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# EXISTENCE RESULTS FOR FRACTIONAL DIFFERENTIAL INCLUSIONS INVOLVING NON-CONVEX VALUED MAPS WITH FOUR-POINT NONLOCAL INTEGRAL BOUNDARY CONDITIONS

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**Abstract**

This paper studies the existence of solutions for fractional differential inclusions of order $q \in (1,2]$ with four-point nonlocal integral boundary conditions. We consider two cases: (a) the multivalued map in the problem is not necessarily convex valued, (b) the multivalued map consists of non-convex values. A nonlinear alternative of Leray Schauder type coupled with the selection theorem of Bressan and Colombo is employed to deal with the first case, while the second case is based on Wegrzyk's fixed point theorem for generalized contractions.

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# 1 Introduction

Boundary value problems with integral boundary conditions constitute a very interesting and important class of problems. Various problems arising in heat conduction, chemical engineering, underground water flow, thermo-elasticity, and plasma physics can be reduced to the nonlocal problems with integral boundary conditions. Another example is that of Goursat's problem with integral boundary conditions. Integral boundary conditions for unsteady biomedical CFD applications are taking much importance these days. For a detailed description of the integral boundary conditions, we refer the reader to the papers [1, 2] and references therein.

Differential equations and inclusions of fractional order have recently been addressed by several researchers for a variety of problems. Fractional calculus has found its applications in a variety of disciplines of science and engineering such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc.([3]-[8]). For some recent development on the subject, for instance, see ([9]-[31]) and the references therein. More recently, Ahmad and Sivasundaram [32] discussed the existence of solutions for a nonlocal four-point integral boundary value problem of nonlinear fractional differential equations.

In this paper, we continue the study initiated in [32] for multivalued case. Precisely, we consider a nonlocal four-point integral boundary value problem of nonlinear fractional differential inclusions given by

$$\begin{cases} {}^cD^q x(t) \in F(t, x(t)), \ \ 0 < t < 1, \ \ 1 < q \le 2, \\ \delta_1 x(0) + \delta_2 x'(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad \delta_1 x(1) + \delta_2 x'(1) = \beta \displaystyle\int_0^{\eta} x(s)ds, \ \ 0 < \xi, \ \eta < 1, \end{cases} \tag{1.1}$$

where ${}^cD$ is the Caputo fractional derivative, $F : [0,1] \times \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map, $\mathcal{P}(\mathbb{R})$ is the family of all nonempty subsets of $\mathbb{R}$, and $\delta_1, \delta_2, \alpha, \beta$ are real numbers.

Our first existence result is based on the nonlinear alternative of Leray Schauder type together with the selection theorem of Bressan and Colombo for lower semi-continuous maps with decomposable values, while the second result relies on Wegrzyk's fixed point theorem for multivalued maps.

# 2 Preliminaries

Let us recall some basic definitions on multi-valued maps.

Let $X$ denote a normed space with the norm $|\cdot|$. A multivalued map $G : X \to \mathcal{P}(X)$ is convex (closed) valued if $G(x)$ is convex (closed) for all $x \in X$. $G$ is bounded on bounded sets if $G(B) = \cup_{x\in B} G(x)$ is bounded in $X$ for all bounded sets $B$ in $X$ (i.e. $\sup_{x\in B}\{|y| : y \in G(x)\} < \infty$). $G$ is called upper semi-continuous (u.s.c.) on $X$ if for each $x_0 \in X$, the set $G(x_0)$ is a nonempty closed subset of $X$, and if for each open set $N$ of $X$ containing $G(x_0)$, there exists an open neighborhood $N_0$ of $x_0$ such that $G(N_0) \subseteq N$. $G$ is said to be completely continuous if $G(B)$ is relatively compact for every bounded set $B$ in $X$. If the multivalued map $G$ is completely continuous with nonempty compact values, then $G$ is u.s.c. if and only if $G$ has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). $G$ has a fixed point if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator $G$ will be denoted by $FixG$.

For more details on multivalued maps, see the books of Aubin and Cellina [33], Aubin and Frankowska [34], Deimling [35], Hu and Papageorgiou [36], and Smirnov [37].

Let $C([0,1],\mathbb{R})$ denote the Banach space of all continuous functions from $[0,1]$ into $\mathbb{R}$ with the norm

$$\|u\| = \sup\{|u(t)| : t \in [0,1]\}.$$

Let $L^1([0,1],\mathbb{R})$ be the Banach space of measurable functions $u : [0,1] \longrightarrow \mathbb{R}$ which are Lebesgue integrable and normed by

$$\|u\|_{L^1} = \int_0^1 |u(t)|dt \quad \text{for all } u \in L^1([0,1],\mathbb{R}).$$

Let $E$ be a Banach space, $X$ a nonempty closed subset of $E$ and $G : X \to \mathcal{P}(E)$ a multivalued operator with nonempty closed values. $G$ is lower semi-continuous (l.s.c.) if the set $\{x \in X : G(x) \cap B \neq \emptyset\}$ is open for any open set $B$ in $E$. Let $A$ be a subset of $[0,1]\times\mathbb{R}$. $A$ is $\mathcal{L}\otimes B$ measurable if $A$ belongs to the $\sigma$-algebra generated by all sets of the form $\mathcal{J}\times D$, where $\mathcal{J}$ is Lebesgue measurable in $[0,1]$ and $D$ is Borel measurable in $\mathbb{R}$. A subset $A$ of $L^1([0,1],\mathbb{R})$ is decomposable if for all $u, v \in A$ and $\mathcal{J} \subset [0,1]$ measurable, the function $u\chi_{\mathcal{J}} + v\chi_{[0,1]\backslash\mathcal{J}} \in A$, where $\chi_{\mathcal{J}}$ stands for the characteristic function of $\mathcal{J}$.

**Definition 2.1.** If $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ is a multivalued map with compact values and $x(\cdot) \in C([0,1],\mathbb{R})$, then $F(\cdot,\cdot)$ is of lower semi-continuous type if

$$S_{F,x} = \{w \in L^1([0,1],\mathbb{R}) : w(t) \in F(t,x(t)) \text{ for } a.e.\ t \in [0,1]\}$$

is lower semi-continuous with closed and decomposable values.

Let $(X,d)$ be a metric space associated with the metric $d$. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \;\; d^*(A,B) = \sup\{d(a,B) : a \in A\},$$

where $d(x,B) = \inf_{y\in B} d(x,y)$ [38].

**Definition 2.2.** ([39]) A function $l : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a strict comparison function if it is continuous, strictly increasing and $\sum_{n=1}^{\infty} l^n(t) < \infty$, for each $t > 0$.

**Definition 2.3.** A multivalued operator $N$ on $X$ with nonempty values in $X$ is called

a) $\gamma$-Lipschitz if and only if there exists $\gamma > 0$ such that

$$d_H(N(x),N(y)) \leq \gamma d(x,y), \quad \text{for each } x,\ y \in X;$$

b) a contraction if and only if it is $\gamma$-Lipschitz with $\gamma < 1$;

c) a generalized contraction if and only if there is a strict comparison function $l : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$d_H(N(x),N(y)) \leq l(d(x,y)) \text{ for each } x,y \in X.$$

The following lemmas will be used in the sequel.

**Lemma 2.4.** *([40]). Let $Y$ be a separable metric space and let $N : Y \to \mathcal{P}(L^1([0,1],\mathbb{R}))$ be a lower semi-continuous multivalued map with closed decomposable values. Then $N(\cdot)$ has a continuous selection; i.e., there exists a continuous mapping (single-valued) $g : Y \to L^1([0,1],\mathbb{R})$ such that $g(y) \in N(y)$ for every $y \in Y$.*

**Lemma 2.5.** *(Wegrzyk's fixed point theorem [41]). Let $(X,d)$ be a complete metric space. If $N : X \to \mathcal{P}(X)$ is a generalized contraction with nonempty closed values, then $FixN \neq \emptyset$.*

**Lemma 2.6.** *(Covitz and Nadler's fixed point theorem [42]). Let $(X,d)$ be a complete metric space. If $N : X \to \mathcal{P}(X)$ is a multivalued contraction with nonempty closed values, then $N$ has a fixed point $z \in X$ such that $z \in N(z)$, i.e., $FixN \neq \emptyset$.*

In order to define the solution of (1.1), we consider the following lemma whose proof is given in [32].

**Lemma 2.7.** *For a given $\sigma \in C[0,1]$, the unique solution of the boundary value problem*

$$\begin{cases} {}^cD^q x(t) = \sigma(t), \ \ 0 < t < 1, \ \ 1 < q \le 2, \\ \delta_1 x(0) + \delta_2 x'(0) = \alpha \displaystyle\int_0^{\xi} x(s)ds, \quad \delta_1 x(1) + \delta_2 x'(1) = \beta \int_0^{\eta} x(s)ds, \ \ 0 < \xi, \ \eta < 1, \end{cases} \tag{2.1}$$

*is given by*

$$\begin{aligned} x(t) \ = \ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}\sigma(s)ds - \alpha a_1(t)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\sigma(m)dm\right)ds \\ & + a_2(t)\Bigg[\beta\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}\sigma(m)dm\right)ds - \delta_1\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}\sigma(s)ds \\ & - \delta_2\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}\sigma(s)ds\Bigg], \end{aligned} \tag{2.2}$$

*where*

$$a_1(t) = \frac{1}{\Delta}\left(\delta_1 + \delta_2 - \frac{\beta\eta^2}{2} - (\delta_1 - \beta\eta)t\right), \tag{2.3}$$

$$a_2(t) = \frac{1}{\Delta}\left(\left(\delta_2 - \frac{\alpha\xi^2}{2}\right) - (\delta_1 - \xi\alpha)t\right), \tag{2.4}$$

$$\Delta = \left[(\delta_1 - \beta\eta)\left(\delta_2 - \frac{\alpha\xi^2}{2}\right) - (\delta_1 - \alpha\xi)\left(\delta_1 + \delta_2 - \frac{\beta\eta^2}{2}\right)\right] \neq 0. \tag{2.5}$$

**Definition 2.8.** A function $x \in AC^1([0,1])$ is a solution of the problem (1.1) if there exists a function $f \in L^1([0,1],\mathbb{R})$ such that $f(t) \in F(t,x(t))$ a.e. on $[0,1]$ and

$$\begin{aligned} x(t) \ = \ & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}f(s)ds - \alpha a_1(t)\int_0^{\xi}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}f(m)dm\right)ds \\ & + a_2(t)\Bigg[\beta\int_0^{\eta}\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}f(m)dm\right)ds - \delta_1\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}f(s)ds \\ & - \delta_2\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}f(s)ds\Bigg]. \end{aligned}$$

# 3 Main results

For the first result, it is assumed that $F$ is not necessarily convex valued and the proof of this result is based on the nonlinear alternative of Leray-Schauder type together with the selection theorem of Bressan and Colombo [40] for lower semi-continuous maps with decomposable values.

For computational convenience, we set

$$\nu = \frac{1}{\Gamma(q+2)}\Big((q+1)\{1+|\delta_1\bar{a}_2|+|\delta_2\bar{a}_2|q\}+|\alpha\bar{a}_1|\xi^{q+1}+|\beta\bar{a}_2|\eta^{q+1}\Big), \tag{3.1}$$

where $\bar{a}_1 = \max_{t\in[0,1]}|a_1(t)|$, $\bar{a}_2 = \max_{t\in[0,1]}|a_2(t)|$.

**Theorem 3.1.** *Assume that:*

$(H_1)$ *there exists a continuous nondecreasing function* $\psi : [0,\infty) \to (0,\infty)$ *and a positive continuous function* $p$ *such that*

$$\|F(t,x)\| := \sup\{|y| : y \in F(t,x)\} \le p(t)\psi(\|x\|) \ \textit{for each}\ (t,x) \in [0,1]\times\mathbb{R};$$

$(H_2)$ *there exists a number* $M > 0$ *such that*

$$\frac{M}{\nu\psi(M)\|p\|} > 1,$$

*where* $\nu$ *is given by (3.1).*

$(H_3)$ $F : [0,1]\times\mathbb{R} \to \mathcal{P}(\mathbb{R})$ *is a nonempty compact-valued multivalued map such that*

$(a)$ $(t,x) \longmapsto F(t,x)$ *is* $\mathcal{L}\otimes B$ *measurable,*

$(b)$ $x \longmapsto F(t,x)$ *is lower semicontinuous for each* $t \in [0,1]$.

*Then the boundary value problem (1.1) has at least one solution on* $[0,1]$.

*Proof.* It follows from $(H_1)$ and $(H_3)$ that $F$ is of l.s.c. type. Then from Lemma 2.4, there exists a continuous function $f : C([0,1],\mathbb{R}) \to L^1([0,1],\mathbb{R})$ such that $f(x) \in S_{F,x}$ for all $x \in C([0,1],\mathbb{R})$.

Consider the problem

$$\begin{cases} {}^cD^q x(t) = f(x)(t), \ t\in[0,1], \ 1<q\le 2, \\ \delta_1 x(0)+\delta_2 x'(0) = \alpha\displaystyle\int_0^\xi x(s)ds, \quad \delta_1 x(1)+\delta_2 x'(1) = \beta\int_0^\eta x(s)ds, \ 0<\xi,\ \eta<1. \end{cases} \tag{3.2}$$

Observe that if $x \in AC^1([0,1],\mathbb{R})$ is a solution of (3.2), then $x$ is a solution to the problem (1.1). In order to transform the problem (3.2) into a fixed point problem, we define the operator $\Omega : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$

$$\begin{aligned}\Omega(x)(t) \;=\; & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(x)(s)ds - \alpha a_1(t)\int_0^\xi\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(x)(m)dm\right)ds \\ & + a_2(t)\Bigg[\beta\int_0^\eta\left(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(x)(m)dm\right)ds - \delta_1\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(x)(s)ds \\ & - \delta_2\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(x)(s)ds\Bigg].\end{aligned}$$

The proof consists of several steps.

(i) $\Omega$ *is continuous.* Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
& |\Omega(y_n)(t) - \Omega(y)(t)| \\
= & \Bigg| \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} [f(y_n)(s) - f(y)(s)] ds \\
& -\alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} [f(y_n)(m) - f(y)(m)] dm \right) ds \\
& + a_2(t) \Bigg[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} [f(y_n)(m) - f(y)(m)] dm \right) ds \\
& -\delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} [f(y_n)(s) - f(y)(s)] ds - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} [f(y_n)(s) - f(y)(s)] ds \Bigg] \Bigg| \\
\le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} \|f(y_n)(s) - f(y)(s)\| ds \\
& + |\alpha a_1(t)| \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} \|f(y_n)(m) - f(y)(m)\| dm \right) ds \\
& + |a_2(t)| \Bigg[ |\beta| \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} \|f(y_n)(m) - f(y)(m)\| dm \right) ds \\
& + |\delta_1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} \|f(y_n)(s) - f(y)(s)\| ds \\
& + |\delta_2| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} \|f(y_n)(s) - f(y)(s)\| ds \Bigg].
\end{aligned}$$

Hence

$$\|\Omega(y_n) - \Omega(y)\| = \sup_{t \in [0,1]} |\Omega(y_n)(t) - \Omega(y)(t)| \to 0 \text{ as } n \to \infty.$$

Thus $\Omega$ is continuous.

(ii) $\Omega$ *maps bounded sets into bounded sets in* $C([0,1],\mathbb{R})$. Indeed, it is enough to show that there exists a positive constant $\nu_1$ such that, for each $x \in B_r = \{x \in C([0,1],\mathbb{R}) : \|x\| \le r\}$, we have $\|\Omega(x)\| \le \nu_1$. From $(H_1)$ we have:

$$\begin{aligned}
|\Omega(x)(t)| \le & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|) ds + |\alpha a_1(t)| \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi(\|x\|) dm \right) ds \\
& + |a_2(t)| \Bigg[ |\beta| \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi(\|x\|) dm \right) ds \\
& + |\delta_1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|) ds + |\delta_2| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} p(s)\psi(\|x\|) ds \Bigg] \\
\le & \frac{1}{\Gamma(q+2)} \Big( (q+1)\{1 + |\delta_1 \bar{a}_2| + q|\delta_2 \bar{a}_2|\} + |\alpha \bar{a}_1| \xi^{q+1} + |\beta \bar{a}_2| \eta^{q+1} \Big) \|p\| \psi(\|x\|).
\end{aligned}$$

Taking norm and using (3.1), we get

$$\|\Omega(x)\| \le \nu \|p\| \psi(r) := \nu_1.$$

(iii) $\Omega$ *maps bounded sets into equicontinuous sets in* $C([0,1],\mathbb{R})$. Let $t_1, t_2 \in [0,1], t_1 < t_2$ and $B_r$ be a bounded set in $C([0,1],\mathbb{R})$. Then

$$\begin{aligned}
& |\Omega(x)(t_2) - \Omega(x)(t_1)| \\
\leq\ & \left|\int_0^{t_1} \frac{(t_2-s)^{q-1} - (t_1-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)\,ds\right| + \left|\int_{t_1}^{t_2} \frac{(t_2-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)\,ds\right| \\
& + |\alpha||a_1(t_2) - a_1(t_1)| \int_0^{\xi}\int_0^{s} \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi(\|x\|)\,dmds \\
& + |a_2(t_2) - a_2(t_1)| \bigg[ |\beta| \int_0^{\eta}\int_0^{s} \frac{(s-m)^{q-1}}{\Gamma(q)} p(m)\psi(\|x\|)\,dmds \\
& + |\delta_1| \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} p(s)\psi(\|x\|)ds + |\delta_2| \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} p(s)\psi(\|x\|)\,ds \bigg].
\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero independently of $x \in B_r$. Therefore it follows by the Arzelá-Ascoli theorem that $\Omega : C([0,1],\mathbb{R}) \to C([0,1],\mathbb{R})$ is completely continuous.

(iv) Finally, we discuss *a priori bounds on solutions*. Let $x$ be a solution of (3.2). In view of $(H_1)$, for each $t \in [0,1]$, we obtain

$$\begin{aligned}
|x(t)| \leq\ & \frac{1}{\Gamma(q+2)}\Big((q+1)\{t^q + |\delta_1 a_2(t)| + q|\delta_2 a_2(t)|\} \\
& + |\alpha a_1(t)|\xi^{q+1} + |\beta a_2(t)|\eta^{q+1}\Big)\|p\|\psi(\|x\|) \\
\leq\ & \frac{1}{\Gamma(q+2)}\Big((q+1)\{1 + |\delta_1 \bar{a}_2| + q|\delta_2 \bar{a}_2|\} \\
& + |\alpha \bar{a}_1|\xi^{q+1} + |\beta \bar{a}_2|\eta^{q+1}\Big)\|p\|\psi(\|x\|),
\end{aligned}$$

which, on taking norm and using (3.1), yields

$$\frac{\|x\|}{\nu\|p\|\psi(\|x\|)} \leq 1.$$

In view of $(H_2)$, there exists $M$ such that $\|x\| \neq M$. Let us set

$$U = \{x \in C([0,1],\mathbb{R}) : \|x\| < M\}.$$

Note that the operator $\Omega : \overline{U} \to C([0,1],\mathbb{R})$ is upper semicontinuous and completely continuous. From the choice of $U$, there is no $x \in \partial U$ such that $x = \mu\Omega(x)$ for some $\mu \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type [43], we deduce that $\Omega$ has a fixed point $x \in \overline{U}$ which is a solution of the problem (3.2). Consequently, it is a solution to the problem (1.1). This completes the proof. □

Now we prove the existence of solutions for the problem (1.1) with a non-convex valued right hand side by applying Lemma 2.5 due to Wegrzyk.

**Theorem 3.2.** *Suppose that:*

$(H_4)$ $F : [0,1]\times\mathbb{R} \longrightarrow \mathcal{P}(\mathbb{R})$ *has nonempty compact values and* $F(\cdot,u)$ *is measurable for each* $u \in \mathbb{R}$;

$(H_5)$ $d_H(F(t,x),F(t,\overline{x})) \le k(t)\ell(|x-\overline{x}|)$ *for almost all* $t \in [0,1]$ *and* $x,\overline{x} \in \mathbb{R}$ *with* $k \in C([0,1],\mathbb{R}_+)$ *and* $d(0,F(t,0)) \le k(t)$ *for almost all* $t \in [0,1]$, *where* $\ell : \mathbb{R}_+ \to \mathbb{R}_+$ *is strictly increasing.*

*Then the BVP (1.1) has at least one solution on* $[0,1]$ *if* $\gamma\ell : \mathbb{R}_+ \to \mathbb{R}_+$ *is a strict comparison function, where* $\gamma = \nu\|k\|$ *(*$\nu$ *is given by (3.1)).*

*Proof.* Suppose that $\gamma\ell : \mathbb{R}_+ \to \mathbb{R}_+$ is a strict comparison function. Observe that by the assumptions $(H_4)$ and $(H_5)$, $F(\cdot,x(\cdot))$ is measurable and has a measurable selection $v(\cdot)$ (see Theorem III.6 [44]). Also $k \in C([0,1],\mathbb{R}_+)$ and

$$\begin{aligned} |v(t)| &\le d(0,F(t,0)) + H_d(F(t,0),F(t,x(t))) \\ &\le k(t) + k(t)\ell(|x(t)|) \\ &\le (1+\ell(\|x\|))k(t). \end{aligned}$$

Thus the set $S_{F,x}$ is nonempty for each $x \in C([0,1],\mathbb{R})$.

Transform the problem (1.1) into a fixed point problem. Consider the operator $\Omega_F : C([0,1],\mathbb{R}) \to \mathcal{P}(C([0,1],\mathbb{R}))$ defined by

$$\Omega_F(x) = \left\{ \begin{array}{l} h \in C([0,1],\mathbb{R}) : \\ h(t) = \left\{ \begin{array}{l} \displaystyle\int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} f(s)ds \\ \displaystyle -\alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m)dm \right) ds \\ \displaystyle +a_2(t) \Big[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} f(m)dm \right) ds \\ \displaystyle -\delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} f(s)ds - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} f(s)ds \Big], \end{array} \right. \end{array} \right\}$$

for $f \in S_{F,x}$. We shall show that the map $\Omega_F$ satisfies the assumptions of Lemma 2.5. To show that the map $\Omega_F(x)$ is closed for each $x \in C([0,1],\mathbb{R})$, let $(x_n)_{n\ge 0} \in \Omega_F(x)$ such that $x_n \longrightarrow \tilde{x}$ in $C([0,1],\mathbb{R})$. Then, $\tilde{x} \in C([0,1],\mathbb{R})$ and there exists $y_n \in S_{F,x_n}$ such that, for each $t \in [0,1]$,

$$\begin{aligned} x_n(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y_n(s)ds - \alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_n(m)dm \right) ds \\ &\quad + a_2(t) \Big[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_n(m)dm \right) ds - \delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_n(s)ds \\ &\quad - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_n(s)ds \Big]. \end{aligned}$$

As $F$ has compact values, we pass onto a subsequence to obtain that $y_n$ converges to $y$ in $L^1([0,1],\mathbb{R})$. Thus, $y \in S_{F,x}$ and for each $t \in [0,1]$,

$$\begin{aligned} x_n(t) \longrightarrow \tilde{x}(t) &= \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y(s)ds - \alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y(m)dm \right) ds \\ &\quad + a_2(t) \Big[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y(m)dm \right) ds - \delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y(s)ds \\ &\quad - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y(s)ds \Big]. \end{aligned}$$

So, $\tilde{x} \in \Omega_F(x)$ and hence $\Omega_F(x)$ is closed.

Next, we show that

$$d_H(\Omega_F(x), \Omega_F(\overline{x})) \leq \gamma \ell(\|x - \overline{x}\|) \text{ for each } x, \overline{x} \in C([0,1], \mathbb{R}).$$

Let $x, \overline{x} \in C([0,1], \mathbb{R})$ and $h_1 \in \Omega_F(x)$. Then, there exists $y_1(t) \in F(t, x(t))$ such that for each $t \in [0,1]$,

$$\begin{aligned} h_1(t) \quad = \quad & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y_1(s)ds - \alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_1(m)dm \right) ds \\ & + a_2(t) \left[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_1(m)dm \right) ds - \delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_1(s)ds \right. \\ & \left. - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_1(s)ds \right]. \end{aligned}$$

From $(H_5)$, it follows that

$$d_H(F(t, x(t)), F(t, \overline{x}(t))) \leq k(t) \ell(|x(t) - \overline{x}(t)|).$$

So, there exists $w \in F(t, \overline{x}(t))$ such that

$$|y_1(t) - w(t)| \leq k(t) \ell(|x(t) - \overline{x}(t)|), \ t \in [0,1].$$

Define $U : [0,1] \to \mathcal{P}(\mathbb{R})$ as

$$U(t) = \{ w \in \mathbb{R} : |y_1(t) - w(t)| \leq k(t) \ell(|x(t) - \overline{x}(t)|) \}.$$

Since the multivalued operator $U(t) \cap F(t, \overline{x}(t))$ is measurable (see Proposition III.4 in [44]), there exists a function $y_2(t)$ which is a measurable selection for $U(t) \cap F(t, \overline{x}(t))$. So, $y_2(t) \in F(t, \overline{x}(t))$, and for each $t \in [0,1]$,

$$|y_1(t) - y_2(t)| \leq k(t) \ell(|x(t) - \overline{x}(t)|).$$

For each $t \in [0,1]$, let us define

$$\begin{aligned} h_2(t) \quad = \quad & \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)} y_2(s)ds - \alpha a_1(t) \int_0^{\xi} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_2(m)dm \right) ds \\ & + a_2(t) \left[ \beta \int_0^{\eta} \left( \int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)} y_2(m)dm \right) ds - \delta_1 \int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)} y_2(s)ds \right. \\ & \left. - \delta_2 \int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)} y_2(s)ds \right]. \end{aligned}$$

Then

$$\begin{aligned}
|h_1(t)-h_2(t)| &\leq \int_0^t \frac{(t-s)^{q-1}}{\Gamma(q)}|y_1(s)-y_2(s)|ds \\
&+|\alpha a_1(t)|\int_0^{\xi}\Big(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}|y_1(m)-y_2(m)|dm\Big)ds \\
&+|a_2(t)|\Bigg[|\beta|\int_0^{\eta}\Big(\int_0^s \frac{(s-m)^{q-1}}{\Gamma(q)}|y_1(m)-y_2(m)|dm\Big)ds \\
&+|\delta_1|\int_0^1 \frac{(1-s)^{q-1}}{\Gamma(q)}|y_1(s)-y_2(s)|ds \\
&+|\delta_2|\int_0^1 \frac{(1-s)^{q-2}}{\Gamma(q-1)}|y_1(s)-y_2(s)|ds\Bigg] \\
&\leq \nu\|k\|\ell(\|x-\overline{x}\|).
\end{aligned}$$

Thus

$$\|h_1-h_2\| \leq \nu\|k\|\ell(\|x-\overline{x}\|) = \gamma\ell(\|x-\overline{x}\|).$$

By an analogous argument, interchanging the roles of $x$ and $\overline{x}$, we obtain

$$d_H(\Omega_F(x),\Omega_F(\overline{x})) \leq \nu\|k\|\ell(\|x-\overline{x}\|) = \gamma\ell(\|x-\overline{x}\|)$$

for each $x,\overline{x}\in C([0,1],\mathbb{R})$. So, $\Omega_F$ is a generalized contraction and thus, by Lemma 2.5, $\Omega_F$ has a fixed point $x$ which is solution to (1.1). This completes the proof. □

*Remark* 3.3. It is imperative to note that Theorem 3.2 holds for several values of the function $\ell$, for example, $\ell(t)=\frac{ln(1+t)}{\mu}$, where $\mu\in(0,1)$; $\ell(t)=t$, etc. Furthermore, condition $(H_5)$ reduces to $d_H(F(t,x),F(t,\overline{x}))\leq k(t)|x-\overline{x}|$ for $\ell(t)=t$, where a contraction principle for multivalued map due to Covitz and Nadler [42] (Lemma 2.6) is applicable under the condition $\nu\|k\|<1$. Thus, our result dealing with a non-convex valued right hand side of (1.1) is more general.

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