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Optimal regularity properties of the Riesz potential operator

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Abstract

We prove continuity of the Riesz potential operator $R^s : E \mapsto CH$, in optimal couples E, CH, where E is a rearrangement invariant function space and CH is the generalized Hölder-Zygmund space generated by a function space H.

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1 Introduction

The Riesz potential operator R^s, $0 < s < n$, $n \geq 1$ is defined by

$$R^s f(x) = \int_\Omega f(y)|x-y|^{s-n}dy,$$

where $f \in L^1(\Omega)$ and Ω is a domain in $\mathbf{R}^n$. In investigating the regularity of the function $R^s f$ we may assume, without any loss of generality, that f is zero outside Ω. For simplicity we suppose that the Lebesgue measure of Ω equals one and that the origin lies in Ω. It is well known that in the super-critical case $s > n/p$,

$$R^s : L^p \mapsto C^{s-n/p},\ s > n/p, \tag{1.1}$$

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where C^γ, $\gamma > 0$, is the Hölder-Zygmund space (see [11]). In the critical case $s = n/p$ the function $R^s f$ may not be even continuous. The result (1.1) is not optimal. We prove that the optimal one is obtained if in (1.1) L^p is replaced by the Marcinkiewicz space $L^{p,\infty}$. In this paper we prove similar optimal results, when $L^{p,\infty}$ is replaced by more general rearrangement invariant spaces E. More precisely, we consider quasi-normed rearrangement invariant spaces E, consisting of functions $f \in L^1(\Omega)$, such that the quasi-norm $\|f\|_E = \rho_E(f^*) < \infty$, where ρ_E is a monotone quasi-norm, defined on M^+ with values in $[0, \infty]$. Here M^+ is the cone of all locally integrable functions $g \geq 0$ on $(0, 1)$ with the Lebesgue measure. Monotonicity means that $g_1 \leq g_2$ implies $\rho_E(g_1) \leq \rho_E(g_2)$. We suppose that $L^\infty(\Omega) \hookrightarrow E \hookrightarrow L^1(\Omega)$, which means continuous embeddings. Here f^* is the decreasing rearrangement of f, given by

$$f^*(t) = \inf\{\lambda > 0 : \mu_f(\lambda) \leq t\},\ t > 0$$

and μ_f is the distribution function of f, defined by

$$\mu_f(\lambda) = \left|\{x \in \mathbf{R}^n : |f(x)| > \lambda\}\right|_n,$$

$|\cdot|_n$ denoting Lebesgue $n-$measure.

Let α_E, β_E be the Boyd indices of E. For example, if $E = L^p$, then $\alpha_E = \beta_E = 1/p$ and the condition $1 > s/n \geq 1/p$ implies $p > 1$, or $\beta_E < 1$. For these reasons, we suppose that for general E, $0 < \alpha_E = \beta_E < 1$ and the case $s/n > \alpha_E$ is called super-critical, while the case $s/n = \alpha_E$ - critical. In the super-critical case the function $R^s f$, $f \in E$, is always continuous, while the spaces in the critical case can be divided into two subclasses: in the first subclass the functions $R^s f$ may not be continuous - then the target space is rearrangement invariant, while these functions in the second subclass are continuous and the target space is the generalized Hölder-Zygmund space CH (see [1], [5] and the definition below). The separating space for these two subclasses is given by the Lorentz space $L^{n/s,1}$.

The main goal of this paper is to prove continuity of the Riesz potential operator $R^s : E \mapsto CH$ in optimal couples E, CH. First we prove that this continuity is equivalent to the continuity of the operator $Sg(t) = \int_0^t u^{s/n-1} g(u)du$. Moreover, in the super-critical case, we can replace S by the operator of multiplication $t^{s/n}g(t)$. This implies a very simple characterization of both optimal target space H and optimal domain space E. Namely, the quasi-norm in the optimal target space $H(E)$ is given by $\rho_E(t^{-s/n}g(t))$ and the quasi-norm in the optimal domain space $E(H)$ is given by $\rho_H(t^{s/n}g(t))$. Note that we do not require ρ_E to be rearrangement invariant. In the critical case, the formula for the optimal target quasi-norm is more complicated. In some cases it can be simplified. To this end, we apply the Σ^q-method of extrapolation ([8]) from the super-critical case. As a byproduct, we also characterize the mapping property $R^s : E \mapsto C^j$, $j < s$, where C^j consists of all functions with bounded and uniformly continuous derivatives up to order j. Namely, this is equivalent to the embedding $E \hookrightarrow L^{n/(s-j),1}$.

The problem of the optimal target rearrangement invariant space for potential type operators is considered in [7] by using L^p-capacities. The problem of the mapping properties of the Riesz potential in optimal couples of rearrangement invariant spaces is treated in [6], [4], [12]. The characterization of the continuous embedding of the generalized Bessel potential spaces into the generalized Hölder-Zygmund spaces CH, when H is a weighted

Lebesgue space, is given in [5]. Our method is different and more general and it could be applied for Bessel potentials as well.

The plan of the paper is as follows. In Section 2 we provide some basic definitions and known results. In Section 3 we characterize the continuity of the Riesz potential operator $R^s : E \mapsto CH$. The optimal quasi-norms are constructed in Section 4.

2 Preliminaries

We use the notations $a_1 \lesssim a_2$ or $a_2 \gtrsim a_1$ for nonnegative functions or functionals to mean that the quotient a_1/a_2 is bounded; also, $a_1 \approx a_2$ means that $a_1 \lesssim a_2$ and $a_1 \gtrsim a_2$. We say that a_1 is equivalent to a_2 if $a_1 \approx a_2$.

Let E be a quasi-normed rearrangement invariant space as in the Introduction. There is an equivalent quasi-norm $\rho_p \approx \rho_E$ that satisfies the triangle inequality $\rho_p^p(g_1+g_2) \le \rho_p^p(g_1) + \rho_p^p(g_2)$ for some $p \in (0,1]$ that depends only on the space E (see [9]. We say that the norm ρ_E satisfies Minkowski's inequality if for the equivalent quasi-norm ρ_p,

$$\rho_p^p\Big(\sum g_j\Big) \lesssim \sum \rho_p^p(g_j),\ g_j \in M^+. \tag{2.1}$$

Usually we apply this inequality for functions $g \in M^+$ with some kind of monotonicity.

Recall the definition of the lower and upper Boyd indices α_E and β_E. Let $g_u(t) = g(t/u)$ if $t < u$ and $g_u(t) = 0$ if $t \ge u$, where $0 < t < 1$, $g \in M^+$, and let

$$h_E(u) = \sup\left\{\frac{\rho_E(g_u^*)}{\rho_E(g^*)} : g \in M^+\right\},\ u > 0,$$

be the dilation function generated by ρ_E. Suppose that it is finite. Then

$$\alpha_E := \sup_{0<t<1} \frac{\log h_E(t)}{\log t} \text{ and } \beta_E := \inf_{1<t<\infty} \frac{\log h_E(t)}{\log t}.$$

The function h_E is submultiplicative, increasing, $h_E(1) = 1$, $h_E(u)h_E(1/u) \ge 1$, hence $0 \le \alpha_E \le \beta_E$. We suppose that $0 < \alpha_E = \beta_E < 1$.

Since $\beta_E < 1$ we have by using Minkowski's inequality that $\rho_E(f^*) \approx \rho_E(\chi_{(0,1)} f^{**})$, where $f^{**}(t) := \frac{1}{t}\int_0^t f^*(s)ds$ and $\chi_{(a,b)}$, $0 < a < b < \infty$, is the characteristic function of the interval (a,b). In particular, $\|f\|_E \approx \rho_E(\chi_{(0,1)} f^{**})$. For example, consider the Gamma spaces $\Gamma^q(w)$, $0 < q \le \infty$, w - positive weight, i.e. a positive function from M^+ , with a quasi-norm $\|f\|_{\Gamma^q(w)} := \rho_{w,q}(f^{**})$, where

$$\rho_{w,q}(g) := \left(\int_0^1 [g(t)w(t)]^q dt/t\right)^{1/q},\ g \in M^+, \tag{2.2}$$

and

$$\left(\int_0^1 w^q(t)dt/t\right)^{1/q} < \infty.$$

Then $L^\infty(\Omega) \hookrightarrow \Gamma^q(w) \hookrightarrow L^1(\Omega)$. If $w(t) = t^{1/p}$, $1 < p < \infty$, we write as usual $L^{p,q}$ instead of $\Gamma^q(t^{1/p})$.

We need the modified dilation function $\tilde{h}_E$, generated by ρ_E, without supposing that ρ_E is rearrangement invariant, as follows

$$\tilde{h}_E(u) = \sup\left\{\frac{\rho_E(g_u)}{\rho_E(g)} : g \in M_1\right\},$$

where

$$M_1 = \{g \in M^+ : t^{s/n-1}g(t) \text{ is decreasing}\}.$$

This function is submultiplicative, $u^{1-s/n}\tilde{h}_E$ is increasing, $\tilde{h}_E(u)\tilde{h}_E(1/u) \geq 1$, $\tilde{h}_E(1) = 1$. Suppose that it is finite. Then

$$\tilde{\alpha}_E := \sup_{0<t<1} \frac{\log \tilde{h}_E(t)}{\log t} \text{ and } \tilde{\beta}_E := \inf_{1<t<\infty} \frac{\log \tilde{h}_E(t)}{\log t}.$$

We have $h_E \leq \tilde{h}_E$ and as a consequence, $\tilde{\alpha}_E \leq \alpha_E \leq \beta_E \leq \tilde{\beta}_E$. We suppose that $\tilde{\alpha}_E = \tilde{\beta}_E$, hence $0 < \tilde{\alpha}_E = \tilde{\beta}_E = \alpha_E = \beta_E < 1$. For example, if $E = L^r$ and $\rho_E(g) = (\int_0^1 g^r(t)dt)^{1/r}$, then $\tilde{\alpha}_E = \tilde{\beta}_E = 1/r$. This technical tool will simplify our investigations. Note that if $E = \Gamma^q(t^\alpha w)$, $0 < \alpha < 1$, where w is slowly varying, then $\alpha_E = \beta_E = \alpha$. Recall that $w \in M^+$ is slowly varying if for all $\varepsilon > 0$ the function $t^\varepsilon w(t)$ is equivalent to an increasing function, and the function $t^{-\varepsilon}w(t)$ is equivalent to a decreasing function.

In order to introduce the Hölder-Zygmund class of spaces, we denote the modulus of continuity of order k by

$$\omega^k(t, f) = \sup_{|h|\leq t} \sup_{x\in\mathbf{R}^n} |\Delta_h^k f(x)|.$$

where $\Delta_h^k f$ are the usual iterated differences of f. When $k = 1$ we simply write $\omega(t, f)$. Let H be a quasi-normed space of locally integrable functions on the interval $(0, 1)$ with the Lebesgue measure, continuously embedded in $L^\infty(0, 1)$ and $\|g\|_H = \rho_H(|g|)$, where ρ_H is a monotone quasi-norm on M^+ which satisfies Minkowski's inequality. The dilation function generated by ρ_H is given by

$$h_H(u) = \sup\left\{\frac{\rho_H(\tilde{g}_u)}{\rho_H(g)} : g \in L\right\},$$

where $\tilde{g}_u(t) = g(ut)$ if $ut < 1$, $\tilde{g}_u(t) = g(1)$ if $ut \geq 1$, $0 < t < 1$, and

$$L := \{g \in M^+ : t^{-1}g(t) \text{ is decreasing}\}.$$

The choice of the space L is motivated by the fact that $\chi_{(0,1)}(t)\omega^n(t^{1/n}, f)$ is equivalent to a function $g \in L$. The function $h_H(u)$ is sub-multiplicative, $u^{-1}h_H(u)$ is decreasing and $h_H(1) = 1$, $h_H(u)h_H(1/u) \geq 1$. Suppose that it is finite. Then the Boyd indices of H are well-defined

$$\alpha_H = \sup_{0<t<1} \frac{\log h_H(t)}{\log t} \quad \text{and} \quad \beta_H = \inf_{1<t<\infty} \frac{\log h_H(t)}{\log t},$$

and they satisfy $0 \leq \alpha_H \leq \beta_H \leq 1$. In what follows, we suppose that $\alpha_H = \beta_H < 1$. For example, let $H = L_*^q(b(t)t^{-\gamma/n})$. Here $0 \leq \gamma < n$ and b is a slowly varying function, and $L_*^q(w)$, or simply L_*^q if $w = 1$, is the weighted Lebesgue space with a quasi-norm $\|g\|_{L_*^q(w)} = \rho_{w,q}(|g|)$, where $\rho_{w,q}$ is given by (2.2). It turns out that $\alpha_H = \beta_H = \gamma/n$.

Definition 2.1. Let $j = 0,1,\dots$ and let C^j stand for the space of all functions f, defined on $\mathbf{R}^n$, that have bounded and uniformly continuous derivatives up to the order j, normed by $\|f\|_{C^j} = \sup\sum_{l=0}^{j}|P^l f(x)|$, where $P^l f(x) = \sum_{|\nu|=l} D^\nu f(x)$.

- If $j/n < \alpha_H < (j+1)/n$ for $j \geq 1$ or $0 \leq \alpha_H < 1/n$ for $j = 0$, then CH is formed by all functions f in C^j having a finite quasi-norm

$$\|f\|_{CH} = \|f\|_{C^j} + \rho_H(\chi_{(0,1)}(t)t^{j/n}\omega(t^{1/n}, P^j f)).$$

- If $\alpha_H = (j+1)/n$, then CH consists of all functions f in C^j having a finite quasi-norm

$$\|f\|_{CH} = \|f\|_{C^j} + \rho_H(\chi_{(0,1)}(t)t^{j/n}\omega^2(t^{1/n}, P^j f)).$$

In particular, if $H = L^\infty(t^{-\gamma/n})$, $\gamma > 0$, then CH coincides with the usual Hölder-Zygmund space C^γ (see [11]). Also, if $H = L^\infty$, then $CH = C^0$.

We shall use the following equivalent quasi-norm (see [1] for an analogous proof):

Theorem 2.2. *(equivalence) Let* $0 \leq \alpha_H = \beta_H < 1$. *If* $\rho_H(\chi_{(0,1)}(t)t^\alpha) < \infty$ *for* $\alpha > \alpha_H$, *then for all* $m \geq n$,

$$\|f\|_{CH} \approx \|f\|_{C^0} + \rho_H(\chi_{(0,1)}(t)\omega^m(t^{1/n}, f)).$$

Note that if $\rho_H(\chi_{(0,1)}(t)t^{m/n}) < \infty$, then CH is a $K-$interpolation space for the couple (C^0, C^m), namely $CH = (C^0, C^m)_{H_1}$, where $\rho_{H_1}(g) = \rho_H(g(t^{m/n}))$. In particular, $CL^1_*(t^{-j}) \hookrightarrow C^j \hookrightarrow CL^\infty(t^{-j})$.

Recall some basic definitions from the theory of interpolation spaces [3]. Let (A_0, A_1) be a couple of two quasi-normed spaces, such that both are continuously embedded in some quasi-normed space and let

$$K(t,f) = K(t,f;A_0,A_1) = \inf_{f=f_0+f_1}\{\|f_0\|_{A_0} + t\|f_1\|_{A_1}\},\ f \in A_0 + A_1,$$

be the $K-$functional of Peetre. By definition, the $K-$interpolation space $A_\Phi = (A_0, A_1)_\Phi$ has a quasi-norm $\|f\|_{A_\Phi} = \|K(t,f)\|_\Phi$, where Φ is a quasi-normed function space with a monotone quasi-norm on $(0,\infty)$ with the Lebesgue measure and such that $\min\{1,t\} \in \Phi$. Then $A_0 \cap A_1 \hookrightarrow A_\Phi \hookrightarrow A_0 + A_1$. If

$$\|g\|_\Phi = \left(\int_0^\infty [w(t)t^{-\theta}g(t)]^q dt/t\right)^{1/q},\ 0 \leq \theta \leq 1,\ 0 < q \leq \infty,\ w \in \mathcal{M}^+,$$

we write $(A_0, A_1)_{wt^{-\theta},q}$ instead of $(A_0, A_1)_\Phi$. Also, if $w = 1$ then we write $(A_0, A_1)_{\theta,q}$. By definition,

$$\|f\|_{A_0\cap A_1} = \|f\|_{A_0} + \|f\|_{A_1},\ \|f\|_{A_0+A_1} = K(1,f;A_0,A_1).$$

It will be convenient to use the following definitions.

Definition 2.3. (admissible couple) We say that the couple ρ_E, ρ_H is admissible for the Riesz potential if

$$\|R^s f\|_{CH} \lesssim \rho_E(f^*),\ f \in L^1(\Omega). \tag{2.3}$$

Moreover, ρ_E (E) is called domain quasi-norm (domain space), and ρ_H (H) is called target quasi-norm (target space).

Let

$$M_0 = \{g \in M^+ : g \text{ is increasing, } t^{-1}g(t) \text{ is decreasing and } g(+0) = 0\}.$$

The choice of M_0 is motivated by the fact that $\omega^n(t^{1/n}, R^s f)$, $0 < t < 1$, is equivalent to a function $g \in M_0$ if $f \in E$ and $E \hookrightarrow L^{n/s,1}$.

Definition 2.4. (optimal target quasi-norm) Given the domain quasi-norm ρ_E, the optimal target quasi-norm, denoted by $\rho_{H(E)}$, is the strongest target quasi-norm on the interval $(0,1)$, i.e.

$$\rho_H(g) \lesssim \rho_{H(E)}(g),\ g \in M_0 \tag{2.4}$$

for any target quasi-norm ρ_H such that the couple ρ_E, ρ_H is admissible. Since $CH(E) \hookrightarrow CH$, we call $CH(E)$ the optimal Hölder-Zygmund space.

Definition 2.5. (optimal domain quasi-norm) Given the target quasi-norm ρ_H, the optimal domain quasi-norm, denoted by $\rho_{E(H)}$, is the weakest domain quasi-norm, i.e.

$$\rho_{E(H)}(f^*) \lesssim \rho_E(f^*),\ f \in L^1(\Omega), \tag{2.5}$$

for any domain quasi-norm ρ_E such that the couple ρ_E, ρ_H is admissible.

Definition 2.6. (optimal couple) The admissible couple ρ_E, ρ_H is said to be optimal if both ρ_E and ρ_H are optimal.

3 Admissible couples

Here we give a characterization of all admissible couples ρ_E, ρ_H. It will be convenient to introduce the classes of the domain and target quasi-norms, where the optimality is investigated. Let N_d consist of all domain quasi-norms ρ_E that are monotone, satisfy Minkowski's inequality, $0 < \alpha_E = \beta_E < 1$, the condition (3.3) below, $L^\infty(\Omega) \hookrightarrow E \hookrightarrow L^1(\Omega)$ and $\rho_E(\chi_{(0,1)}t^{-\alpha}) < \infty$ if $\alpha < \alpha_E$. Let N_t consist of all target quasi-norms ρ_H that are monotone, satisfy Minkowski's inequality, $0 \le \alpha_H = \beta_H < 1$, $\rho_H(\chi_{(0,1)}(t)t^\alpha) < \infty$ if $\alpha > \alpha_H$ and

$$\sup g(t) \lesssim \rho_H(g),\ g \in M^+. \tag{3.1}$$

We start with the main estimate.

Theorem 3.1. *Let $f \in L^1(\Omega)$. Then*

$$\omega^m(t^{1/n}, R^s f) \lesssim S(f^*)(t),\ s < m, \tag{3.2}$$

where

$$S g(t) = \int_0^t u^{s/n-1} g(u)du,\ g \in M^+.$$

Proof. Let

$$R^s f(x) = f_{1t}(x) + f_{2t}(x),\ f_{jt}(x) = \int_{\mathbf{R}^n} f(y)\psi_{jt}(|x-y|)|x-y|^{s-n}dy,$$

where $\psi_{1t} \in C_0^\infty(-a,a)$, $a = ct^{1/n}$, $\psi_{1t}(u) = 1$ if $u \in (-b,b)$, $b = c_1 t^{1/n}$, $c_1 < c$, $0 \le \psi_{1t} \le 1$ and let $\psi_{2t} = 1 - \psi_{1t}$. Then

$$R^s f(x) = f_{1t}(x) + f_{2t}(x),\ f_{jt}(x) = \int_{\mathbf{R}^n} f(y)\psi_{jt}(|x-y|)|x-y|^{s-n}dy.$$

We have for appropriate c and using the Hardy-Littlewood inequality,

$$|\Delta_h^m f_{1t}(x)| \lesssim \int_{\mathbf{R}^n} |\Delta_h^m f(x-y)|\psi_{1t}(|y|)|y|^{s-n}dy \lesssim S f^*(t),$$

since $h_t^*(u) \approx u^{s/n-1}\chi_{(0,t)}(u)$, if $h_t(y) = |y|^{s-n}$ for $|y| \le ct^{1/n}$ and $h_t(y) = 0$ otherwise.

On the other hand, using the formula (4.16), p. 336 [2], we can write for $|h| \le t^{1/n}$,

$$|\Delta_h^m f_{2t}(x)| \lesssim \int_{-\infty}^{\infty} \sum_{j=0}^{m} t^{j/n} g_j(x+uh)M_m(u)du,$$

where

$$g_j(z) = \int_{B_j} |f(y)||z-y|^{s-n-j}dy,$$

$$B_j = \{y : c_1 t^{1/n} \le |z-y| \le ct^{1/n}\},\ \text{if } 0 \le j < m,\ B_m = \{y : |z-y| > ct^{1/n}\}.$$

Hence

$$g_j(z) \lesssim t^{-j/n} S f^*(t),\ 0 \le j < m.$$

Also

$$g_m(z) \lesssim \int_0^\infty f^*(u)(u+t)^{s/n-1-m/n}du \lesssim t^{-m/n} S f^*(t),$$

since $\int_t^\infty f^*(u)(u+t)^{s/n-1-m/n}du \lesssim t^{\frac{s-m}{n}} f^*(t) \lesssim t^{-m/n} S f^*(t)$ for $m > s$. Thus (3.2) follows. □

Now we discuss the mapping property $R^s : E \mapsto C^0$.

Theorem 3.2. *A necessary and sufficient condition for the mapping $R^s : E \mapsto C^0$ is the following one*

$$\int_0^1 t^{s/n-1} g(t)dt \lesssim \rho_E(g),\ g \in M_1. \tag{3.3}$$

Proof. Using the Hardy-Littlewood inequality

$$\int_{\mathbf{R}^n} |f(x)g(x)|dx \le \int_0^\infty f^*(t)g^*(t)dt,$$

we get the well known mapping property

$$R^s : \Gamma^1(t^{s/n}) \mapsto L^\infty.$$

From (3.3) it follows

$$R^s : E \to L^\infty. \tag{3.4}$$

To prove that $R^s(E) \subset C^0$, it remains to show that $\lim_{t\to 0}\omega(t^{1/n}, R^s f) = 0$ if $f \in E$. By Marchaud's inequality (see [2], Theorem 5.4.4), we have

$$\omega(t^{1/n}, R^s f) \lesssim t^{1/n}\int_t^\infty u^{-1/n}\omega^m(u^{1/n}, R^s f)\frac{du}{u}.$$

By Lopital's rule, it is enough to check that $\lim_{t\to 0}\omega^m(t^{1/n}, R^s f) = 0$ if $f \in E$. But this follows from (3.2) and (3.3).

Before proving the reverse, note that (3.3) is always satisfied if $s/n > \alpha_E$. To see this, we need the estimate

$$g(u) \lesssim \tilde{h}_E(1/u)\rho_E(g),\ g \in M_1. \tag{3.5}$$

Indeed, since $0 < u < 1$, we have

$$\rho_E(\chi_{(0,1)}(t)t^{1-s/n})g(u) \le \rho_E(g(tu)) \le \tilde{h}_E(1/u)\rho_E(g).$$

Hence for $0 < \varepsilon < s/n - \alpha_E$,

$$\int_0^1 u^{s/n-1}g(u)du \lesssim \rho_E(g)\int_0^1 u^{s/n-\alpha_E-\varepsilon}du/u \lesssim \rho_E(g),\ g \in M_1.$$

It remains to prove that if $R^s : E \mapsto C^0$ then (3.3) is true for $\alpha_E = s/n$. To this end, we choose a test function h as follows. Let $g \in M_1$ and

$$h(x) = \int_0^1 g(u)\varphi(|x|u^{-1/n})\frac{du}{u}, \tag{3.6}$$

where $\varphi \ge 0$ is a smooth function with compact support in $(-c^{-1/n}, c^{-1/n})$ such that if $\psi = R^s\varphi$ then $\psi(0) > 0$. Note that h has a compact support, $h(x) \lesssim \int_{c|x|^n}^1 g(u)du/u$ and for appropriate $c > 0$, $h^*(t) \lesssim \int_t^1 g(u)du/u$. Now Minkowski's inequality gives $\rho_E(h^*) \lesssim \rho_E(g)$ since $\alpha_E > 0$. Also $R^s h(0) = \psi(0)/n\int_0^1 u^{s/n-1}g(u)du \lesssim \|h\|_E \lesssim \rho_E(g)$. Thus (3.3) is proved.

□

Remark 3.3. Similar arguments show that $R^s : E \mapsto C^j$, $j < s$, if and only if $E \hookrightarrow L^{n/(s-j),1}$.

Theorem 3.4. *The couple $\rho_E \in N_d$, $\rho_H \in N_t$ is admissible if and only if*

$$\rho_H(Sg) \lesssim \rho_E(g),\ g \in M_1. \tag{3.7}$$

Proof. It is clear that (2.3) follows from (3.7), (3.2) and (3.4). Now we prove that (2.3) implies (3.7). To this end we choose the test function in the form $f(x) = R^s h(x)$, where h is given by (3.6). Note that $\psi(u) \lesssim u^{s-n}$ for $u > c$.

Let $|h| = Ct^{1/n}$. We split $f = f_{1t} + f_{2t}$, $0 < t < 1$,

$$f_{1t}(x) = \int_0^t u^{s/n}g(u)\psi(|x|u^{-1/n})du/u,\ f_{2t}(x) = \int_t^1 u^{s/n}g(u)\psi(|x|u^{-1/n})du/u,$$

First we prove that for some large $C > 0$,

$$\omega^m(Ct^{1/n}, f_{1t}) \geq \frac{1}{2}\psi(0)Sg(t),\ 0 < t < 1. \tag{3.8}$$

Indeed, we have $\omega^m(Ct^{1/n}, f_{1t}) \geq |(\Delta_h^m f_{1t})(0)|$ and $\psi(jCt^{1/n}u^{-1/n}) \lesssim C^{s-n} < \psi(0)/2$ for $1 \leq j \leq m$. Hence (3.8) follows. Further,

$$\omega^m(t^{1/n}, f) \geq \omega^m(t^{1/n}, f_{1t}) - \omega^m(t^{1/n}, f_{2t}),\ 0 < t < 1. \tag{3.9}$$

Since

$$\omega^m(t^{1/n}, f_{2t}) \lesssim t^{m/n}\|P^m f_{2t}\|_{L^\infty} \lesssim t^{m/n}\int_t^1 u^{(s-m)/n}g(u)du/u$$

and $g \in M_1$, we get $\omega^m(t^{1/n}, f_{2t}) \lesssim t^{m/n}\int_t^1 u^{-m/n}Sg(u)du/u$. Therefore

$$Sg(t) \leq c_1\omega^m(t^{1/n}, R^s h) + ct^{m/n}\int_t^1 u^{-m/n}Sg(u)du/u,\ 0 < t < 1 \tag{3.10}$$

and

$$Sg(t) \leq c_1\omega^m(t^{1/n}, R^s h) + ct^{m/n}\int_t^1 u^{-m/n}g(u)du/u,\ 0 < t < 1 \tag{3.11}$$

To solve the integral inequality (3.10) for $p(t) := t^{-m/n}Sg(t)$, we set $q(t) = c_1 t^{-m/n}\omega^m(t^{1/n}, R^s h)$ and rewrite it as $p(t) \leq q(t) + c\int_t^1 p(u)du/u$, $0 < t < 1$. If $r(t) = \int_t^1 p(u)du/u$ then we get the differential inequality $0 \leq tr'(t) + cr(t) + q(t)$. If $r(t) = t^{-c}v(t)$, then $0 \leq v'(t) + t^{c-1}q(t)$, whence $v(t) \leq \int_t^1 u^{c-1}q(u)du$. Therefore

$$\chi_{(0,1/2)}(t)Sg(t) \lesssim t^{m/n-c}\int_t^1 u^{c-m/n}\omega^m(u^{1/n}, R^s h)du/u.$$

Hence by using Minkowski's inequality and choosing m large enough, we obtain

$$\rho_H(\chi_{(0,1/2)}Sg) \lesssim \rho_H(\chi_{(0,1)}(t)\omega^m(t^{1/n}, R^s h).$$

On the other hand, from (3.11) it follows that

$$\rho_H(\chi_{(1/2,1)}Sg) \leq \rho_H(\chi_{(0,1)}(t)\omega^m(t^{1/n}, R^s h)) + \int_0^1 g(u)du.$$

Hence, using also (3.3), we get

$$\rho_H(Sg) \lesssim \rho_H(\chi_{(0,1)}(t)\omega^m(t^{1/n}, R^s h) + \rho_E(g). \tag{3.12}$$

On the other hand, as above

$$\rho_E(h^*) \lesssim \rho_E(g),\ \alpha_E > 0,\ g \in M_1. \tag{3.13}$$

Thus, if (2.3) is given, then (3.12), (3.13) imply (3.7).

□

4 Optimal quasi-norms

Here we give a characterization of the optimal domain and optimal target quasi-norms.

4.1 Optimal domain quasi-norms

We can construct an optimal domain quasi-norm $\rho_{E(H)}$ by Theorem 3.4 as follows.

Definition 4.1. (construction of an optimal domain quasi-norm) For a given target quasi-norm $\rho_H \in N_t$, we set

$$\rho_{E(H)}(g) := \rho_H(Sg),\ g \in M^+. \tag{4.1}$$

Note that $S(g_u) = u^{s/n}(\widetilde{Sg})_{1/u}$ and $Sg \in L$ if $g \in M_1$. Hence $\alpha_{E(H)} = \beta_{E(H)} = s/n - \alpha_H$.

Theorem 4.2. *The quasi-norm $\rho_{E(H)}$ belongs to N_d, the couple $\rho_{E(H)}$,ρ_H is admissible, the domain quasi-norm $\rho_{E(H)}$ is optimal. Moreover, the target quasi-norm ρ_H is also optimal and*

$$\rho_{E(H)}(g) \approx \rho_H(t^{s/n}g),\ g \in M_1 \ \text{ if }\ \alpha_H > 0. \tag{4.2}$$

Proof. It is easy to check that $\rho_{E(H)} \in N_d$. Further, the couple $\rho_{E(H)}$,ρ_H is admissible since $\rho_H(Sg) = \rho_{E(H)}(g),\ g \in M_1$. Moreover, $\rho_{E(H)}$ is optimal, since for any admissible couple $\rho_{E_1} \in N_d, \rho_H$ we have $\rho_H(Sg) \lesssim \rho_{E_1}(g)$, where $g \in M_1$. Therefore,

$$\rho_{E(H)}(f^*) = \rho_H(S(f^*)) \lesssim \rho_{E_1}(f^*),\ f \in L^1(\Omega).$$

To prove that ρ_H is also optimal, let $\rho_{E(H)}, \rho_{H_1} \in N_t$ be an arbitrary admissible couple. Then

$$\rho_{H_1}(Sg) \lesssim \rho_{E(H)}(g),\ g \in M_1.$$

We have to show that

$$\rho_{H_1}(g) \lesssim \rho_H(g),\ g \in M_0. \tag{4.3}$$

Since $g \in M_0$ is quasi-concave, it is equivalent to a concave one, hence $g(t) \approx \int_0^t h_1(u)du$ for some decreasing $h_1 \in M^+$. Let $h(t) = t^{1-s/n}h_1(t)$. Then $h \in M_1$ and $g \approx Sh$. Therefore

$$\rho_{H_1}(g) \lesssim \rho_{H_1}(Sh) \lesssim \rho_{E(H)}(h) \lesssim \rho_H(Sh) \lesssim \rho_H(g).$$

Thus (4.3) is proved. To prove the equivalence (4.2), we use $t^{s/n}g(t) \lesssim Sg(t),\ g \in M_1$, hence $t^{s/n}g(t) \in L$, and Minkowski's inequality as follows:

$$\rho_H^p(Sg) \lesssim \sum_{k=-\infty}^{0} h_H^p(2^k)\rho_H^p(t^{s/n}g(t)),\ g \in M_1, \alpha_H > 0,$$

whence $\rho_{E(H)}(g) \lesssim \rho_H(t^{s/n}g(t)),\ g \in M_1$.

□

Example 4.3. Consider the space $H = L^1_*(v)$, where $\rho_H(g) = \int_0^1 v(t)g(t)dt/t$ and $\rho_H \in N_t$. Using Theorem 4.2, we can construct an optimal domain E, where

$$\rho_E(g) = \rho_H(Sg) = \int_0^1 t^{s/n} w(t)g(t)dt/t, \; g \in M^+$$

and $w(t) = \int_t^1 v(u)du/u$. Hence $E = \Gamma^1(t^{s/n}w)$ and this couple is optimal. Also $\alpha_E = \beta_E = s/n$ if v is slowly varying.

Example 4.4. Let $H = L^\infty(v)$, where $\rho_H(g) = \sup v(t)g(t)$ and $\rho_H \in N_t$ and let

$$\rho_E(g) = \sup v(t) \int_0^t u^{s/n} g(u)du/u, \; g \in M^+.$$

Then by Theorem 4.2, the domain E is optimal and the couple is optimal. In particular, the couple $L^{n/s,1}$, C^0 is optimal.

4.2 Optimal target quasi-norms

Definition 4.5. (construction of the optimal target quasi-norm) For a given domain quasi-norm $\rho_E \in N_d$, we set

$$\rho_{H(E)}(g) := \inf\{\rho_E(h) : g \leq Sh, \; h \in M_1\}, \; g \in M^+. \tag{4.4}$$

Note that $\alpha_{H(E)} = \beta_{H(E)} = s/n - \alpha_E$.

Theorem 4.6. *The target quasi-norm $\rho_{H(E)}$ belongs to N_t, the couple $\rho_E, \rho_{H(E)}$ is admissible and the target quasi-norm is optimal.*

Proof. The property "$\rho_{H(E)}(g) = 0$, $g \in M^+$, implies $g = 0$" follows from (3.3). Also, since $\rho_E \in N_d$ it is easy to check that $\rho_{H(E)} \in N_t$. The couple is admissible since $\rho_{H(E)}(Sh) \leq \rho_E(h)$, $h \in M_1$. Suppose that the couple $\rho_E, \rho_{H_1} \in N_t$ is admissible. Then $\rho_{H_1}(Sh) \lesssim \rho_E(h)$, $h \in M_1$. Therefore if $g \leq Sh$, $h \in M_1$, then $\rho_{H_1}(g) \leq \rho_{H_1}(Sh) \lesssim \rho_E(h)$, whence $\rho_{H_1}(g) \lesssim \rho_{H(E)}(g)$, $g \in M_0$. Hence $\rho_{H(E)}$ is optimal.

□

Theorem 4.7. *If $\alpha_E < s/n$, then*

$$\rho_{H(E)}(g) \approx \rho_E(t^{-s/n} g(t)), \; g \in M_0.$$

Moreover, the couple ρ_E,$\rho_{H(E)}$ is optimal.

Proof. If $g \leq Sh$, $h \in M_1$, then by Minkowski's inequality,

$$\rho_E(t^{-s/n} g(t)) \leq \rho_E(t^{-s/n} Sh(t)) \lesssim \rho_E(h), \; h \in M_1, \; s/n > \alpha_E.$$

Hence, taking the infimum, we get $\rho_E(t^{-s/n} g(t)) \lesssim \rho_{H(E)}(g)$.

On the other hand, for $g \in M_0$, we have $g \lesssim Sh$, $h(t) = t^{-s/n} g(t)$. Since $h \in M_1$ it follows $\rho_{H(E)}(g) \lesssim \rho_E(t^{-s/n} g(t))$.

The domain quasi-norm ρ_E is also optimal since

$$\rho_{E(H(E))}(f^*) = \rho_{H(E)}(Sf^*) \approx \rho_E(t^{-s/n} Sf^*(t)) \gtrsim \rho_E(f^*), \; f \in L^1(\Omega).$$

□

Example 4.8. Consider the space $E = \Gamma^q(w)$, $0 < q \le \infty$, $s/n > \beta_E = \alpha_E > 0$. Then by Theorem 4.7 the couple ρ_E, ρ_H, $H = L^q_*(t^{-s/n}w)$ is optimal. In particular, the couple $L^{p,\infty}$, $C^{s-n/p}$, $s > n/p$, $1 < p < \infty$, is optimal.

In the critical case we do not know how to simplify the optimal target quasi-norm, defined in (4.4). Instead, we can construct a large class of domain quasi-norms and the corresponding optimal target quasi-norms by using extrapolation from the super-critical case. Recall some basic definitions and results from the extrapolation theory [8]. Let (A_0, A_1) be a couple of quasi-Banach spaces. The sigma extrapolation space $\Sigma^q(M(\sigma)(A_0,A_1)_{a(t)t^{-\sigma},q})$, a - positive weight, $0 < \sigma < \sigma_0$, $0 < q \le \infty$, M - positive decreasing weight, consists of all $f \in A_0 + A_1$ such that $f = \sum_{j=l}^{\infty} g_j$, $g_j \in A_j$, $A_j := (A_0, A_1)_{a(t)t^{-\frac{1}{2^j}},q}$, with a quasi-norm

$$\|f\|_{\Sigma^q(M(\sigma)(A_0,A_1)_{a(t)t^{-\sigma},q})} = \inf\left(\sum_{j=l}^{\infty}\left[M(2^{-j})\|g_j\|_{A_j}\right]^q\right)^{1/q},$$

where the infimum is taken with respect to all representations $f = \sum_{j=l}^{\infty} g_j$.

This space can be characterized as an interpolation space.

Theorem 4.9. *([8]) Let $a(t) = t^{-\theta}b(t)$, b - slowly varying, $0 < \theta < 1$. Then*

$$\Sigma^q(M(\sigma)(A_0,A_1)_{a(t)t^{-\sigma},q}) = (A_0,A_1)_{w,q},$$

where

$$\frac{1}{w(t)} = \frac{1}{a(t)}\left(\int_0^{\sigma_0}\left[\frac{t^{\sigma}}{M(\sigma)}\right]^r\frac{d\sigma}{\sigma}\right)^{1/r} \tag{4.5}$$

and $1/r + 1/q = 1$ if $q > 1$, $r = \infty$ if $0 < q \le 1$.

Our main result is the following one.

Theorem 4.10. *Let $E = \Gamma^q(t^{s/n}c(t)(1-\ln t))$, $0 < s < n$, c - slowly varying weight, $c(+0) = \infty$, $c(t^2) \approx c(t)$, $0 < q \le \infty$, $H = L^q_*(c)$. We suppose that $\rho_E \in N_d$ and $\rho_H \in N_t$. Then this couple is admissible and the target quasi-norm is optimal.*

Proof. Step 1 (admissibility). Since $\alpha_E = \beta_E = s/n < 1$, it will be enough to check that

$$\rho_H(S(g^{**})) \lesssim \rho_E(g^{**}), \tag{4.6}$$

where

$$\rho_E(g) = \left(\int_0^1 [t^{s/n}c(t)(1-\ln t)g(t)]^q dt/t\right)^{1/q}, \ \rho_H(g) = \left(\int_0^1 [c(t)g(t)]^q dt/t\right)^{1/q}.$$

Applying Minkowski's inequality we obtain for $0 < \sigma < \sigma_0 < s/n$, b -slowly varying weight,

$$\sigma\|S(g^{**})\|_{L^q_*(b(t)t^{-\sigma})} \lesssim \|g\|_{\Gamma^q(t^{s/n-\sigma}b(t))}.$$

In order to extrapolate these inequalities, we write

$$\Gamma^q(t^{s/n-\sigma}b(t)) = (L^1, L^\infty)_{b(t)t^{s/n-1-\sigma},q}$$

and

$$L_*^q(t^{-\sigma}b(t)) = (L_*^q(t^{1/2}b(t)), L_*^q(t^{-1/2}b(t)))_{1/2+\sigma,q},\ \sigma_0 < 1/2.$$

This is true since

$$K(t,g;L_*^q(w_0),L_*^q(w_1)) \approx \left(\int_0^1 [g(u)\min(w_0(u),tw_1(u))]^q du/u\right)^{1/q},\ 0<t<1.$$

Let $\sigma = 2^{-j}$ and $g = \sum g_j$ (convergence in L^1), where $g_j \in L^\infty$. Then $g^{**} \le \sum g_j^{**}$, whence $S(g^{**}) \le \sum S(g_j^{**})$ and for $M(\sigma) = \sigma^{-2}$, $p = \min(q,1)$, we have

$$K^p(t,S(g^{**});B_0,B_1) \le C_v := \sum_{j\ge l} K^p(t,S(g_j^{**});B_0,B_1),$$

where $B_0 = L_*^q(t^{1/2}b(t))$, $B_1 = L_*^q(t^{-1/2}b(t))$. We can write

$$C_v = \sum_{j\ge l}[t^{-1/2-2^{-j}}2^{-j}M(2^{-j})K(t,S(g_j^{**});B_0,B_1)]^p\left[\frac{t^{1/2+2^{-j}}}{2^{-j}M(2^{-j})}\right]^p$$

and using also Hölder's inequality if $q>1$, we get

$$[v(t)]^p C_v \le \sum_{j\ge l}[t^{-1/2-2^{-j}}2^{-j}M(2^{-j})K(t,S(g_j^{**});B_0,B_1)]^p,$$

where

$$\frac{1}{v(t)} = \left(\sum_{j\ge l}\left[\frac{t^{1/2+2^{-j}}}{2^{-j}M(2^{-j})}\right]^r\right)^{1/r}.$$

Hence

$$\|S(g^{**})\|_{(B_0,B_1)_{v,q}} \lesssim \left(\sum_{j\ge l}\left[2^{-j}M(2^{-j})\|S(g_j^{**})\|_{(B_0,B_1)_{1/2+2^{-j},q}}\right]^q\right)^{1/q}.$$

Since

$$2^{-j}\|S(g_j^{**})\|_{(B_0,B_1)_{1/2+2^{-j},q}} \lesssim \|g_j\|_{\Gamma^q(t^{s/n-2^{-j}}b(t))},$$

we get

$$\|S(g^{**})\|_{(B_0,B_1)_{v,q}} \lesssim \|g\|_{\Sigma^q(M(\sigma)(L^1,L^\infty)_{b(t)t^{s/n-1-\sigma},q})},$$

whence

$$S : ((L^1,L^\infty)_{w,q} \mapsto (L_*^q(t^{1/2}b(t)), L_*^q(t^{-1/2}b(t))_{v,q},$$

where w is given by (4.5) with $a(t) = b(t)t^{s/n-1}$ and $M(\sigma) = \sigma^{-2}$. It is easy to calculate these weights, see [8]. We have

$$w(t) \approx b(t)t^{s/n-1}(1-\ln t)^2,\ v(t) \approx t^{-1/2}(1-\ln t),\ 0<t<1.$$

Then for $b(t) = c(t)(1-\ln t)^{-1}$ we get

$$\Gamma^q(t^{s/n}c(t)(1-\ln t)) \hookrightarrow (L^1,L^\infty)_{w,q}$$

and

$$(L^q_*(t^{1/2}b(t)), L^q_*(t^{-1/2}b(t))_{\nu,q} \hookrightarrow L^q_*(c).$$

Hence (4.6) is proved.

Step 2 (optimality of the target quasi-norm). We want to prove that ρ_H is an optimal target quasi-norm. It is sufficient to see that

$$\rho_{H(E)}(g) \lesssim \rho_H(g),\ g \in M_0,$$

where $\rho_{H(E)}$ is defined by (4.4). To this end for any such g we construct an $h \in M_1$ such that $g \lesssim Sh$ and $\rho_E(h) \lesssim \rho_H(g)$. Let $t^{s/n}(1-\ln t)h(t) = g(\sqrt{te})$. Then $h \in M_1$ and $\rho_E(h) \lesssim \rho_H(g)$. On the other hand,

$$Sh(t) \geq \int_{t^2/e}^{t} \frac{g(\sqrt{eu})}{1-\ln u}\frac{du}{u} \gtrsim g(t),$$

since $\int_{t^2/e}^{t}(1-\ln u)^{-1}du/u = \ln 2$. Then by the definition of $\rho_{H(E)}$ we get

$$\rho_{H(E)}(g) \lesssim \rho_E(h) \lesssim \rho_H(g).$$

□

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