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Nonlinear Fractional Order Riemman-Liouville Volterra-Stieltjes Partial Integral Equations on Unbounded Domains

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Abstract

This paper deals with the existence and the stability of solutions of a class of fractional order functional Riemann-Liouville Volterra-Stieltjes partial integral equations. Our results are obtained by using an extension of the Burton-Kirk fixed point theorem in the case of an unbounded domain.

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1 Introduction

Integral equations are one of the most useful mathematical tools in both pure and applied analysis. This is particularly true of problems in mechanical vibrations and the related fields of engineering and mathematical physics. We can find numerous applications of differential and integral equations of fractional order in viscoelasticity, electrochemistry, control, porous media, electromagnetism, etc., [12, 16, 25]. There has been a significant development in ordinary and partial fractional differential and integral equations in recent

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years; see the monographs of Abbas *et al.* [7], Kilbas *et al.* [19], Miller and Ross [20], Podlubny [22], Samko *et al.* [24], and the papers of Abbas *et al.* [1, 2, 3, 4, 5, 6, 8, 9], Ahmad *et al.* [10], Banaś and Zając [14], Darwish *et al.* [15], Diethelm and Ford [17] and the references therein.

In [5], Abbas *et al.* used the Schauder fixed point theorem in Banach spaces, for the study of the existence of solutions to the following nonlinear quadratic Volterra integral equation of Riemann-Liouville fractional order,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \tfrac{1}{\Gamma(r)}\int_0^{\beta(t)}(\beta(t)-s)^{r-1} \\ &\quad \times g(t,x,s,u(s,x),u(\gamma(s),x))ds,\ (t,x)\in\mathbb{R}_+\times[0,b], \end{aligned} \tag{1.1}$$

where $b>0,\ r\in(0,\infty),\ \alpha,\ \beta,\ \gamma:\mathbb{R}_+\to\mathbb{R}_+,\ f:\mathbb{R}_+\times[0,b]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ and $g:\mathbb{R}_+\times[0,b]\times\mathbb{R}_+\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions and $\Gamma(\cdot)$ is the (Euler's) Gamma function defined by

$$\Gamma(\xi)=\int_0^\infty t^{\xi-1}e^{-t}dt,\ \xi>0.$$

Motivated by that paper, this work deals with the existence and the stability of solutions to the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic partial integral equations,

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \int_0^{\beta(t)}\int_0^x \tfrac{(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\quad \times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s);\ (t,x)\in J, \end{aligned} \tag{1.2}$$

where $J=\mathbb{R}_+\times[0,b],\ b>0,\ \mathbb{R}_+=[0,\infty),\ r_1,r_2\in(0,\infty),\ \alpha,\ \beta,\ \gamma:\mathbb{R}_+\to\mathbb{R}_+,\ f:J\times\mathbb{R}\times\mathbb{R}\to\mathbb{R},\ g:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R},\ h:J'\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ are given continuous functions, $\lim_{t\to\infty}\alpha(t)=\infty$, and $J'=\{(t,x,s,y)\in J^2:\ s\le t,\ y\le x\}$. We use an extension of the Burton-Kirk fixed point theorem in the case of an unbounded domain for the existence of solutions of the equation (1.2), and we prove that all solutions are globally asymptotically stable. Finally, we present an example illustrating the applicability of the imposed conditions.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts which are used throughout this paper. By $L^1([0,p]\times[0,q])$, for $p,q>0$, we denote the space of Lebesgue-integrable functions $u:[0,p]\times[0,q]\to\mathbb{R}$ with the norm

$$\|u\|_1=\int_0^p\int_0^q|u(t,x)|dxdt.$$

As usual, $AC(J)$ is the space of absolutely continuous functions from J into $\mathbb{R}$, and $C(J)$ is the space of all continuous functions from J into $\mathbb{R}$.

Definition 2.1. [26] Let $r=(r_1,r_2)\in(0,\infty)\times(0,\infty),\ \theta=(0,0)$ and $u\in L^1([0,p]\times[0,q])$. The left-sided mixed Riemann-Liouville integral of order r of u is defined by

$$(I_\theta^r u)(t,x)=\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^t\int_0^x(t-\tau)^{r_1-1}(x-s)^{r_2-1}u(s,t)dsd\tau.$$

In particular,

$$(I_\theta^\theta u)(t,x) = u(t,x),\ (I_\theta^\sigma u)(t,x) = \int_0^t \int_0^x u(\tau,s)dsd\tau;$$

for almost all $(t,x) \in [0,p]\times[0,q]$, where $\sigma = (1,1)$.
For instance, $I_\theta^r u$ exists for all $r_1, r_2 \in (0,\infty)$, when $u \in L^1([0,p]\times[0,q])$. Note also that when $u \in C([0,p]\times[0,q])$, then $(I_\theta^r u) \in C([0,p]\times[0,q])$. Moreover

$$(I_\theta^r u)(t,0) = (I_\theta^r u)(0,x) = 0;\ t \in [0,p],\ x \in [0,q].$$

Example 2.2. Let $\lambda,\omega \in (-1,\infty)$ and $r = (r_1,r_2) \in (0,\infty)\times(0,\infty)$, then

$$I_\theta^r t^\lambda x^\omega = \frac{\Gamma(1+\lambda)\Gamma(1+\omega)}{\Gamma(1+\lambda+r_1)\Gamma(1+\omega+r_2)} t^{\lambda+r_1} x^{\omega+r_2},\ \text{for almost all } (t,x) \in [0,p]\times[0,q].$$

If u is a real function defined on the interval $[a,b]$, then the symbol $\bigvee_a^b u$ denotes the variation of u on $[a,b]$. We say that u is of bounded variation on the interval $[a,b]$ whenever $\bigvee_a^b u$ is finite. If $w : [a,b]\times[c,b] \to \mathbb{R}$, then the symbol $\bigvee_{t=p}^q w(t,s)$ indicates the variation of the function $t \to w(t,s)$ on the interval $[p,q] \subset [a,b]$, where s is arbitrarily fixed in $[c,d]$. In the same way we define $\bigvee_{s=p}^q w(t,s)$. For the properties of functions of bounded variation we refer to [21].

If u and φ are two real functions defined on the interval $[a,b]$, then under some conditions (see [21]) we can define the Stieltjes integral (in the Riemann-Stieltjes sense)

$$\int_a^b u(t)d\varphi(t)$$

of the function u with respect to φ. In this case we say that u is Stieltjes integrable on $[a,b]$ with respect to φ. Several conditions are known guaranteeing Stieltjes integrability [21]. One of the most frequently used requires that u is continuous and φ is of bounded variation on $[a,b]$.

In what follows we use the following properties of the Stieltjes integral ([23], section 8.13).
If u is Stieltjes integrable on the interval $[a,b]$ with respect to a function φ of bounded variation, then

$$\left|\int_a^b u(t)d\varphi(t)\right| \le \int_a^b |u(t)|d\left(\bigvee_a^t \varphi\right).$$

If u and v are Stieltjes integrable functions on the interval $[a,b]$ with respect to a nondecreasing function φ such that $u(t) \le v(t)$ for $t \in [a,b]$, then

$$\int_a^b u(t)d\varphi(t) \le \int_a^b v(t)d\varphi(t).$$

In the sequel we also consider Stieltjes integrals of the form

$$\int_a^b u(t)d_s g(t,s),$$

and Riemann-Liouville Stieltjes integrals of fractional order of the form

$$\frac{1}{\Gamma(r)}\int_0^t (t-s)^{r-1}u(s)d_sg(t,s),$$

where $g:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R},\ r\in(0,\infty)$ and the symbol d_s indicates the integration with respect to s.

Let X be a Fréchet space with a family of semi-norms $\{\|\cdot\|_n\}_{n\in\mathbb{N}^*}$. We assume that the family of semi-norms $\{\|\cdot\|_n\}$ verifies:

$$\|x\|_1\le\|x\|_2\le\ \|x\|_3\le\ldots\quad for\ every\ x\in X.$$

Let $Y\subset X$, we say that Y is bounded if for every $n\in\mathbb{N}$, there exists $\overline{M}_n>0$ such that

$$\|y\|_n\le\overline{M}_n\quad for\ all\ y\in Y.$$

To X we associate a sequence of Banach spaces $\{(X^n,\|\cdot\|_n)\}$ as follows: For every $n\in\mathbb{N}$, we consider the equivalence relation $\sim_n$ defined by: $x\sim_n y$ if and only if $\|x-y\|_n=0$ for $x,y\in X$. We denote $X^n=(X|_{\sim_n},\|\cdot\|_n)$ the quotient space, the completion of X^n with respect to $\|\cdot\|_n$. To every $Y\subset X$, we associate a sequence $\{Y^n\}$ of subsets $Y^n\subset X^n$ as follows : For every $x\in X$, we denote $[x]_n$ the equivalence class of x of subset X^n and we defined $Y^n=\{[x]_n : x\in Y\}$. We denote $\overline{Y^n}$, $int_n(Y^n)$ and ∂_nY^n, respectively, the closure, the interior and the boundary of Y^n with respect to $\|\cdot\|_n$ in X^n. For more information about this subject see [18].

For each $p\in\mathbb{N}$ we consider following set, $C_p=C([0,p]\times[0,b])$, and we define in $C(J)$ the semi-norms by

$$\|u\|_p=\sup_{(t,x)\in[0,p]\times[0,b]}\|u(t,x)\|.$$

Then $C(J)$ is a Fréchet space with the family of semi-norms $\{\|u\|_p\}$.

Definition 2.3. Let X be a Fréchet space. A function $N:X\longrightarrow X$ is said to be a contraction if for each $n\in\mathbb{N}^*$ there exists $k_n\in[0,1)$ such that

$$\|N(u)-N(v)\|_n\le k_n\|u-v\|_n\ \text{ for all } u,v\in X.$$

We need the following extension of the Burton-Kirk fixed point theorem in the case of a Fréchet space.

Theorem 2.4. *[11] Let $(X,\|.\|_n)$ be a Fréchet space and let $A,B:X\to X$ be two operators such that*

(a) *A is a compact operator;*

(b) *B is a contraction operator with respect to a family of seminorms $\{\|.\|_n\}$;*

(c) *the set $\left\{x\in X : x=\lambda A(x)+\lambda B\left(\frac{x}{\lambda}\right),\ \lambda\in(0,1)\right\}$ is bounded.*

Then the operator equation $A(u)+B(u)=u$ has a solution in X.

Let $\emptyset \neq \Omega \subset C(J)$, and let $G : \Omega \to \Omega$, and consider the solutions of equation

$$(Gu)(t,x) = u(t,x). \tag{2.1}$$

Now we introduce the concept of attractivity of solutions for our equations.

Definition 2.5. ([13]) Solutions of equation (2.1) are locally attractive if there exists a ball $B(u_0,\eta)$ in the space $C(J)$ such that, for arbitrary solutions $v = v(t,x)$ and $w = w(t,x)$ of equation (2.1) belonging to $B(u_0,\eta) \cap \Omega$, we have that, for each $x \in [0,b]$,

$$\lim_{t\to\infty} \Big(v(t,x) - w(t,x) \Big) = 0. \tag{2.2}$$

When the limit (2.2) is uniform with respect to $B(u_0,\eta) \cap \Omega$, solutions of equation (2.1) are said to be uniformly locally attractive (or equivalently that solutions of (2.1) are locally asymptotically stable).

Definition 2.6. ([13]) The solution $v = v(t,x)$ of equation (2.1) is said to be globally attractive if (2.2) holds for each solution $w = w(t,x)$ of (2.1). If condition (2.2) is satisfied uniformly with respect to the set Ω, solutions of equation (2.1) are said to be globally asymptotically stable (or uniformly globally attractive).

3 Main Results

In this section, we are concerned with the existence and the stability of solutions for the equation (1.2). Let us start by defining what we mean by a solution of the equation (1.2).

Definition 3.1. We mean by a solution of equation (1.2), every function $u \in C(J)$ such that u satisfies the equation (1.2) on J.

The following hypotheses will be used in the sequel.

(H_1) There exist continuous functions $l,k : J \to \mathbb{R}_+$ such that

$$|f(t,x,u_1,u_2) - f(t,x,v_1,v_2)| \leq \frac{l(t,x)|u_1 - v_1| + k(t,x)|u_2 - v_2|}{1 + |u_1 - v_1| + |u_2 - v_2|},$$

for $(t,x) \in J$ and $u_1,u_2,v_1,v_2 \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} l(t,x) = \lim_{t\to\infty} k(t,x) = 0; \; for \; x \in [0,b], \tag{3.1}$$

and the function $t \to f(t,x,0,0)$ is bounded on J with $f^* = \sup_{(t,x)\in J} f(t,x,0,0)$.

(H_2) For all $t_1,t_2 \in \mathbb{R}_+$ such that $t_1 < t_2$ the function $s \mapsto g(t_2,s) - g(t_1,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_3) The function $s \mapsto g(0,s)$ is nondecreasing on $\mathbb{R}_+$.

(H_4) The functions $s \mapsto g(t,s)$ and $t \mapsto g(t,s)$ are continuous on $\mathbb{R}_+$ for each fixed $t \in \mathbb{R}_+$ or $s \in \mathbb{R}_+$, respectively.

(H_5) There exist continuous functions $P,Q : J' \to \mathbb{R}_+$ such that

$$|h(t,x,s,y,u,v)| \leq \frac{P(t,x,s,y)|u| + Q(t,x,s,y)|v|}{1+|u|+|v|};$$

for $(t,x,s,y) \in J'$, $u,v \in \mathbb{R}$. Moreover, assume that

$$\lim_{t\to\infty} \int_0^{\beta(t)} \frac{P(t,x,s,y)+Q(t,x,s,y)}{(\beta(t)-s)^{1-r_1}} d_s\left(\bigvee_{k=0}^{s} g(t,k)\right) = 0. \tag{3.2}$$

Theorem 3.2. *Assume that hypotheses $(H_1)-(H_5)$ hold. If*

$$l_p + k_p < 1, \tag{3.3}$$

where

$$l_p = \sup_{(t,x)\in[0,p]\times[0,b]} l(t,x),\ k_p = \sup_{(t,x)\in[0,p]\times[0,b]} k(t,x);\ p \in \mathbb{N}^*,$$

then the equation (1.2) has at least one solution in the space $C(J)$. Moreover, solutions of equation (1.2) are globally asymptotically stable.

Proof. Let us define the operators $A,B : C(J) \to C(J)$ defined by

$$\begin{array}{rcl}(Au)(t,x) & = & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ & & \times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_s g(t,s);\ (t,x)\in J,\end{array} \tag{3.4}$$

$$(Bu)(t,x) = f(t,x,u(t,x),u(\alpha(t),x));\ (t,x)\in J. \tag{3.5}$$

We shall show that operators A and B satisfied all the conditions of Theorem 2.4. The proof will be given in several steps.

Step 1: *A is compact.*
To this aim, we must prove that A is continuous and it transforms every bounded set into a relatively compact set. Recall that $M \subset C(J)$ is bounded if and only if

$$\forall p \in \mathbb{N}^*,\ \exists \ell_p > 0 : \forall u \in M,\ \|u\|_p \leq \ell_p,$$

and $M = \{u(t,x);\ (t,x)) \in J\} \subset C(J)$ is relatively compact if and only if for any $p \in \mathbb{N}^*$, the family $\{u(t,x)|_{(t,x)]\in[0,p]\times[0,b]}\}$ is equicontinuous and uniformly bounded on $[0,p]\times[0,b]$. The proof will be given in several claims.

Claim 1: *A is continuous.*
Let $\{u_n\}_{n\in\mathbb{N}}$ be a sequence such that $u_n \to u$ in $C(J)$. Then, for each $(t,x)\in J$, we have

$$\begin{array}{rl} & |(Au_n)(t,x)-(Au)(t,x)| \\ \leq & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ & \times|h(t,x,s,y,u_n(s,y),u_n(\gamma(s),y)) - h(t,x,s,y,u(s,y),u(\gamma(s),y))|dyd_s g(t,s) \\ \leq & \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^{\beta(t)} \int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ & \times|h(t,x,s,y,u_n(s,y),u_n(\gamma(s),y)) \\ & -h(t,x,s,y,u(s,y),u(\gamma(s),y))|dyd_s\left(\bigvee_{k=0}^{s} g(t,k)\right).\end{array} \tag{3.6}$$

If $(t,x)\in[0,p]\times[0,b];\ p\in\mathbb{N}^*$, then, since $u_n\to u$ as $n\to\infty$ and g,h are continuous, (3.6) gives

$$\|N(u_n)-N(u)\|_p\to 0 \quad \text{as } n\to\infty.$$

Claim 2: *A maps bounded sets into bonded sets in C(J).*
Let M be a bounded set in $C(J)$, then, for each $p\in\mathbb{N}^*$, there exists $\ell_p>0$, such that for all $u\in C(J)$ we have $\|u\|_p\le\ell_p$. Then, for arbitrarily fixed $(t,x)\in[0,p]\times[0,b]$ we have

$$\begin{aligned}
|(Au)(t,x)| =\le & \Big|\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\
&\times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big|\\
\le & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\
&\times\frac{P(t,x,s,y)|u(s,y)|+Q(t,x,s,y)|u(\gamma(s),y)|}{1+|u(s,y)|+|u(\gamma(s),y)|}dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
\le & \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}\\
&\times(P(t,x,s,y)+Q(t,x,s,y))dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right)\\
\le & (P_p+Q_p)\|u\|_p,
\end{aligned}$$

where

$$P_p:=\sup_{(t,x)\in(t,x)\in[0,p]\times[0,b]}\int_0^{\beta(t)}\int_0^x\frac{P(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)(\beta(t)-s)^{1-r_1}(x-y)^{1-r_2}}dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right),$$

and

$$Q_p:=\sup_{(t,x)\in(t,x)\in[0,p]\times[0,b]}\int_0^{\beta(t)}\int_0^x\frac{Q(t,x,s,y)}{\Gamma(r_1)\Gamma(r_2)(\beta(t)-s)^{1-r_1}(x-y)^{1-r_2}}dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right).$$

Thus

$$\|A(u)\|_p\le(P_p+Q_p)\ell_p:=\ell_p'.$$

Claim 3: *A maps bounded sets into equicontinuous sets in C(J).*
Let $(t_1,x_1),(t_2,x_2)\in[0,p]\times[0,b];\ p\in\mathbb{N}^*$, $t_1<t_2,\ x_1<x_2$ and let $u\in M$. Also without loss

of generality, suppose that $\beta(t_1) \le \beta(t_2)$. Then we have

$$
\begin{aligned}
&|(Au)(t_2,x_2)-(Au)(t_1,x_1)| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times[h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y)) \\
&-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))]dyd_sg(t,s)\Big| \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s) \\
&-\int_0^{\beta(t_1)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big| \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\Big|\int_0^{\beta(t_1)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s) \\
&-\int_0^{\beta(t_1)}\int_0^{x_1}(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))dyd_sg(t,s)\Big|.
\end{aligned}
$$

Thus, we obtain

$$
\begin{aligned}
&|(Au)(t_2,x_2)-(Au)(t_1,x_1)| \\
&\le \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times\Big|h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y)) \\
&-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_0^{x_1}\Big|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}-(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\Big| \\
&\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_{x_1}^{x_2}|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}| \\
&\times\Big|h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right).
\end{aligned}
$$

Hence, we get

$$
\begin{aligned}
&|(Au)(t_2,x_2)-(Au)(t_1,x_1)| \\
&\leq \frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times\Big|h(t_2,x_2,s,y,u(s,y),u(\gamma(s),y)) \\
&-h(t_1,x_1,s,y,u(s,y),u(\gamma(s),y))\Big|dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_{\beta(t_1)}^{\beta(t_2)}\int_0^{x_2}(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1} \\
&\times\Big(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_0^{x_1}\Big|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}-(\beta(t_1)-s)^{r_1-1}(x_1-y)^{r_2-1}\Big| \\
&\times\Big(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right) \\
&+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t_1)}\int_{x_1}^{x_2}|(\beta(t_2)-s)^{r_1-1}(x_2-y)^{r_2-1}| \\
&\times\Big(P(t_1,x_1,s,y)+Q(t_1,x_1,s,y)\Big)dyd_s\left(\bigvee_{k=0}^{s}g(t,k)\right).
\end{aligned}
$$

From continuity of α,β,g,h,P,Q and as $t_1\to t_2$ and $x_1\to x_2$, the right-hand side of the above inequality tends to zero. As a consequence of claims 1 to 3 together with the Arzelá-Ascoli theorem, we can conclude that A is continuous and compact.

Step 2: *B is a contraction.*
Consider $v,w\in C(J)$. Then, by (H_2), for any $p\in\mathbb{N}$ and each $(t,x)\in[0,p]\times[0,b]$, we have

$$
\begin{aligned}
|(Bv)(t,x)-(Bw)(t,x)| &\leq |f(t,x,v(t,x),v(\alpha(t),x))-f(t,x,w(t,x),w(\alpha(t),x))| \\
&\leq \frac{l(t,x)|v(t,x)-w(t,x)|+k(t,x)|v(\alpha(t),x)-w(\alpha(t),x)|}{1+\alpha(t)} \\
&\leq (l(t,x)+k(t,x))|v-w|.
\end{aligned}
$$

Thus

$$\|B(v)-B(w)\|_p\leq(l_p+k_p)\|v-w\|_p.$$

By (3.3), we conclude that B is a contraction.

Step 3: *The set* $\mathcal{E}:=\left\{u\in C(J): u=\lambda A(u)+\lambda B\left(\frac{u}{\lambda}\right),\ \lambda\in(0,1)\right\}$ *is bounded.*
Let $u\in C(J)$, such that $u=\lambda A(u)+\lambda B\left(\frac{u}{\lambda}\right)$ for some $\lambda\in(0,1)$. Then, for any $p\in\mathbb{N}^*$ and each

$(t,x) \in [0,p]\times[0,b]$, we have

$$\begin{aligned} |u(t,x)| &\leq \lambda|A(u)| + \lambda|B\left(\frac{u}{\lambda}\right)| \\ &\leq P(t,x) + Q(t,x) + f^* + l(t,x) + k(t,x) \\ &\leq P_p + Q_p + f^* + l_p + k_p. \end{aligned}$$

Thus

$$\|u\|_p \leq P_p + Q_p + f^* + l_p + k_p =: \ell_p^*.$$

Hence, the set $\mathcal{E}$ is bounded.

Step 4: *The uniform global attractivity of solutions of the equation (1.2).*
Let u and v be any two solutions of equation (1.2), then for each $(t,x) \in J$ we have

$$\begin{aligned} &|u(t,x) - v(t,x)| \leq |f(t,x,u(t,x),u(\alpha(t),x)) - f(t,x,v(t,x),v(\alpha(t),x))| \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ &\times|h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,v(s,y),v(\gamma(s),y))|dyd_sg(t,s) \\ \leq\ & l(t,x)|u(t,x)-v(t,x)|| + k(t,x)|u(\alpha(t),x) - v(\alpha(t),x)| \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ &\times|h(t,x,s,y,u(s,y),u(\gamma(s),y)) - h(t,x,s,y,v(s,y),v(\gamma(s,y)))|dyd_s\left(\bigvee_{k=0}^{s} g(t,k)\right) \\ \leq\ & l(t,x)|u(t,x)-v(t,x)| + k(t,x)|u(\alpha(t),x) - v(\alpha(t),x)| \\ &+\frac{1}{\Gamma(r_1)\Gamma(r_2)}\int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1} \\ &\qquad\times(P(t,x,s,y)+Q(t,x,s,y))dyd_s\left(\bigvee_{k=0}^{s} g(t,k)\right). \end{aligned} \tag{3.7}$$

By using (3.3), (3.7) and the fact that $\alpha(t) \to \infty$ as $t \to \infty$, we deduce that

$$\begin{aligned} &\lim_{t\to\infty}|u(t,x)-v(t,x)| \leq \lim_{t\to\infty}\frac{1}{\Gamma(r_1)\Gamma(r_2)(1-l(t,x)-k(t,x))} \\ &\times \int_0^{\beta(t)}\int_0^x (\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}(P(t,x,s,y)+Q(t,x,s,y))dyd_sg(t,s) \\ &\leq \lim_{t\to\infty}\frac{1}{(1-l(t,x)-k(t,x))\Gamma(r_1)\Gamma(r_2)} \\ &\quad\times\int_0^x (x-y)^{r_2-1}\left(\int_0^{\beta(t)}\frac{P(t,x,s,y)+Q(t,x,s,y)}{(\beta(t)-s)^{1-r_1}}d_s\left(\bigvee_{k=0}^{s} g(t,k)\right)\right)dy. \end{aligned} \tag{3.8}$$

Hence, by (3.1), (3.2) and (3.8), we deduce that

$$\lim_{t\to\infty}(u(t,x)-v(t,x)) = 0.$$

Consequently, the equation (1.2) has a solution and all solutions are globally asymptotically stable. □

4 An Example

As an application of our results we consider the following nonlinear fractional order Riemann-Liouville Volterra-Stieltjes quadratic partial integral equation of the form

$$\begin{aligned} u(t,x) &= f(t,x,u(t,x),u(\alpha(t),x)) + \int_0^{\beta(t)}\int_0^x \frac{(\beta(t)-s)^{r_1-1}(x-y)^{r_2-1}}{\Gamma(r_1)\Gamma(r_2)} \\ &\quad \times h(t,x,s,y,u(s,y),u(\gamma(s),y))dyd_s g(t,s);\ (t,x)\in J, \end{aligned} \tag{4.1}$$

where $J=\mathbb{R}_+\times[0,1],\ r_1=\frac{1}{4},\ r_2=\frac{1}{2},\ \alpha(t)=\beta(t)=\gamma(t)=t;\ t\in\mathbb{R}_+,$

$$f(t,x,u,v)=\frac{xe^{-t-p}}{8(1+t)(1+|u|+2|v|)};\ p\in\mathbb{N}^*,\ (t,x)\in J \text{ and } u,v\in\mathbb{R},$$

$$g(t,s)=s,\ (t,s)\in\mathbb{R}_+^2,$$

$$\begin{cases} h(t,x,s,y,u,v)=\dfrac{cxs^{\frac{-3}{4}}(1+|u|)\sin\sqrt{t}\sin s}{(1+y^2+t^2)(1+|u|+|v|)}; \\ \quad if\ (t,x,s,y)\in J',\ s\neq 0,\ y\in[0,1]\ and\ u,v\in\mathbb{R}, \\ h(t,x,0,y,u,v)=0;\ if\ (t,x)\in J,\ y\in[0,1]\ and\ u,v\in\mathbb{R}, \end{cases}$$

$$c=\frac{\pi}{8e\Gamma(\frac{1}{4})}\ and\ J'=\{(t,x,s,y)\in J^2: s\le t\ and\ x\le y\}.$$

First, we can see that $\lim\limits_{t\to 0}\alpha(t)=0$. Next, the function f is a continuous, and

$$|f(t,x,u_1,u_2)-f(t,x,v_1,v_2)|\le\frac{xe^{-t-p}(|u_1-v_1|+2|u_2-v_2|)}{8(1+t)(1+|u_1-v_1|+|u_2-v_2|)};\ (t,x)\in J,\ u,v\in\mathbb{R}.$$

Then, the assumption (H_1) is satisfied with

$$l(t,x)=\frac{xe^{-t-p}}{8(1+t)},\ k(t,x)=\frac{xe^{-t-p}}{4(1+t)},\ l_p=\frac{e^{-p}}{8},\ k_p=\frac{e^{-p}}{4},\ f^*=\frac{1}{8}.$$

Also, we can easily see that the function g satisfies the hypotheses $(H_2)-(H_4)$.
The function h satisfies the assumption (H_5). Indeed, h is continuous and

$$|h(t,x,s,y,u,v)|\le\frac{P(t,x,s,y)|u|+Q(t,x,s,y)|v|}{1+|u|+|v|};\ (t,x,s,y)\in J',\ u,v\in\mathbb{R},$$

and

$$\begin{cases} P(t,x,s,y)=Q(t,x,s,y)=\dfrac{cxs^{\frac{-3}{4}}\sin\sqrt{t}\sin s}{1+y^2+t^2};\ (t,x,s,y)\in J',\ y\in[0,1],\ s\neq 0, \\ P(t,x,0,y)=Q(t,x,0,y)=0; \qquad (t,x)\in J,\ y\in[0,1]. \end{cases}$$

Then,

$$\begin{aligned}\left|\int_0^t (t-s)^{r-1}P(t,x,s,y)d_s g(t,s)\right| &\leq \int_0^t (t-s)^{\frac{-3}{4}} cxs^{\frac{-3}{4}} |\sin\sqrt{t}\sin s| d_s\left(\bigvee_{k=0}^{s} g(t,k)\right)\\ &\leq cx|\sin\sqrt{t}|\int_0^t (t-s)^{\frac{-3}{4}} s^{\frac{-3}{4}} ds\\ &\leq \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi}}\left|\frac{\sin\sqrt{t}}{\sqrt{t}}\right|\\ &\leq \frac{cx\Gamma^2(\frac{1}{4})}{\sqrt{\pi t}} \longrightarrow 0 \; as \; t\to\infty.\end{aligned}$$

We shall show that condition (3.3). Indeed, for each $p\in\mathbb{N}^*$, we get $l_p+k_p=\frac{3e^{-p}}{8}<1$. Consequently, by Theorem 3.2, the equation (4.1) has a solution defined on $\mathbb{R}_+\times[0,1]$ and solutions of this equation are globally asymptotically stable.

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