitely many accumulation points with* $\liminf x(n) = 0$. *In the case where* $z(n) < 0$, *then* $(x(n))$ *tends to infinity.*

(iiid) $c \geq 0$, *then* $(x(n))$ *tends to zero.*

(iv) *If* $\lim_{n\to\infty} z(n) = A > 0$ *and:*

(iva) $c \leq -1$, *then* (E_1) *or* (E_m) *has no positive solution.*

(ivb) $-1 < c < 0$, *then* $\liminf x(n) \geq \frac{A}{1+c}$ *and if* $(x(n))$ *has a real accumulation point greater than* $\frac{A}{1+c}$, *it will have infinitely many real accumulation points including* $\frac{A}{1+c}$.

(ivc) $c = 0$, *then* $\lim_{n\to\infty} x(n) = A$.

(ivd) $c > 0$, *then* $(x(n))$ *is bounded.*

(v) *If* $\lim_{n\to\infty} z(n) = +\infty$ *and:*

(va) $c \leq -1$, *then* (E_1) *or* (E_m) *has no positive solution.*

(vb) $-1 < c \leq 0$, *then* $(x(n))$ *tends to infinity.*

(vc) $c > 0$, *then* $(x(n))$ *is unbounded.*

Proof. Part (i): Assume that $\lim_{n\to\infty} z(n) = -\infty$.

If $c < 0$, then by (2.1) we have $\lim_{n\to\infty}[x(n) + cx(n+a)] = -\infty$, which guarantees that $(x(n+a))$ tends to infinity, and consequently $(x(n))$ tends to infinity.

If $c \geq 0$, clearly $z(n) = x(n) + cx(n+a) \geq x(n) > 0$ which contradicts $\lim_{n\to\infty} z(n) = -\infty$. Therefore, Eq. (E_1) or (E_m) has no positive solution. The proof of Part (i) of the lemma is complete.

Part (ii): Assume that $\lim_{n\to\infty} z(n) = A < 0$.

If $c < -1$, then

$$\lim_{n\to\infty}\left[\frac{x(n-a)}{c} + x(n)\right] = \frac{A}{c}.$$

Thus, for every $\epsilon > 0$, there exists $n_1 \geq a$ such that

$$\frac{x(n-a)}{-c} + \frac{A}{c} - \epsilon < x(n) < \frac{x(n-a)}{-c} + \frac{A}{c} + \epsilon, \quad \forall n \geq n_1.$$

Hence

$$\frac{\frac{x(n-2a)}{-c} + \frac{A}{c} - \epsilon}{-c} + \frac{A}{c} - \epsilon < x(n) < \frac{\frac{x(n-2a)}{-c} + \frac{A}{c} + \epsilon}{-c} + \frac{A}{c} + \epsilon,$$

or

$$\frac{x(n-2a)}{(-c)^2} - \frac{A}{c^2} + \frac{\epsilon}{c} + \frac{A}{c} - \epsilon < x(n) < \frac{x(n-2a)}{(-c)^2} - \frac{A}{c^2} - \frac{\epsilon}{c} + \frac{A}{c} + \epsilon.$$

Applying this procedure, we obtain

$$\frac{x(n_s)}{(-c)^{n_k}} + \frac{1 - \left(\frac{1}{c}\right)^{n_k}}{1+c}(A - c\epsilon) < x(n) < \frac{x(n_s)}{(-c)^{n_k}} + \frac{1 - \left(\frac{1}{c}\right)^{n_k}}{1+c}(A + c\epsilon), \tag{2.2}$$

where $n_s = n - n_k a > 0$. The last inequality guarantees that $(x(n))$ tends to $\frac{A}{1+c}$.

If $c = -1$, then

$$\lim_{n\to\infty}[x(n) - x(n-a)] = -A > 0.$$

Thus, for every $\epsilon > 0$ with $0 < \epsilon < -A$, there exists $n_2 \geq a$ such that $x(n) > x(n-a) - A - \epsilon$, $\forall n \geq n_2$. Hence

$$\begin{aligned} x(n) &> x(n-2a) - 2A - 2\epsilon \\ &> \ldots > x(n_\lambda) + n_\ell(-A-\epsilon) \to +\infty \text{ as } n \to \infty, \end{aligned}$$

where $n_\lambda = n - n_\ell a > 0$. This means that $(x(n))$ tends to infinity.

If $-1 < c < 0$, then $\lim_{n\to\infty}\left[\frac{x(n-a)}{c} + x(n)\right] = \frac{A}{c} > 0$. This means that $\frac{x(n-a)}{c} + x(n) > 0$ eventually. Hence

$$x(n) > \frac{x(n-a)}{-c} > \frac{x(n-2a)}{(-c)^2} > \ldots > \frac{x(n_s)}{(-c)^{n_k}} \to +\infty \quad \text{as } n \to \infty,$$

which guarantees that $(x(n))$ tends to infinity.

If $c \geq 0$, clearly $z(n) = x(n) + cx(n+a) \geq x(n) > 0$ which contradicts $A < 0$. Therefore, Eq. (E_1) or (E_m) has no positive solution. The proof of Part (ii) of the lemma is complete.

Part (iii): Assume that $\lim_{n\to\infty} z(n) = 0$.

If $c < -1$, then for every $\epsilon > 0$, there exists $n_3 \geq a$ such that (2.2) holds (for $A = 0$), i.e.,

$$\frac{x(n_s)}{(-c)^{n_k}} - c\epsilon\frac{1-\left(\frac{1}{c}\right)^{n_k}}{1+c} < x(n) < \frac{x(n_s)}{(-c)^{n_k}} + c\epsilon\frac{1-\left(\frac{1}{c}\right)^{n_k}}{1+c}.$$

The last inequality guarantees that $(x(n))$ tends to zero.

If $c = -1$, then $\lim_{n\to\infty}[x(n) - x(n+a)] = 0$, which means that $(x(n))$ tends to zero or to infinity or $(x(n)), (x(n+a))$ have the same set of accumulation points.

Suppose that $z(n) > 0$. Then $x(n) - x(n+a) > 0$ or $x(n-a) - x(n) > 0$ or $x(n) < x(n-a)$. Repeating this procedure, we obtain

$$x(n) < x(n-a) < x(n-2a) < \ldots < x(n_\lambda),$$

which means that $(x(n))$ is bounded.

Suppose that $z(n) < 0$. Then $x(n) - x(n+a) < 0$ or $x(n-a) - x(n) < 0$ or $x(n) > x(n-a)$. Applying this procedure, we obtain

$$x(n) > x(n-a) > x(n-2a) > \ldots > x(n_\lambda),$$

which means that $\liminf x(n) > 0$.

If $-1 < c < 0$, we have $\lim_{n\to\infty}\left[\frac{x(n-a)}{c} + x(n)\right] = 0$ which means that $(x(n))$ tends to zero or to infinity or $(x(n))$ has infinitely many accumulation points. Indeed, in the case where $(x(n))$ does not tend to zero or to infinity, let $L > 0$ be an accumulation point of $(x(n))$. Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = L$. Hence $\lim_{n\to\infty} x(\theta(n) - a) = -cL$. Similarly,

$$\lim_{n\to\infty} x(\theta(n) - 2a) = (-c)^2 L.$$

Repeating this procedure, we can construct a sequence $(b_n)_{n\geq 1}$ of accumulation points with $b_n = (-c)^n L$. Notice that this sequence of accumulation points converges to zero, and therefore $\liminf x(n) = 0$.

Suppose that $z(n) < 0$. Then $x(n) + cx(n+a) < 0$ or $x(n) > \frac{x(n-a)}{-c}$. Applying this procedure, we obtain

$$x(n) > \frac{x(n-2a)}{(-c)^2} > \ldots > \frac{x(n_s)}{(-c)^{n_k}} \to +\infty \quad \text{as } n \to \infty,$$

which guarantees that $(x(n))$ tends to infinity.

If $c \geq 0$, then $\lim_{n\to\infty} [x(n) + cx(n+a)] = 0$, which means that $(x(n))$ tends to zero. The proof of Part (iii) of the lemma is complete.

Part (iv): Assume that $\lim_{n\to\infty} z(n) = A > 0$.

If $c < -1$, then $\lim_{n\to\infty} \left[\frac{x(n-a)}{c} + x(n)\right] = \frac{A}{c} < 0$ which, for sufficiently large n, means that $\frac{x(n-a)}{c} + x(n) < 0$. Thus

$$x(n) < \frac{x(n-a)}{-c} < \frac{x(n-2a)}{(-c)^2} < \ldots < \frac{x(n_s)}{(-c)^{n_k}} \to 0 \quad \text{as } n \to \infty,$$

which guarantees that $(x(n))$ tends to zero, and consequently $(z(n))$ tends to zero. This contradicts $A > 0$. Therefore, Eq. (E_1) or (E_m) has no positive solution.

If $c = -1$, then $\lim_{n\to\infty} [x(n) - x(n-a)] = -A$. Thus, for every $\epsilon > 0$ with $0 < \epsilon < A$, there exists n_4 such that $x(n) < x(n-a) - A + \epsilon$, $\forall n \geq n_4$. Applying this procedure, we obtain

$$\begin{aligned} x(n) &< x(n-2a) - 2(A-\epsilon) \\ &< \ldots < x(n_s) - n_k(A-\epsilon) \to -\infty \quad \text{as } n \to \infty. \end{aligned}$$

This contradicts $x(n) > 0$. Therefore Eq. (E_1) or (E_m) has no positive solution.

If $-1 < c < 0$, then $\lim_{n\to\infty} \left[x(n) + \frac{x(n-a)}{c}\right] = \frac{A}{c} < 0$. Hence for every $\epsilon > 0$ there exists n_5 such that

$$x(n) < \frac{x(n-a)}{-c} + \frac{A}{c} + \epsilon, \qquad \forall n \geq n_5.$$

Repeating this procedure, we obtain

$$\begin{aligned} x(n) &< \frac{1}{-c}\left[\frac{x(n-2a)}{-c} + \frac{A}{c} + \epsilon\right] + \frac{A}{c} + \epsilon \\ &= \frac{1}{(-c)^2} x(n-2a) - \frac{A}{c^2} - \frac{\epsilon}{c} + \frac{A}{c} + \epsilon \\ &< \ldots < \frac{1}{(-c)^{n_k}} x(n_s) + \left(\frac{A}{c} + \epsilon\right) \frac{1 - \left(\frac{1}{-c}\right)^{n_k}}{1 + \frac{1}{c}} \\ &= \frac{1}{(-c)^{n_k}} \left[x(n_s) - \frac{\frac{A}{c} + \epsilon}{1 + \frac{1}{c}}\right] + \frac{\frac{A}{c} + \epsilon}{1 + \frac{1}{c}} \\ &= \frac{1}{(-c)^{n_k}} \left[x(n_s) - \frac{A + c\epsilon}{1 + c}\right] + \frac{A + c\epsilon}{1 + c}. \end{aligned}$$

Since $x(n) > 0$, clearly $x(n_s) \geq \frac{A+c\epsilon}{1+c}$, or eventually $x(n-a) \geq \frac{A+c\epsilon}{1+c}$. Therefore $\liminf x(n) \geq \frac{A}{1+c} > 0$.

Clearly $\frac{A}{1+c}$ could be an accumulation point of $(x(n))$. Let $L > \frac{A}{1+c}$ be an accumulation point of $(x(n))$. Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = L$. Hence $\lim_{n\to\infty}[x(\theta(n)) + cx(\theta(n)+a)] = A$, or

$$\lim_{n\to\infty} x(\theta(n)+a) = \frac{A}{c} + \frac{L}{-c}.$$

In view of this, we have $\lim_{n\to\infty}[x(\theta(n)+a) + cx(\theta(n)+2a)] = A$, or

$$\lim_{n\to\infty} x(\theta(n)+2a) = \frac{A}{c} + \frac{A}{(-c)^2} + \frac{L}{(-c)^2}.$$

Applying this procedure, we can construct a sequence $(\lambda_n)_{n\geq 1}$ of accumulation points with

$$\lambda_n = \frac{A}{c} + \frac{A}{(-c)^2} + \frac{A}{(-c)^3} + \ldots + \frac{A}{(-c)^n} + \frac{L}{(-c)^n}, \quad n \geq 1.$$

Notice that this sequence of accumulation points converges to $\frac{A}{1+c}$.

If $c = 0$, clearly $\lim_{n\to\infty} z(n) = \lim_{n\to\infty} x(n) = A$.

If $c > 0$, then $(x(n))$ is bounded since $\lim_{n\to\infty} z(n) = A$. The proof of Part (iv) of the lemma is complete.

Part (v): Assume that $\lim_{n\to\infty} z(n) = +\infty$.

If $c < -1$, then $\lim_{n\to\infty}\left[x(n) + \frac{x(n-a)}{c}\right] = -\infty$. Hence $x(n) + \frac{x(n-a)}{c} < 0$ eventually, i.e.,

$$x(n) < \frac{x(n-a)}{-c} < \frac{x(n-2a)}{(-c)^2} < \ldots < \frac{x(n_s)}{(-c)^{n_k}} \to 0 \quad \text{as } n \to \infty,$$

which contradicts $x(n) > 0$. Therefore, Eq. (E_1) or (E_m) has no positive solution.

If $c = -1$, then $\lim_{n\to\infty}[x(n) - x(n-a)] = -\infty$. Thus, for sufficiently large n, we have $x(n) - x(n-a) < 0$. Hence

$$x(n) < x(n-a) < x(n-2a) < \ldots < x(n_s),$$

which means that $(x(n))$ is bounded, and consequently $(z(n))$ is bounded. This contradicts $\lim_{n\to\infty} z(n) = +\infty$. Therefore, Eq. (E_1) or (E_m) has no positive solution.

If $-1 < c < 0$, then $\lim_{n\to\infty}\left[x(n) + \frac{x(n-a)}{c}\right] = -\infty$, i.e., $\lim_{n\to\infty} x(n-a) = +\infty$, which guarantees that $(x(n))$ tends to infinity.

If $c = 0$, clearly $\lim_{n\to\infty} x(n) = \lim_{n\to\infty} z(n) = +\infty$.

If $c > 0$, clearly $(x(n))$ is unbounded since $\lim_{n\to\infty} z(n) = +\infty$. The proof of Part (v) of the lemma is complete.

The proof of the lemma is complete.

To prove the main results we will also need the following two lemmas that recently have been established by Chatzarakis et al. [5].

Lemma 2.2 [5] *Assume that $(z(n))$ is a sequence of real numbers and $m \in \mathbb{N}$. Then the following statements hold:*

(i) *If* $\lim_{n\to\infty} \Delta^m z(n) > 0$ *then* $(z(n))$ *tends to* $+\infty$.

(ii) *If* $\lim_{n\to\infty} \Delta^m z(n) < 0$ *then* $(z(n))$ *tends to* $-\infty$.

(iii) *If* $\lim_{n\to\infty} \Delta^m z(n) = 0$ *and* $\Delta^{m+1} z(n) \geq 0$, $\forall n$ *or* $\lim_{n\to\infty} \Delta^m z(n) = 0$ *and* $\Delta^{m+1} z(n) \leq 0$, $\forall n$ *then the sequence* $(z(n))$ *is monotone and therefore its limit exists.*

Lemma 2.3 [5] *Assume that* $(z(n))$ *is a sequence of real numbers and* $\mathbb{N} \ni m \geq 2$. *Then the following statements hold:*

(i) *If* m *is even and* $\Delta^m z(n) \leq 0$, *then* $(z(n))$ *tends to* $\pm\infty$ *or it is increasing.*
(ii) *If* m *is even and* $\Delta^m z(n) \geq 0$ *then* $(z(n))$ *tends to* $\pm\infty$ *or it is decreasing.*
(iii) *If* m *is odd and* $\Delta^m z(n) \leq 0$, *then* $(z(n))$ *tends to* $\pm\infty$ *or it is decreasing.*
(iv) *If* m *is odd and* $\Delta^m z(n) \geq 0$, *then* $(z(n))$ *tends to* $\pm\infty$ *or it is increasing.*

3 Main results for 1^{st}-order NADE

3.1 $p(n) \geq 0$

The asymptotic behavior of the solutions of the equation (E_1) is described by the following theorem.

Theorem 3.1. *Assume that* $p(n) \geq 0, \forall n \geq 0$. *Then for every nonoscillatory solution* $(x(n))$ *of Eq.* (E_1) *the following statements hold:*

(I) *If* $c < -1$, *then* $(x(n))$ *tends to infinity or tends to a finite limit.*
(II) *If* $c = -1$, *then* $(x(n))$ *tends to infinity or it is bounded.*
(III) *If* $-1 < c < 0$, *then* $(x(n))$ *has a unique or infinitely many accumulation points.*
(IV) *If* $c = 0$, *then* $(x(n))$ *tends to a finite limit.*
(V) *If* $c > 0$, *then* $(x(n))$ *tends to zero or it is bounded.*

Proof. Assume that a solution $(x(n))_{n\geq 0}$ of (E_1) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq 0}$ is also a solution of (E_1), we may restrict ourselves to the case where $x(n) > 0$ for all large n. Let n_0 be a natural number such that $x(n) > 0$ for all $n \geq n_0 \geq a$.

In view of (2.1), Eq.(E_1) becomes $\Delta z(n) = -p(n)x(\sigma(n))$. Therefore, for sufficiently large n and since $p(n) \geq 0$, we have $\Delta z(n) \leq 0$. This means that the sequence $(z(n))$ is eventually decreasing, regardless of the value of the real constant c. Consequently $\lim_{n\to\infty} z(n) = -\infty$ or $\lim_{n\to\infty} z(n) = A \in \mathbb{R}$.

Part (I): $c < -1$

If $\lim_{n\to\infty} z(n) = -\infty$, then by Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A < 0$, then by Part (iia) of Lemma 2.1 we have that $(x(n))$ tends to $\frac{A}{1+c}$.

If $A = 0$, then by Part (iiia) of Lemma 2.1 we have that $(x(n))$ tends to zero.

If $A > 0$, then by Part (iva) of Lemma 2.1 we have that this is false. The proof of Part (I) of the theorem is complete.

Part (II): $c = -1$

If $\lim_{n\to\infty} z(n) = -\infty$, then by Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A < 0$, then by Part (iib) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A = 0$, clearly $z(n) > 0$ since $(z(n))$ is eventually decreasing. By Part (iiib) of Lemma 2.1, we have that $(x(n))$ is bounded.

If $A > 0$, then by Part (iva) of Lemma 2.1 we have that this is false. The proof of Part (II) of the theorem is complete.

Part (III): $-1 < c < 0$

If $\lim_{n\to\infty} z(n) = -\infty$, then by Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A < 0$, then by Part (iib) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A = 0$, then by Part (iiic) of Lemma 2.1 we have that $(x(n))$ tends to zero or to infinity or $(x(n))$ has infinitely many accumulation points with $\liminf x(n) = 0$.

If $A > 0$, then by Part (ivb) of Lemma 2.1 we have that $\liminf x(n) \geq \frac{A}{1+c}$ and if $(x(n))$ has a real accumulation point greater than $\frac{A}{1+c}$, it will have infinitely many real accumulation points including $\frac{A}{1+c}$. The proof of Part (III) of the theorem is complete.

Part (IV): $c = 0$

If $\lim_{n\to\infty} z(n) = -\infty$, then by Part (ib) of Lemma 2.1 we have that this is false.

If $A < 0$, then by Part (iic) of Lemma 2.1 we have that this is false.

If $A = 0$, then by Part (iiid) of Lemma 2.1 we have that $(x(n))$ tends to zero.

If $A > 0$, then by Part (ivc) of Lemma 2.1 we have that $(x(n))$ tends to A. The proof of Part (IV) of the theorem is complete.

Part (V): $c > 0$

If $\lim_{n\to\infty} z(n) = -\infty$, then by Part (ib) of Lemma 2.1 we have that this is false.

If $A < 0$, then by Part (iic) of Lemma 2.1 we have that this is false.

If $A = 0$, then by Part (iiid) of Lemma 2.1 we have that $(x(n))$ tends to zero.

If $A > 0$, then by Part (ivd) of Lemma 2.1 we have $(x(n))$ is bounded. The proof of Part (V) of the theorem is complete. The proof of the theorem is complete.

Remark 3.1. *Assume that the coefficients $p(i)$ are always nonpositive or nonnegative, $\sum_{i=n_0}^{\infty} p(i) = \pm\infty$ and $\liminf x(n) > 0$. Then $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = \pm\infty$, respectively.*

Remark 3.2. *Assume that the coefficients $p(i)$ are always nonpositive or nonnegative, $\sum_{i=n_0}^{\infty} p(i) = \pm\infty$ and $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \in \mathbb{R}$. Then $\liminf x(n) = 0$.*

Next we present an auxiliary lemma. On the basis of this lemma and Theorem 3.1, we postulate the Corollary 3.1 for the case where $\sum_{i=n_0}^{\infty} p(i) = +\infty$.

Auxiliary Lemma. *Assume that the coefficients $p(i)$ are always nonpositive or nonnegative, $\sum_{i=n_0}^{\infty} p(i) = \pm\infty$ and $c \in (0,1)\cup(1,+\infty)$. Then the sequence $(z(n))$ cannot tend to a real positive limit.*

Proof. Assume, for the sake of contradiction, that $\lim_{n\to\infty} z(n) = A > 0$.

Now, summing up $\Delta z(n) = -p(n)x(\sigma(n))$ from n_0 to n, $n \geq n_0$ we obtain

$$z(n+1) = z(n_0) - \sum_{i=n_0}^{n} p(i)x(\sigma(i)). \tag{3.1}$$

Let $0 < c < 1$. Since $A \in \mathbb{R}$, (3.1) guarantees that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) \in \mathbb{R}$. By Remark 3.2, we have that $\liminf x(n) = 0$. Then there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = 0$. Thus $\lim_{n\to\infty} x(\theta(n)+a) = \frac{A}{c}$. Consequently,

$$\lim_{n\to\infty} [x(\theta(n)+a) + cx(\theta(n)+2a)] = A,$$

or

$$\lim_{n\to\infty} x(\theta(n)+2a) = \frac{A - \frac{A}{c}}{c} < 0, \text{ since } 0 < c < 1$$

which contradicts $x(n) > 0$.

Let $c > 1$. As in the previous case, there exists a subsequence $(x(\theta(n)))$ of $(x(n))$ such that $\lim_{n\to\infty} x(\theta(n)) = 0$. Thus $\lim_{n\to\infty} x(\theta(n) - a) = A$. Consequently,

$$\lim_{n\to\infty} [x(\theta(n) - 2a) + cx(\theta(n) - a)] = A,$$

or

$$\lim_{n\to\infty} x(\theta(n) - 2a) = A - cA < 0, \text{ since } c > 1$$

which contradicts $x(n) > 0$.

The proof of the auxiliary lemma is complete.

Corollary 3.1. *Assume that* $p(n) \geq 0, \forall n \geq 0$ *and* $\sum_{i=n_0}^{\infty} p(i) = +\infty$. *Then for every nonoscillatory solution* $(x(n))$ *of Eq.* (E_1) *the following statements hold:*

(i) *If* $c < -1$*, then* $(x(n))$ *tends to infinity or tends to zero.*

(ii) *If* $c = -1$*, then* $(x(n))$ *tends to infinity or it is bounded with* $\liminf x(n) = 0$.

(iii) *If* $-1 < c < 0$*, then* $(x(n))$ *tends to infinity or has infinitely many accumulation points with* $\liminf x(n) = 0$.

(iv) *If* $c = 0$*, then* $(x(n))$ *tends to zero.*

(v) *If* $c > 0$ *and* $c \neq 1$*, then* $(x(n))$ *tends to zero.*

(vi) *If* $c = 1$*, then* $(x(n))$ *tends to zero or it is bounded with* $\liminf x(n) = 0$.

Proof. By Part (I) of Theorem 3.1 we have that $(x(n))$ tends to infinity or tends to finite limit.

We shall show that $\lim_{n\to\infty} x(n) = \ell \in \mathbb{R}_+$ is false. Indeed, in this case $\lim_{n\to\infty} z(n) = (1 + c)\ell = A < 0$. Therefore for every $\varepsilon > 0$, with $\varepsilon < \ell$, there exists n_1 such that

$$x(n) > \ell - \varepsilon, \quad \forall n \geq n_1. \tag{3.2}$$

Thus, for every n_2 with $\sigma(n_2) \geq n_1$, by (3.2) and (3.1) we obtain

$$z(n + 1) < z(n_2) - (\ell - \varepsilon) \sum_{i=n_2}^{n} p(i) \to -\infty \quad \text{as } n \to \infty,$$

which guarantees that $\lim_{n\to\infty} z(n) = -\infty$. This contradicts $A < 0$. The proof of Part (i) of the corollary is complete.

Part (ii): $c = -1$

By Part (II) of Theorem 3.1 we have that $(x(n))$ tends to infinity or it is bounded.

We shall show that if $(x(n))$ is bounded.then $\liminf x(n) = 0$. Indeed, if $\liminf x(n) > 0$ then by Remark 3.1 we have that $\sum_{i=n_0}^{\infty} p(i)x(\sigma(i)) = +\infty$. By (3.1), clearly $\lim_{n\to\infty} z(n) = -\infty$ which contradicts $(x(n))$ is bounded. The proof of Part (ii) of the corollary is complete.

Part (iii): $-1 < c < 0$

By Part (III) of Theorem 3.1 we have that $(x(n))$ has a unique or infinitely many accumulation points.

We shall show that if $(x(n))$ has infinitely many accumulation point or a unique real limit, then $\liminf x(n) = 0$. Indeed, if $\liminf x(n) > 0$, then as in previous case we are led to a contradiction. The proof of Part (iii) of the corollary is complete.

Part (iv): $c = 0$

By Part (IV) of Theorem 3.1 we have that $(x(n))$ tends to a finite limit.

We shall show that $\lim_{n\to\infty} x(n) = \ell \in \mathbb{R}_+$ is false. Indeed, in this case $\lim_{n\to\infty} z(n) = \lim_{n\to\infty} x(n) = \ell > 0$. As in the proof of Part (i) we conclude that $\lim_{n\to\infty} z(n) = -\infty$ which contradicts $\lim_{n\to\infty} z(n) = \ell > 0$. The proof of Part (iv) of the corollary is complete.

Part (v): $c > 0$ and $c \neq 1$

In view of auxiliary lemma, the case $A > 0$ is false. The rest of the proof follows directly from Part (V) of Theorem 3.1.

Part (vi): $c = 1$

By Part (V) of Theorem 3.1 we have that $(x(n))$ tends to zero or it is bounded.

Suppose that $(x(n))$ is bounded and does not tend to zero. Then $A > 0$. By Remark 3.2 we have $\liminf x(n) = 0$. The proof of Part (vi) of the corollary is complete.

The proof of the corollary is complete.

3.2 $p(n) \leq 0$

The asymptotic behavior of the solutions of the equation (E_1) is described by the following theorem:

Theorem 3.2. *Assume that* $p(n) \leq 0, \forall n \geq 0$. *Then for every nonoscillatory solution* $(x(n))$ *of Eq.* (E_1) *the following statements hold:*

(I) *If* $c < -1$, *then* $(x(n))$ *tends to a finite limit.*

(II) *If* $c = -1$, *then* $(x(n))$ *tends to infinity or* $\liminf x(n) > 0$.

(III) *If* $-1 < c < 0$, *then* $(x(n))$ *tends to infinity or has infinitely many accumulation points with* $\liminf x(n) > 0$.

(IV) *If* $c = 0$, *then* $(x(n))$ *tends to infinity or tends a non-zero real limit.*

(V) *If* $c > 0$, *then* $(x(n))$ *cannot tend to zero.*

Proof. Assume that a solution $(x(n))_{n\geq 0}$ of (E_1) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq 0}$ is also a solution of (E_1), we may restrict ourselves to the case where $x(n) > 0$ for all large n. Let n_0 be a natural number such that $x(n) > 0$ for all $n \geq n_0 \geq a$.

In view of (2.1), Eq.(E_1) becomes $\Delta z(n) = -p(n)x(\sigma(n))$. Therefore, for sufficiently large n and since $p(n) \leq 0$, we have $\Delta z(n) \geq 0$. This means that the sequence $(z(n))$ is eventually increasing, regardless of the value of the real constant c. Consequently $\lim_{n\to\infty} z(n) = A \in \mathbb{R}$ or $\lim_{n\to\infty} z(n) = +\infty$.

Part (I): $c < -1$

If $A < 0$, then by Part (iia) of Lemma 2.1 we have that $(x(n))$ tends to $\frac{A}{1+c} > 0$.

If $A = 0$, then by Part (iiia) of Lemma 2.1 we have that $(x(n))$ tends to zero.

If $A > 0$, then by Part (iva) of Lemma 2.1 this is false.

If $\lim_{n\to\infty} z(n) = +\infty$, then by Part (va) of Lemma 2.1 we have that this is false. The proof of Part (I) of the theorem is complete.

Part (II): $c = -1$

If $A < 0$, then by Part (iib) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

f $A = 0$, clearly $z(n) < 0$ since $(z(n))$ is eventually increasing. By Part (iiib) of Lemma 2.1 we have that $\liminf x(n) > 0$.

If $A > 0$, then by Part (iva) of Lemma 2.1 this is false.

If that $\lim_{n\to\infty} z(n) = +\infty$, then by Part (va) of Lemma 2.1 we have that this is false. The proof of Part (II) of the theorem is complete.

Part (III): $-1 < c < 0$.

If $A < 0$, then by Part (iib) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A = 0$, clearly $z(n) < 0$ since $(z(n))$ is eventually increasing. By Part (iiic) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

If $A > 0$, then by Part (ivb) of Lemma 2.1 we have that $\liminf x(n) \geq \frac{A}{1+c}$ and if $(x(n))$ has a real accumulation point greater than $\frac{A}{1+c}$, it will have infinitely many real accumulation points including $\frac{A}{1+c}$.

If $\lim_{n\to\infty} z(n) = +\infty$, then by Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity. The proof of Part (III) of the theorem is complete.

Part (IV): $c = 0$

If $A < 0$, then by Part (iic) of Lemma 2.1 this is false.

If $A = 0$, clearly $z(n) < 0$ since $(z(n))$ is eventually increasing. This contradicts $z(n) = x(n) > 0$.

If $A > 0$, then by Part (ivc) of Lemma 2.1 we have that $(x(n))$ tends to A.

If $\lim_{n\to\infty} z(n) = +\infty$, then by Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity. The proof of Part (IV) of the theorem is complete.

Part (V): $c > 0$

If $A < 0$, then by Part (iic) of Lemma 2.1 this is false.

If $A = 0$, clearly $z(n) < 0$ since $(z(n))$ is eventually increasing. This contradicts $z(n) = x(n) + cx(n+a) > x(n) > 0$.

If $A > 0$, then by Part (ivd) of Lemma 2.1 we have that $(x(n))$ is bounded.

If $\lim_{n\to\infty} z(n) = +\infty$, then by Part (vc) of Lemma 2.1 we have that $(x(n))$ is unbounded. The proof of Part (V) of the theorem is complete.

The proof of the theorem is complete.

Next, on the basis of auxiliary lemma and Theorem 3.2, we postulate the Corollary 3.2 for the case where $\sum_{i=n_0}^{\infty} p(i) = -\infty$.

Corollary 3.2. *Assume that $p(n) \leq 0, \forall n \geq 0$ and $\sum_{i=n_0}^{\infty} p(i) = -\infty$. Then for every solution $(x(n))$ of Eq.* (E_1) *the following statements hold:*

(i) *If $c < -1$, then every nonoscillatory solution tends to zero.*

(ii) *If $c = -1$, then every solution oscillates.*

(iii) *If $-1 < c < 0$, then every nonoscillatory solution tends to infinity.*

(iv) *If $c = 0$, then every nonoscillatory solution tends to infinity.*

(v) *If $c > 0$ and $c \neq 1$, then every nonoscillatory solution is unbounded.*

(vi) *If $c = 1$, then every nonoscillatory solution is bounded and* $\liminf x(n) = 0$ *or it is unbounded.*

Proof. By Part (I) of Theorem 3.2 we have that $(x(n))$ tends to a finite limit.

We shall show that $\lim_{n\to\infty} x(n) = \ell \in \mathbb{R}_+$ is false. Indeed, in this case $\lim_{n\to\infty} z(n) = (1+c)\ell = A < 0$. Therefore for every $\varepsilon > 0$, with $\varepsilon < \ell$, there exists n_2 such that

$$x(n) > \ell - \varepsilon, \quad \forall n \geq n_2. \tag{3.3}$$

Thus, for every n_3 with $\sigma(n_3) \geq n_2$, by (3.3) and (3.1) we obtain

$$z(n+1) < z(n_2) - (\ell - \varepsilon) \sum_{i=n_2}^{n} p(i) \to +\infty \quad \text{as } n \to \infty,$$

which guarantees that $\lim_{n\to\infty} z(n) = +\infty$. This contradicts $A < 0$. The proof of Part (i) of the corollary is complete.

Part (ii): $c = -1$

If $A < 0$, then by Part (iib) of Lemma 2.1 we have that $(x(n))$ tends to infinity. By (3.1) we conclude that $\lim_{n\to\infty} z(n) = +\infty$ which contradicts $A < 0$.

If $A = 0$, then by Part (iiib) of Lemma 2.1 we have that $\liminf x(n) > 0$. By (3.1) we conclude that $\lim_{n\to\infty} z(n) = +\infty$ which contradicts $A = 0$. The rest of the proof is direct from Part (II) of Theorem 3.2. Therefore $(x(n))$ oscillates. The proof of Part (ii) of the corollary is complete.

Part (iii): $-1 < c < 0$

If $A \le 0$, then by Parts (iib) and (iiic) of Lemma 2.1 we have that $(x(n))$ tends to infinity. By (3.1) we conclude that $\lim_{n\to\infty} z(n) = +\infty$ which contradicts $A \le 0$.

If $A > 0$, then by Part (ivb) of Lemma 2.1 we have that $\liminf x(n) \ge \frac{A}{1+c}$. By (3.1) we conclude that $\lim_{n\to\infty} z(n) = +\infty$ which contradicts $A > 0$. The rest of the proof follows directly from Part (III) of Theorem 3.2. The proof of Part (iii) of the corollary is complete.

Part (iv): $c = 0$

By Part (IV) of Theorem 3.2 we have that $(x(n))$ tends to infinity or to a non-zero real limit.

We shall show that $\lim_{n\to\infty} x(n) = \ell \in \mathbb{R}_+$ is false. Indeed, in this case $\lim_{n\to\infty} z(n) = \ell > 0$. Then by (3.1), clearly $\lim_{n\to\infty} z(n) = +\infty$ which contradicts $\ell > 0$. The proof of Part (iv) of the corollary is complete.

Part (v): $c > 0$ and $c \ne 1$

In view of auxiliary lemma, the case $A > 0$ is false.

The rest of the proof follows directly from Part (V) of Theorem 3.2. The proof of Part (v) of the corollary is complete.

Part (vi): $c = 1$

By Part (V) of Theorem 3.2 we have that $(x(n))$ cannot tend to zero.

Suppose that $(x(n))$ is bounded. Then $A > 0$. By Remark 3.2 we have $\liminf x(n) = 0$. The rest of the proof follows directly from Part (IV) of Theorem 3.2. The proof of Part (vi) of the corollary is complete.

The proof of the corollary is complete.

4 Main results for m^{th}-order NADE

4.1 $p(n) \ge 0$

The asymptotic behavior of the solutions of the neutral type difference equation (E_m) is described by the following theorem:

Theorem 4.1. *Assume that $p(n) \ge 0$, $\forall n \ge 0$. Then for Eq.* (E_m) *the following statements hold:*

(i) *If $c < -1$, then every nonoscillatory solution tends to infinity or tends to a finite limit.*

(ii) *If $c = -1$, then every nonoscillatory solution:*

(iia) *tends to infinity or $\liminf x(n) > 0$, if m is even.*

(iib) *tends to infinity or it is bounded, if m is odd.*

(iii) *If* $-1 < c < 0$*, then every nonoscillatory solution:*

(iiia) *tends to infinity or has infinitely many accumulation points with* $\liminf x(n) > 0$*, if m is even.*

(iiib) *has a unique or infinitely many accumulation points, if m is odd.*

(iv) *If* $c = 0$*, then every nonoscillatory solution:*

(iva) *tends to infinity or tends to a non-zero real limit, if m is even.*

(ivb) *tends to infinity or tends to a finite limit, if m is odd.*

(v) *If* $c > 0$*, then every nonoscillatory solution:*

(va) *cannot tend to zero, if* m is even.

(vb) *has no restriction in its behavior, if m is odd.*

Proof. Assume that a solution $(x(n))_{n\geq 0}$ of (E_m) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq 0}$ is also a solution of (E_m), we may restrict ourselves to the case where $x(n) > 0$ for all large n. Let n_0 be a natural number such that $x(n) > 0$ for all $n \geq n_0 \geq a$.

In view of (2.1), Eq.(E_m) becomes $\Delta^m z(n) = -p(n)x(\sigma(n))$. Therefore, for sufficiently large n and since $p(n) \geq 0$, we have $\Delta^m z(n) \leq 0$.

Part (i): $c < -1$

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by Part (i) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (I) of Theorem 3.2.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by Part (iii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (va) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (I) of Theorem 3.1.

Part (ii): $c = -1$

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by Part (i) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (II) of Theorem 3.2.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by Part (iii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (va) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (II) of Theorem 3.1.

Part (iii): $-1 < c < 0$

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by Part (i) of Lemma 2.1 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (III) of Theorem 3.2.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by Part (iii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (III) of Theorem 3.1.

Part (iv): $c = 0$

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by Part (i) of Lemma 2.1 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ib) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (IV) of Theorem 3.2.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by Part (iii) of Lemma 2.1 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (IV) of Theorem 3.1.

Part (v): $c > 0$

Assume that m is even. Since $\Delta^m z(n) \leq 0$, by Part (i) of Lemma 2.1 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ib) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (V) of Theorem 3.2.

Assume that m is odd. Since $\Delta^m z(n) \leq 0$, by Part (iii) of Lemma 2.1 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vc) of Lemma 2.1 we have that $(x(n))$ is unbounded.

The rest of the proof is identical to the proof of Part (V) of Theorem 3.1.

The proof of the theorem is complete.

4.2 $p(n) \leq 0$

The asymptotic behavior of the solutions of the neutral type difference equation (E_m) is described by the following theorem:

Theorem 4.2. *Assume that* $p(n) \leq 0$, $\forall n \geq 0$. *Then for every nonoscillatory solution* $(x(n))$ *of Eq.* (E_m) *the following statements hold:*

(i) *If* $c < -1$, *then* $(x(n))$ *tends to infinity or tends to a finite limit.*

(ii) *If* $c = -1$, *then:*

(iia) $(x(n))$ *tends to infinity or it is bounded, if* m *is even.*

(iib) $(x(n))$ *tends to infinity or* $\liminf x(n) > 0$, *if* m *is odd.*

(iii) *If* $-1 < c < 0$, *then:*

(iiia) $(x(n))$ *has a unique or infinitely many accumulation points, if* m *is even.*

(iiib) $(x(n))$ *tends to infinity or has infinitely many accumulation points with* $\liminf x(n) > 0$, *if* m *is odd.*

(iv) *If* $c = 0$, *then:*

(iva) *tends to infinity or tends to a finite limit, if* m *is even.*

(ivb) *tends to infinity or tends to a non-zero real limit, if* m *is odd.*

(v) *If* $c > 0$, *then:*

(va) $(x(n))$ *has no restriction in its behavior, if* m *is even.*

(vb) $(x(n))$ *cannot tend to zero, if* m *is odd.*

Proof. Assume that a solution $(x(n))_{n\geq 0}$ of (E_m) is nonoscillatory. Then it is either eventually positive or eventually negative. As $(-x(n))_{n\geq 0}$ is also a solution of (E_m), we may restrict ourselves to the case where $x(n) > 0$ for all large n. Let n_0 be a natural number such that $x(n) > 0$ for all $n \geq n_0 \geq a$.

In view of (2.1), Eq.(E_m) becomes $\Delta^m z(n) = -p(n)x(\sigma(n))$. Therefore, for sufficiently large n and since $p(n) \leq 0$, we have $\Delta^m z(n) \geq 0$.

Part (i): $c < -1$

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by Part (ii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (va) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (I) of Theorem 3.1.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by Part (iv) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (I) of Theorem 3.2.

Part (ii): $c = -1$

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by Part (ii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (va) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (II) of Theorem 3.1.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by Part (iv) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (II) of Theorem 3.2.

Part (iii): $-1 < c < 0$

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by Part (ii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (III) of Theorem 3.1.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by Part (iv) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ia) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (III) of Theorem 3.2.

Part (iv): $c = 0$

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by Part (ii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vb) of Lemma 2.1 we have that $(x(n))$ tends to infinity.

The rest of the proof is identical to the proof of Part (IV) of Theorem 3.1.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by Part (iv) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ib) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (IV) of Theorem 3.2.

Part (v): $c > 0$

Assume that m is even. Since $\Delta^m z(n) \geq 0$, by Part (ii) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is decreasing.

If $\lim_{n\to\infty} z(n) = +\infty$, then in view of Part (vc) of Lemma 2.1 we have that $(x(n))$ unbounded.

The rest of the proof is identical to the proof of Part (V) of Theorem 3.1.

Assume that m is odd. Since $\Delta^m z(n) \geq 0$, by Part (iv) of Lemma 2.3 we have that $(z(n))$ tends to $\pm\infty$ or it is increasing.

If $\lim_{n\to\infty} z(n) = -\infty$, then in view of Part (ib) of Lemma 2.1 this is false.

The rest of the proof is identical to the proof of Part (V) of Theorem 3.2.

The proof of the theorem is complete.

5 Examples

Example 5.1. Consider the difference equation

$$\Delta(x(n) - 2x(n+2)) + p(n)x(n+3) = 0, \quad n \geq 4, \tag{5.1}$$

where $p(n) = \frac{n^2-n-8}{(n+1)(n+2)(n+3)^2}$.

It is easy to see that all conditions of Part (I) of Theorem 3.1 are satisfied. Thus every nonoscillatory solution $(x(n))$ of (5.1) tends to infinity or tends to a (finite) limit. In fact $(x(n)) = (\frac{n}{n+1})$ is one such solution, since it satisfies (5.1) for all $n \geq 4$ and $\lim_{n\to\infty} x(n) = 1$. Note that $\sum_{i=4}^{\infty} p(i) < +\infty$.

Example 5.2. Consider the difference equation

$$\Delta(x(n) - x(n+2)) + p(n)x(n+4) = 0, \quad n \geq 0, \tag{5.2}$$

where $p(n) = \frac{4}{(n+4)^2}$.

All conditions of Part (II) of Theorem 3.1 are satisfied. Therefore every nonoscillatory solution $(x(n))$ of (5.2) tends to infinity, or it is bounded. In fact $(x(n)) = (n^2)$ is one such solution, since it satisfies (5.2) for all $n \geq 0$ and $\lim_{n\to\infty} x(n) = +\infty$. Note that $\sum_{i=0}^{\infty} p(i) < +\infty$.

Example 5.3. Consider the difference equation

$$\Delta\left(x(n) + \frac{1}{2}x(n+2)\right) + p(n)x(n^2+1) = 0, \quad n \geq 2, \tag{5.3}$$

where $p(n) = \frac{(3n^2+11n+12)(n^2+1)}{2n(n+1)(n+2)(n+3)}$.

Clearly, $\sum_{i=2}^{\infty} p(i) = +\infty$. All conditions of Part (v) of Corollary 3.1 are satisfied. Therefore every nonoscillatory solution $(x(n))$ of (5.3) tends to zero. In fact $(x(n)) = (\frac{1}{n})$ is one such solution, since it satisfies (5.3) for all $n \geq 2$ and $\lim_{n\to\infty} x(n) = 0$.

Example 5.4. Consider the difference equation

$$\Delta(x(n) - x(n+2)) + p(n)x(n+4) = 0, \quad n \geq 1, \tag{5.4}$$

where $p(n) = \frac{\sqrt{n}+\sqrt{n+3}-\sqrt{n+1}-\sqrt{n+2}}{\sqrt{n+4}} \le 0,\ \forall n \ge 1$.

All conditions of Part (II) of Theorem 3.2 are satisfied. Thus every nonoscillatory solution $(x(n))$ of (5.4) tends to infinity or $\liminf x(n) > 0$. In fact $(x(n)) = (\sqrt{n})$ is one such solution, since it satisfies (5.4) for all $n \ge 1$ and $\lim_{n\to\infty} x(n) = +\infty$. Note that $\sum_{n=1}^{\infty} p(n) > -\sum_{n=1}^{\infty} \frac{1}{n^2} > -\infty$, which can be observed in the graphs of the functions $p(x), -1/x^2$ in $[1,+\infty)$, using software.

Example 5.5. Consider the difference equation

$$\Delta(x(n) - x(n+3)) + (-4)\,x(n^2+3) = 0, \quad n \ge 1. \tag{5.5}$$

Clearly, $\sum_{i=1}^{\infty} p(i) = -\infty$. All conditions of Part (II) of Corollary 3.2 are satisfied. Therefore every solution $(x(n))$ of (5.5) oscillates. In fact $(x(n)) = (-1)^n$ is one such solution, since it satisfies (5.5) for all $n \ge 1$ and $(x(n))$ oscillates.

Example 5.6. Consider the difference equation

$$\Delta^2(x(n) - x(n+2)) + p(n)x(n+4) = 0, \quad n \ge 1, \tag{5.6}$$

where $p(n) = \frac{3n^3+4n^2+13n-12}{16n(n+1)(n+3)} > 0,\ \forall n \ge 1$.

All conditions of Part (iia) of Theorem 4.1 are satisfied. Thus every nonoscillatory solution $(x(n))$ of (5.6) tends to infinity, or $\liminf x(n) > 0$. In fact $(x(n)) = \left(\frac{2^n}{5n}\right)$ is one such solution, since it satisfies (5.6) for all $n \ge 1$ and $\lim_{n\to\infty} x(n) = +\infty$.

Example 5.7. Consider the difference equation

$$\Delta^3(x(n) - x(n+3)) + p(n)x(n^2+1) = 0, \quad n \ge 2, \tag{5.7}$$

where $p(n) = \frac{\ln \frac{n(n+6)(n+2)^3(n+4)^3}{(n+1)^3(n+3)^2(n+5)^3}}{\ln\left(n^2+1\right)} \le 0,\ \forall n \ge 2$.

All conditions of Part (iib) of Theorem 4.2 are satisfied. Therefore every nonoscillatory solution $(x(n))$ of (5.7) tends to infinity, or $\liminf x(n) > 0$. In fact $(x(n)) = (\ln n)$ is one such solution, since it satisfies (5.7) for all $n \ge 2$ and $\lim_{n\to\infty} x(n) = +\infty$. Note that $\sum_{n=2}^{\infty} p(n) > -\sum_{n=2}^{\infty} \frac{1}{n^2} > -\infty$, which can be observed in the graphs of the functions $p(x), -1/x^2$ in $[2,+\infty)$, using software.

Remark. Similar to the above, one can construct examples to illustrate other parts of Theorems 3.1-3.2, 4.1-4.2 and Corollaries 3.1-3.2.

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