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On Bang-bang Controls for Some Nonlinear Systems

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Abstract

In the paper we consider the class of nonlinear n-dimensional control systems that can be mapped to linear ones by change of variables and an additive change of control (A-linearizable systems). We show that for sufficiently small initial points the transferring to the origin is possible by means of bang-bang controls with no more than $n-1$ points of switching. Moreover in some cases such a transferring is extremal in the sense of time optimality. These results are based on technique of the power Markov min-problem. An algorithm of searching the mentioned above bang-bang controls is also given.

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1 Introduction

One of the main tasks and the final goal of the control theory is a direct construction of control functions solving particular control problems. This construction is a difficult problem especially if we consider nonlinear systems. One of the important approaches allowing us to solve such a problem for some class of nonlinear systems is based on the mapping of the systems to linear ones (linearization). This approach was originated by V.I. Korobov [1] for the class of triangular systems. We cite here the following remarkable linearization theorem from [1].

Theorem 1.1. [1] *Consider a triangular system of the form*

$$\begin{aligned} \dot{x}_i &= f_i(x_1,\dots,x_{i+1}), \quad i=1,\dots,n-1, \\ \dot{x}_n &= f_n(x_1,\dots,x_n,u) \end{aligned} \tag{1.1}$$

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and suppose that

$$f_i(x_1,\dots,x_{i+1}) \in C^{n-i+1}(\mathbb{R}^{i+1}),\ i = 1,\dots,n-1, \quad f_n(x_1,\dots,x_n,u) \in C^1(\mathbb{R}^{n+1}) \tag{1.2}$$

and

$$\left|\frac{\partial f_i(x_1,\dots,x_{i+1})}{\partial x_{i+1}}\right| \ge \alpha,\ i = 1,\dots,n-1, \quad \left|\frac{\partial f_n(x_1,\dots,x_n,u)}{\partial u}\right| \ge \alpha \tag{1.3}$$

for any $x_1,\dots,x_n,u \in \mathbb{R}$ where $\alpha > 0$. Then there exist a change of variables $z = F(x) \in C^2(\mathbb{R}^n)$ and a change of the control $v = g(x,u) \in C^1(\mathbb{R}^{n+1})$ reducing the system (1.1) *to the linear form*

$$\dot z_i = z_{i+1}, \quad i = 1,\dots,n-1, \qquad \dot z_n = v. \tag{1.4}$$

The concept of mappability considered in Korobov's theorem (by means of a change of variables and control $v = g(x,u)$) is called the feedback linearization. A further progress in the study of this problem was achieved in the works [2], [3], [4] for the class of C^∞-smooth nonlinear systems. It was shown that a system

$$\dot x = f(x,u),\ x \in \mathbb{R}^n,\ u \in \mathbb{R}^1$$

is locally feedback linearizable if and only if it is of the form

$$\dot x = f(x,u) = a(x) + b(x)\psi(x,u),$$

$a(x)$, $b(x) \in C^\infty(U(0))$ and besides the vector fields $a(x)$, $b(x)$ as well as their Lie brackets satisfy some conditions of involutivity.

In our recent work [5] the conditions of feedback linearizability were generalized to the case of C^1-smooth nonlinear systems. We note that all the mentioned results are a direct development of the above-cited theorem.

Let us observe that feedback linearizability of a nonlinear system allows us to reduce various control problems for this system to the similar problems for a linear system, which are much more investigated.

First of all, this concerns problems without restrictions on control, when $u \in \mathbb{R}^1$. But in the case when some restriction is required, $u \in \Omega \subset \mathbb{R}^1$, the substitution of the control $v = g(x,u)$ does not allow us to check it for the initial system using the solutions of the linearized system. Moreover, in the case when the set Ω of possible controls is discrete (for example, finite), feedback linearizability becomes absolutely unusable. In this case it is natural to use linearizability by change of variables only, without change of control (pure linearizability). But the set of pure linearizable systems is rather poor (see [6], [7]). In the work [8] it was proposed to consider linearizability with a special, so-called additive, change of the control

$$v = u + h(x).$$

This type of linearizability (further we call it A-linearizability) is equivalent to mappability of systems to the systems of the form:

$$\begin{cases} \dot z_i = z_{i+1},\ i = 1,\dots,n-1, \\ \dot z_n = g(z) + u, \quad g(0) = 0, \end{cases} \tag{1.5}$$

by a change of variables $z = F(x)$. The subclass of A-linearizable triangular systems is described in [8]. In the case of general nonlinear systems of class C^1 the problem of A-linearizability is solved in [5].

It is well known that in the theory of linear systems an important role is played by bang-bang controls, i.e., by controls switching between two different states. In particular, for the system (1.4), whose control constraint is of the form

$$|v(t)| \le 1$$

and the number of switchings is at most $(n-1)$, bang-bang controls realize the time-optimal transfer from a point to the origin by virtue of the system. Of course, this fact is extended to pure linearizable systems. The main goal of the present paper is to show that for an A-linearizable system linearizable by an additive change of variables the transfer from a point to the origin by means of bang-bang controls is also possible and in some cases can be optimal.

The paper is organized as follows. In Section 2 we recall the results of [5], [8] on the description of the class of nonlinear A-linearizable systems or, equivalently, of the class of systems mappable to systems of the form (1.5) by the change of variables only.

In Section 3 we consider the question on the existence of piecewise constant controls $u = \pm 1$ with at most $n-1$ points of switching (further we call them n-bang-bang controls) realizing a transfer of the initial point z^0 to 0 by virtue of system (1.5). We show that such a control exists for any sufficiently small z^0 under the Lipschitz condition on the function $g(z)$ in a neighborhood of the origin. Combining this result with the result of Section 2, we prove the existence of n-bang-bang controls realizing steering to 0 for A-linearizable systems. We also discuss the possibility for the obtained controls to be time-optimal.

In Section 4 we propose a numerical algorithm for searching n-bang-bang controls based on successive solving of power Markov min-problem [9] and also give an example.

2 *A*-linearizable Systems

In this section we recall the results from [5], [8] where the conditions for a nonlinear system to be A-linearizable are given.

Given a vector function $F = \left(F_i(x)\right)_{i=1}^n \in C^1(\mathbb{R}^n)$, let $F_x = \left(\partial F_i/\partial x_j\right)_{i,j=1}^n$.

Definition 2.1. [5] We say that a nonlinear system of the form

$$\dot{x} = f(x,u), \quad x \in Q \subset \mathbb{R}^n, \quad u \in \mathbb{R}, \tag{2.1}$$

where $f(x,u) \in C^1(Q \times \mathbb{R})$, is locally A-linearizable in the domain Q if there exist a change of variables

$$z = F(x) \in C^2(Q), \quad \det F_x(x) \neq 0, \quad x \in Q, \tag{2.2}$$

and a change of control

$$v = u + h(x), \quad h(x) \in C^1(Q), \tag{2.3}$$

which reduce system (2.1) to the form

$$\dot{z} = A_0 z + b_0 v, \tag{2.4}$$

where

$$A_0 = \begin{pmatrix} 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 1 \\ 0 & \dots & \dots & 0 \end{pmatrix}, \qquad b_0 = \begin{pmatrix} 0 \\ \dots \\ 0 \\ 1 \end{pmatrix}.$$

The following theorem gives A-linearizability conditions for the class of triangular systems. Denote by Q_k the projection of $Q \times \mathbb{R}^1$ on $\mathbb{R}^k$, i.e. $Q_k = \{(y_1, \dots, y_k) : (y_1, \dots, y_{n+1}) \in Q \times \mathbb{R}^1\}$, $k = 1, \dots, n$.

Theorem 2.2. [8] *System* (1.1) *with functions* $f_k \in C^{n-k+1}(Q_{k+1})$, $k = 1, \dots, n$, *is locally A-linearizable in the domain* Q *if and only if for any* $x \in Q$ *and* $u \in \mathbb{R}^1$ *the following equality holds:*

$$\frac{\partial f_n(x_1, \dots, x_n, u)}{\partial u} \cdot \frac{\partial f_{n-1}(x_1, \dots, x_n)}{\partial x_n} \cdot \ldots \cdot \frac{\partial f_1(x_1, x_2)}{\partial x_2} = c(x_1),$$

where the function $c(x_1)$ *is n times differentiable and* $|c(x_1)| \geq \alpha > 0$ *in* Q_1.

Consider the problem in a general form. Suppose that system (1.1) is A-linearizable. Then

$$\dot{z} = F_x(x) f(x,u) = A_0 F(x) + b_0(u + h(x)), \tag{2.5}$$

which implies the equality $f(x,u) = a(x) + b(x)u$, i.e., system (1.1) is of the form

$$\dot{x} = a(x) + b(x)u, \tag{2.6}$$

where $a(x), b(x) \in C^1(Q)$.

Given vector functions $a = (a_i(x))_{i=1}^n$, $b = (b_i(x))_{i=1}^n$ of the class $C^1(\mathbb{R}^n)$, let $[a,b]$ denote the Lie bracket $[a,b] = a_x b - b_x a$, where $a_x = \left(\partial a_i/\partial x_j\right)_{i,j=1}^n$ and $b_x = \left(\partial b_i/\partial x_j\right)_{i,j=1}^n$. Let L_a denote the Lie derivative along the vector field $a(x)$, $(L_a b)(x) = (b_x a)(x)$ for any vector function $b(x)$.

The conditions of A-linearizability are given in the following

Theorem 2.3. [5] *Nonlinear system* (2.1) *is locally A-linearizable in the domain Q if and only if it satisfies the following conditions.*

1) The system is affine (linear by control), i.e. of the form (2.6)*.*

2) There exists a set of scalar continuous in Q functions $\mu_{ij}(x)$, $i = 1,\dots,n-1$, $j = 0,1,\dots,i-1$, such that the vector-functions defined by

$$\chi^0(x) = b(x), \quad \chi^k(x) = \left[a(x), \chi^{k-1}(x)\right] + \sum_{j=0}^{k-1} \mu_{kj}(x)\chi^j(x), \quad k = 1,\dots,n-1, \tag{2.7}$$

exist and belong to the class $C^1(Q)$; vector functions $\chi^0(x),\dots,\chi^{n-1}(x)$ are linearly independent in the domain Q, and the set $\{\chi^0(x),\dots,\chi^{n-2}(x)\}$ satisfies the involutivity condition

$$\left[\chi^i(x), \chi^j(x)\right] = \chi^i_x \chi^j - \chi^j_x \chi^i = \sum_{k=0}^{n-2} \lambda_k^{i,j}(x)\chi^k(x), \quad i,j = 0,\dots,n-2,$$

where $\lambda_k^{i,j}(x)$ are certain continuous functions.

3) Among the solutions of the system of partial differential equations

$$\varphi_x(x)\chi^j(x) = 0, \quad j = 0,\dots,n-2$$

there exists a solution $\varphi(x)$ such that the functions $L_a^{i-1}\varphi(x)$, $i = 1,\dots,n$ (where $L_a^0\varphi = \varphi$, $L_a^i\varphi = (L_a^{i-1}\varphi)_x a(x)$), belong to the class $C^2(Q)$ and the following equality holds

$$\varphi_x(x)\chi^{n-1}(x) = c(\varphi(x)), \quad x \in Q,$$

where $\chi^0(x),\dots,\chi^{n-1}(x)$ are defined by (2.7) *and $c(\tau)$ belongs to the class C^n and $c(\tau) \neq 0$, $\tau \in \varphi(Q)$.*

Under conditions 1)-3) the change of variables $z = F(x)$ and the control $v = u + h(x)$ realizing the mapping can be found by the formulas

$$z_i = F_i(x) = L_a^{i-1}\widetilde{\varphi}(x), \quad i = 1,\dots,n,$$

$$h(x) = L_a^n \widetilde{\varphi}(x),$$

where $\widetilde{\varphi}(x) = \Phi(\varphi(x))$ and

$$\Phi(t) = (-1)^{n-1} \int \frac{1}{c(t)}\, dt \in C^{n+1}(\varphi(Q)).$$

Remark 2.4. In the case when vector fields $a(x)$, $b(x)$ are in $C^n(Q)$ one can put $\chi^i(x) = \mathrm{ad}^i_{a(x)} b(x)$, $i = 1,\dots,n-1$, where $\mathrm{ad}^0_{a(x)} b(x) = b(x)$, $\quad \mathrm{ad}^i_{a(x)} b(x) = \left[a(x), \mathrm{ad}^{i-1}_{a(x)} b(x)\right]$, $i \geq 1$.

Remark 2.5. If the nonlinear system in Theorem 2.2 has the triangular form (1.1) then the function $\varphi(x)$ can be chosen as $\varphi(x) = x_1$.

Note that A-linearizability is equivalent to the local mappability (in a neighborhood of an arbitrary point x) of system (2.1) to a system of the form

$$\dot{z} = A_0 z + b_0(u + g(z)), \tag{2.8}$$

by means of a change of variables (2.2) only.

Remark 2.6. One can observe that if $0 \in \operatorname{int} Q$ and $f(0,0) = 0$ then the mapping $F(x)$ can be chosen in such a way that $F(0) = 0$ and $g(0) = 0$. Indeed, from (2.5) we have

$$0 = F_x(0) f(0,0) = A_0 F(0) + b_0 g(F(0)).$$

This yields

$$A_0 F(0) = 0, \quad g(F(0)) = 0.$$

If we put now

$$F_1(x) = F(x) - F(0)$$

and consider the change of variables

$$z = F_1(x)$$

then we obtain the required properties.

3 Existence of *n*-Bang-bang Controls

In this section we consider a system of the form

$$\dot{z} = Az + b(u + g(z)), \quad |u| \le 1, \tag{3.1}$$

where A is an $(n\times n)$-matrix, $b \in \mathbb{R}^n$, $g(x)$ satisfies the Lipschitz condition with a constant L in the ball $\{z : \| z \| \le \gamma\}$, $\gamma > 0$, and $g(0) = 0$. We also assume that the pair A, b satisfies the Kalman condition:

$$\operatorname{rank}(b, Ab, \dots, A^{n-1}b) = n.$$

Our goal is to prove the existence of n-bang-bang controls transferring an initial point z^0 from a certain neighborhood of the origin to 0 by virtue of system (3.1). Assume that the control $u(t)$ transfers a point z^0, $\|z^0\| \le \gamma$ to 0 for the time $\theta \ge 0$ along the trajectory $z(t)$, $t \in [0,\theta]$ of (3.1) . Then

$$z^0 + \int_0^\theta e^{-At} b\,(u(t) + g(z(t)))\,dt = 0. \tag{3.2}$$

Let G be the Kalman matrix

$$G = (b, Ab, \dots, A^{n-1}b)$$

and $\ell(t)$ be the vector with coordinates

$$\ell_k(t) = (-1)^{k-1} \frac{t^{k-1}}{(k-1)!}, \quad k = 1, \dots, n.$$

Then

$$e^{-At} b = G\ell(t) + R(t), \tag{3.3}$$

where

$$R(t)=\sum_{i=n}^{\infty}\frac{1}{i!}A^{i}b(-t)^{i}=\overline{o}(t^{n-1}),\quad t\to 0.$$

Since $\operatorname{rank} G=n$, equality (3.2) can be rewritten as

$$\xi=-G^{-1}z^{0}-G^{-1}\int_{0}^{\theta}R(t)(u(t)+g(z(t)))dt-\eta,$$

where the coordinates ξ_k,η_k, of the vectors ξ and η satisfy the relations

$$\int_{0}^{\theta}\frac{(-1)^{k-1}}{(k-1)!}t^{k-1}u(t)dt=\xi_k,\quad k=1,\ldots,n;\quad |u(t)|\le 1,\quad \theta\to\min, \tag{3.4}$$

and

$$\int_{0}^{\theta}\frac{(-1)^{k-1}}{(k-1)!}t^{k-1}g(z(t))dt=\eta_k,\quad k=1,\ldots,n. \tag{3.5}$$

Relations (3.4) pose the Markov power $(-1,1)$ moment problem. If we consider this problem on the minimal possible interval (Markov min-problem, see [9])

$$\int_{0}^{\theta}\frac{(-1)^{k-1}}{(k-1)!}t^{k-1}u(t)dt=\xi_k,\quad k=1,\ldots,n;\quad |u(t)|\le 1,\quad \theta\to\min, \tag{3.6}$$

then the solution $(\theta_\xi,u_\xi(t))$ exists, is unique, and $u_\xi(t)$ is n-bang-bang.

Following [9] we introduce operators D and P defined as:

$$D:\mathbb{R}^n\to\mathbb{R}\times L_\infty[0,\infty),\quad D(\xi)=(\theta_\xi,\widehat{u}_\xi(t)),$$

where the pair $(\theta_\xi,u_\xi(t))$ is the solution of min-problem (3.6)

$$\widehat{u}_\xi(t))=\begin{cases} u(t), & t\in[0,\theta_\xi],\\ 0, & t>\theta_\xi,\end{cases}$$

and

$$P:\ \mathbb{R}\times L_\infty[0,\infty)\to\mathbb{R}^n,\quad P(y,w(t))=-G^{-1}\int_{0}^{y}R(t)w(t)dt.$$

Along with these operators we consider operators P_1 and P_2 given by

$$P_1:\ \mathbb{R}\times L_\infty[0,\infty)\to\mathbb{R}^n,\quad P_1(y,w(t))=-G^{-1}\int_{0}^{y}R(t)g(z(t))dt,$$

$$P_2:\ \mathbb{R}\times L_\infty[0,\infty)\to\mathbb{R}^n,\quad P_2(y,w(t))=-\eta,$$

where components of vector η are given by formula (3.5) in which $\theta=y$ and $z(t)$ is the solution of the Cauchy problem

$$\dot{z}=Az+b(w(t)+g(z)),\quad z(0)=z^0. \tag{3.7}$$

The domain of the operators P_1, P_2 is the set of the pairs $(y, w(t))$ such that the solution of the Cauchy problem (3.7) is uniquely defined on the interval $[0, y]$.

With the notation introduced above the existence of an n-bang-bang control transferring z^0 to 0 by virtue of system (3.1) is equivalent to the existence of a fixed point of the operator

$$F_{z^0}: \ \mathbb{R}^n \to \mathbb{R}^n, \ F_{z^0}(\xi) = -G^{-1}z^0 + (P + P_1 + P_2) * D(\xi),$$

where $*$ means a composition. Moreover, if ξ^0 is a fixed point of F_{z^0} then such an n-bang-bang control $u(t)$, $t \in [0, \theta]$ can be chosen as

$$u(t) = \widehat{u}(t), \quad t \in [0, \theta], \quad \text{where} \quad (\theta, \widehat{u}(t)) = D(\xi^0).$$

Now we prove

Theorem 3.1. *There exists a neighborhood of the origin U such that for any $z^0 \in U$ the operator F_{z^0} has at least one fixed point.*

Proof. Denote by U_δ, $\delta > 0$, the parallelepiped of the form

$$U_\delta = \left\{\xi \in \mathbb{R}^n : \ |\xi_k| \le \delta^k / k!, \ k = 1, \ldots, n\right\}.$$

Let $C = \sup\limits_{\xi \in U_1} \theta_\xi > 0$, where $(\theta_\xi, u_\xi) = D(\xi)$. Then $\sup\limits_{\xi \in U_\delta} \theta_\xi = C\delta$. Let us choose $\delta_0 > 0$ so small that for any initial point $z^0 \in U_{\delta_0}$ and for any measurable function $u(t)$ such that $|u(t)| \le 1$ the solution of the Cauchy problem

$$\dot{z} = Az + b(u + g(z)), \quad z(0) = z^0 \tag{3.8}$$

exists on the interval $[0, C\delta_0]$. Then for every $z^0 \in U_{\delta_0}$ the operator F_{z^0} is defined.
Next we choose a number $\delta_1 : \ 0 < \delta_1 \le \min\{\delta_0, 1/C\}$ such that for $t \in [0, C\delta_1]$ the following estimates hold

1. $|R_k(t)/t^{n-1}| \le 1/(4q\|G^{-1}\|(n-1)!)$, $k = 1, \ldots, n$;

2. $\|z(t)\| \le 1/(4qL)$ and therefore $|g(z(t))| \le 1/(4q)$,

where $R_k(t)$ are k-th entry of the vector function $R(t)$, $q = \max\{C^n, 1\}$ and $z(t)$ is a solution of (3.8) with an arbitrary initial state z^0 such that $4G^{-1}z^0 \in U_{\delta_1}$.

Then for $z^0 \in U = \{z : \ 4G^{-1}z \in U_{\delta_1}\}$ and for any $\xi \in U_{\delta_1}$ the components of the vectors $P * D(\xi)$, $P_1 * D(\xi)$, $P_2 * D(\xi)$ satisfy the following estimates:

$$\begin{aligned} &|(P * D(\xi))_k| \le C^n \delta_1^n / (4qn!) \le \delta_1^k / (4k!), \\ &|(P_1 * D(\xi))_k| \le C^n \delta_1^n / (16q^2 n!) \le \delta_1^k / (4k!), \qquad k = 1,\ 2, \ldots, n. \\ &|(P_2 * D(\xi))_k| \le C^k \delta_1^k / (4qk!) \le \delta_1^k / (4k!), \end{aligned}$$

Therefore $F_{z^0}(\xi) \in U_{\delta_1}$, i.e. for $z^0 \in U$ the operator F_{z^0} maps the parallelepiped U_{δ_1} to itself. Let us prove that this operator is continuous. Continuity of the operator $P * D$ is established in [9, 10]. So we need to prove the continuity of $P_1 * D$, $P_2 * D$. Let $\xi_m \to \xi$ as $m \to \infty$, $D(\xi) = (\theta_\xi, \widehat{u}_\xi(t))$, $D(\xi_m) = (\theta_{\xi_m}, \widehat{u}_{\xi_m}(t))$.

Then [10] $\theta_{\xi_m} \to \theta_\xi$ and $\operatorname{mes} E_m \to 0$ as $m \to \infty$, where $E_m = \left\{t \in [0, \theta_\xi] : \ \widehat{u}_{\xi_m(t)} \ne \widehat{u}_\xi(t)\right\}$. Let $z(t)$, $z_m(t)$ be solutions of the Cauchy problem (3.8) with the controls $u(t) = u_\xi(t)$ and $u(t) = u_{\xi_m}(t)$, respectively. Then the following estimate holds:

$$\|z_m(t) - z(t)\| \le 2\|b\| \operatorname{mes} E_m + M \int_0^t \|z_m(s) - z(s)\| ds, \quad t \in [0, \min\{\theta_\xi, \theta_{\xi_m}\}], \quad M = \|A\| + L\|b\|.$$

Then applying the Gronwall-Bellman Lemma we conclude that the sequence $z_m(t)$ uniformly converges to $z(t)$ on every interval $[0,\hat{\tau}]$, $0<\hat{\tau}<\theta_\xi$. As a consequence, $g(z_m(t))$ uniformly converges to $g(z(t))$, $t\in[0,\hat{\tau}]$. This gives $P_i * D(\xi_m)\to P_i * D(\xi)$, $i=1,2$. The continuity is proved. Thus, for any $z\in U$ the operator F_z is a continuous operator transferring the convex compact set U_{δ_1} to itself. We complete the proof by applying the Schauder Theorem. □

Corollary 3.2. *For any vector z^0 from a neighborhood of the origin there exists an n-bang-bang control transferring z^0 to* 0 *by virtue of system* (3.1)

Now let us consider the question on the extremal property of n-bang-bang controls, i.e. on the property to satisfy a necessary condition of the time optimality given by the Pontryagin Maximum Principle. In our case, if a control $u(t)$, $|u(t)|\le 1$ transferring the point z^0 to 0 for the time θ, by virtue of system (3.1), is time optimal then it has the form

$$u(t)=\operatorname{sign}\langle\psi(t),b\rangle,\quad t\in[0,\theta], \tag{3.9}$$

where $\psi(t)$ is a nonzero solution of the conjugate system:

$$\dot{\psi}=-\left(A+b\frac{d}{dz}g(z(t))\right)^*\psi, \tag{3.10}$$

$z(t)$ is the solution of the Cauchy problem

$$\dot{z}=Az+b(u(t)+g(z)),\quad z(0)=z^0$$

(below we assume that the function $g(z)$ is continuously differentiable). Let $\Phi(t)$, $(\Phi(0)=I)$ be the fundamental matrix of the system

$$\dot{y}=\left(A+b\frac{d}{dz}g(z(t))\right)y.$$

Then a general solution of system (3.10) has the form $\psi(t)=\Phi^{-1^*}(t)\psi^0$, $\psi^0\in\mathbb{R}^n$. Hence, condition (3.9) can be written as

$$u(t)=\operatorname{sign}\langle\psi^0,\Phi^{-1}(t)b\rangle,\quad t\in[0,\theta],\quad \psi^0\neq 0. \tag{3.11}$$

Let us assume that the components of the vector $\Phi^{-1}(t)b$ form a Chebyshev system (T-system) on the interval $[0,\theta]$. Then $\operatorname{sign}\langle\psi^0,\Phi^{-1}(t)b\rangle$ runs over the set of all n-bang-bang controls on $[0,\theta]$ while the vector ψ^0 runs over the space $\mathbb{R}^n\backslash\{0\}$. Thus, if an n-bang-bang control $u(t)$ transfers z^0 to 0 and the components of the vector $\Phi^{-1}(t)b$ form a T-system on $[0,\theta]$, then $u(t)$ satisfies necessary conditions of time optimality.

Let us express the matrix $\Phi(t)$ via the matrix exponential e^{At}. We have

$$\Phi(t)=e^{At}(I+K(t)),\quad t\in[0,\theta],$$

where $K(t)$ is the solution of the matrix Cauchy problem

$$\dot{K}(t)=e^{-At}b\frac{d}{dz}g(z(t))e^{At}(I+K(t)),\quad K(0)=0.$$

For a sufficiently small $\theta>0$ one can write

$$\Phi^{-1}(t)=\sum_{m=0}^{\infty}(-1)^m K^m(t)e^{-At}. \tag{3.12}$$

Let us denote by $\varphi_i(t)$, $q_i(t)$, $i=1,\dots,n$ the components of vectors $\Phi^{-1}(t)b$, $e^{-At}b$, respectively, and let $K^m(t)=\left(k_{ij}^m(t)\right)_{i,j=1}^n$, $m=1,2,\dots$. Then (3.12) yields

$$\varphi_i(t)=q_i(t)+\sum_{m=1}^{\infty}(-1)^m\sum_{j=1}^{n}k_{ij}^m(t)q_j(t),\quad i=1,\dots,n. \tag{3.13}$$

We summarize our arguments as the following

Statement 3.3. *Let an n-bang-bang control $u(t)$ transfer a point z^0 (from a small enough neighborhood of the origin) to* 0 *for the time θ by virtue of system* (3.1)*. Let functions $\varphi_i(t)$, $i = 1,\ldots,n$ given by formula* (3.13) *form a T-system on the interval* $[0,\theta]$*. Then the control $u(t)$ is extremal in the sense of time optimality.*

4 Bang-bang Controls for A-linearizable Systems

Let us assume that system (2.1) is locally A-linearizable in the domain Q and $0 \in \operatorname{int} Q$ is a stationary point, i.e. $f(0,0) = 0$. Then there exists some neighborhood of the origin $Q' \subset Q$ in which (2.1) is mapped to a system of the form (2.5) by means of an invertible mapping $z = F(x) \in C^2(Q')$ and moreover (see Remark 2.4) $F(0) = 0$ and $g(0) = 0$. This means that $F(Q')$ is also a neighborhood of the origin and one can use the results of Section 3 for the system (2.5). Namely, combining Theorems 2.3 and 3.1 we obtain

Theorem 4.1. *Let system* (2.1) *satisfy all assumptions of Theorem* 2.3*. Then for some neighborhood of the origin $Q'' \subset Q$ the following property holds: for any $x^0 \in Q''$ there exists an n-bang-bang control $u(t)$ transferring x^0 to* 0 *by virtue of system* (2.1).

Consider now the time-optimal problem for A-linearizable system (see (2.1), (2.6))

$$\dot{x} = f(x,u) = a(x) + b(x)u, \quad a(0) = 0, \quad |u| \le 1,$$
$$x(0) = x^0 \in Q', \quad x(\theta) = 0, \quad \theta \to \min.$$

It is obvious that the optimal control for this problem is the same as in the time-optimal problem for system (2.5)

$$\dot{z} = A_0 z + b_0(u + g(z)) = A_0 z + b_0 \left(u + h(F^{-1}(z))\right), \quad |u| \le 1,$$
$$z(0) = z^0 = F(x_0), \quad z(\theta) = 0, \quad \theta \to \min,$$

where $F(x)$, $h(x)$ are given in Theorem 2.3. Thus, the analysis of time optimality for A-linearizable systems is reduced to the case of systems of the form (2.5).

It is well known [9] that if a function $g(z)$ is linear then n-bang-bang controls transferring z^0 to 0 are time optimal for sufficiently small z^0. Hence in the case of linear $g(z)$ there exists a neighborhood of the origin $\bar{Q}$ such that the time-optimal transferring from $x^0 \in \bar{Q}$ to 0 by virtue of the initial system is realized by n-bang-bang controls.

In the general case the question on possible optimality of n-bang-bang controls can be analyzed by using Statement 3.3. Below we give one more case where the extremality of n-bang-bang controls is established directly.

Example 4.2. Let in system (2.5) function g have the special form $g(z) = g(z_n)$. In this case system (3.10) takes the form

$$\dot{\psi}_1 = 0,\ \dot{\psi}_k = -\psi_{k-1},\ k = 2,\ldots,n-1; \quad \dot{\psi}_n = -\psi_{n-1} - g'(z_n)\psi_n$$

and the optimal control (3.9) is

$$u(t) = \operatorname{sign}\langle \psi(t), b_0\rangle = \operatorname{sign}\psi_n(t), \quad t \in [0,\theta].$$

Let us show that for any nontrivial solution $\psi(t) = (\psi_1(t),\ldots,\psi_n(t))$ function $\psi_n(t)$ has no more than $(n-1)$ real zeros. Indeed,

$$\psi_n(t) = e^{-g'(z_n(t))}\left(\psi_n^0 - \int_0^t e^{g'(z_n(\tau))}\psi_{n-1}(\tau)d\tau\right)$$

and $\psi_{n-1}(t) = \sum_{k=0}^{n-2} c_k t^k$. If $\psi_{n-1}(t) \equiv 0$ then $\psi_n^0 \neq 0$ and $\psi_n(t)$ has no real zeros. Let $\psi_{n-1}(t) \not\equiv 0$. We observe that all real zeros of $\psi_n(t)$ are also zeros of the function

$$\varphi(t) = \psi_n^0 - \int_0^t e^{g'(z_n(\tau))}\psi_{n-1}(\tau)d\tau.$$

But the derivative of φ is

$$\varphi'(t) = e^{g'(z_n(t))}\psi_{n-1}(t)$$

and hence it has no more than $(n-2)$ real zeros. Therefore $\varphi(t)$ has no more than $(n-1)$ zeros. Finally, $\psi_n(t)$ has no more than $(n-1)$ zeros, hence $u(t)$ is n-bang-bang.

This means that any n-bang-bang control realizing transferring from z^0 to 0 is extremal.

Finally we give a numerical algorithm of construction of an n-bang-bang control for system (2.8) (and, therefore for A-linearizable system (2.1)). The algorithm is based on solving power Markov min-problem [9], [10]. The constructive analytic solution of this problem was first given in [11]. A development of the topic can be found in [12], [13].

Algorithm. Let a control $u(t)$ transfer the point z^0 from some neighborhood of the origin to 0 by virtue of the system (2.5) for the time θ. Then

$$z^0 + \int_0^\theta e^{-A_0\tau} b_0\Big(u(\tau) + g(z(\tau))\Big)\, d\tau = 0. \tag{4.1}$$

Let us denote

$$\ell(\tau) = e^{-A_0\tau} b_0 = \begin{pmatrix} (-1)^n \frac{\tau^{n-1}}{(n-1)!} \\ \cdots \\ -\tau \\ 1 \end{pmatrix},$$

then

$$-z^0 = \int_0^\theta \ell(\tau)u(\tau)\, d\tau + \int_0^\theta \ell(\tau)g(z(\tau))\, d\tau. \tag{4.2}$$

We denote

$$r_1(\theta, u(\cdot)) = \int_0^\theta \ell(\tau)g(z(\tau))\, d\tau$$

and rewrite equation (4.2) as

$$-z^0 = \int_0^\theta \ell(\tau)u(\tau)\, d\tau + r_1(\theta, u(\cdot)). \tag{4.3}$$

Let us consider the following method of successive approximations for finding θ and $u(t)$. On the first step we omit the reminder in the equation (4.3) and find the optimal time θ and the control $u(t)$ as a solution of the Markov min-problem

$$-z^0 = \int_0^\theta \ell(\tau)u(\tau)\, d\tau, \quad |u(t)| \leq 1, \quad \theta \to \min.$$

In scalar form the moment equalities are as follows:

$$(-1)^k(k-1)!z^0_{n-k+1} = \int_0^\theta \tau^{k-1}u(\tau)\,d\tau, \quad k = 1,\dots,n. \tag{4.4}$$

It is well known [11] that this problem has the unique solution θ_1, $u_1(t)$, and, moreover, the control $u_1(t)$ is n-bang-bang.

Once we know θ_1, $u_1(t)$, we put

$$r_1(\theta_1,u_1(\cdot)) = \int_0^{\theta_1} \ell(\tau)g(z^{(1)}(\tau))\,d\tau, \tag{4.5}$$

where $z^{(1)}(\tau)$ is the solution of the following Cauchy problem:

$$\dot z^{(1)}(t) = A_0 z^{(1)}(t) + b_0(g(z^{(1)}(t)) + u_1(t)), \quad z^{(1)}(0) = z^0.$$

Using the equality (4.5), we rewrite (4.2) as follows

$$-z^0 = \int_0^\theta \ell(\tau)u(\tau)\,d\tau + r_1(\theta_1,u_1(\cdot)) + \Big(\int_0^\theta \ell(\tau)g(z(\tau))\,d\tau - r_1(\theta_1,u_1(\cdot))\Big) = \int_0^\theta \ell(\tau)u(\tau)\,d\tau + r_1(\theta_1,u_1(\cdot)) + r_2(\theta,u(\cdot)),$$

where

$$r_2(\theta,u(\cdot)) = \int_0^\theta \ell(\tau)g(z(\tau))\,d\tau - r_1(\theta_1,u_1).$$

Now we solve min-problem

$$-z^0 - r_1(\theta_1,u_1(\cdot)) = \int_0^\theta \ell(\tau)u(\tau)\,d\tau. \tag{4.6}$$

Let θ_2 and $u_2(t)$ be a solution of this problem. Let us find $z^{(2)}(t)$ as the solution of the Cauchy problem

$$\dot z^{(2)}(t) = A_0 z^{(2)}(t) + b_0(g(z^{(2)}(t)) + u_2(t)), \quad z^{(2)}(0) = z^0,$$

and put

$$r_2(\theta_2,u_2(\cdot)) = \int_0^{\theta_2} \ell(\tau)g(z^{(2)}(\tau))\,d\tau - r_1(\theta_1,u_1(\cdot)).$$

We rewrite equality (4.2) as

$$-z^0 = \int_0^\theta \ell(\tau)u(\tau)\,d\tau + r_1(\theta_1,u_1(\cdot)) + r_2(\theta_2,u_2(\cdot)) + r_3(\theta,u(\cdot)),$$

where

$$r_3(\theta,u(\cdot)) = \int_0^\theta \ell(\tau)g(z(\tau))\,d\tau - r_1(\theta_1,u_1(\cdot)) - r_2(\theta_2,u_2(\cdot)).$$

Further, we solve the min-problem

$$-z^0 - r_1(\theta_1, u_1(\cdot)) - r_2(\theta_2, u_2(\cdot)) = \int_0^{\theta} \ell(\tau)u(\tau)\,d\tau \tag{4.7}$$

and find θ_3 and $u_3(t)$. We put

$$r_3(\theta_3, u_3(\cdot)) = \int_0^{\theta_3} \ell(\tau)g(z^{(3)}(\tau))\,d\tau - r_1(\theta_1, u_1(\cdot)) - r_2(\theta_2, u_2(\cdot)),$$

and so on. This iterative procedure may be written as follows. Let $r_1(\theta_1, u_1(\cdot))$, ..., $r_m(\theta_m, u_m(\cdot))$ be found, then we find θ_{m+1} and $u_{m+1}(t)$ as the solution of the min-problem

$$-z^0 - \sum_{i=1}^{m} r_i(\theta_i, u_i(\cdot)) = \int_0^{\theta} \ell(\tau)u(\tau)\,d\tau \tag{4.8}$$

and put

$$r_{m+1}(\theta_{m+1}, u_{m+1}(\cdot)) = \int_0^{\theta_{m+1}} \ell(\tau)g(z^{(m+1)}(\tau))\,d\tau - \sum_{i=1}^{m} r_i(\theta_i, u_i(\cdot)),$$

where $z^{(m+1)}(t)$ is the solution of the Cauchy problem

$$\dot{z}^{(m+1)}(t) = A_0 z^{(m+1)}(t) + b_0\Big(g(z^{(m+1)}(t)) + u_{m+1}(t)\Big), \quad z^{(m+1)}(0) = z^0.$$

Realizing this algorithm, on each step we solve the Cauchy problem of the form

$$\dot{z}(t) = A_0 z(t) + b_0(g(z(t)) + \widetilde{u}(t)), \quad z(0) = z^0,\ t \in [0, \widetilde{\theta}], \tag{4.9}$$

where $\widetilde{\theta}$ and $\widetilde{u}(t)$ are given, and calculate $\int_0^{\widetilde{\theta}} \ell(\tau)g(z(\tau))\,d\tau$. Let us show that the value of this integral depends only on the value of the solution of the Cauchy problem at the final time moment $\widetilde{\theta}$. Indeed, let $z(t)$ be the solution of (4.9), then

$$z(\widetilde{\theta}) = e^{A_0\widetilde{\theta}}\left(z^0 + \int_0^{\widetilde{\theta}} \ell(\tau)g(z(\tau))\,d\tau + \int_0^{\widetilde{\theta}} \ell(\tau)\widetilde{u}(\tau)\,d\tau\right).$$

Therefore,

$$\int_0^{\widetilde{\theta}} \ell(\tau)g(z(\tau))\,d\tau = e^{-A_0\widetilde{\theta}}z(\widetilde{\theta}) - z^0 - \int_0^{\widetilde{\theta}} \ell(\tau)\widetilde{u}(\tau)\,d\tau.$$

Thus,

$$r_{m+1}(\theta_{m+1}, u_{m+1}(\cdot)) = e^{-A_0\theta_{m+1}}z^{(m+1)}(\theta_{m+1}) - z^0 - \int_0^{\theta_{m+1}} \ell(\tau)u_{m+1}(\tau)\,d\tau - \sum_{i=1}^{m} r_i(\theta_i, u_i(\cdot)).$$

However, since θ_{m+1} and $u_{m+1}(t)$ are the solutions of min-problem (4.8), the last equality may be written as

$$r_{m+1}(\theta_{m+1}, u_{m+1}(\cdot)) = e^{-A_0\theta_{m+1}}z^{(m+1)}(\theta_{m+1}).$$

Finally, the iterative procedure has the following form: under known $z^{(1)}(\theta_1), \dots, z^{(m)}(\theta_m)$ to find θ_{m+1} and $u_{m+1}(t)$ as the solution of the min-problem

$$-\Big(z^0 + \sum_{i=1}^{m} e^{-A_0\theta_i} z^{(i)}(\theta_i)\Big) = \int_0^{\theta_{m+1}} \ell(\tau) u_{m+1}(\tau)\, d\tau, \quad m = 1, 2, \dots, \tag{4.10}$$

and then to find $z^{(m+1)}(\theta_{m+1})$ as the solution of the Cauchy problem

$$\dot z^{(m+1)}(t) = A_0 z^{(m+1)}(t) + b_0\Big(g(z^{(m+1)}(t)) + u_{m+1}(t)\Big), \quad z^{(m+1)}(0) = z^0, \quad t \in [0, \theta_{m+1}].$$

In the present paper we do not discuss the convergence of the proposed algorithm. In [10] it is proved, that an equivalent algorithm for a linear system of the form (2.8) converges. In our forthcoming work we are going to prove the convergence for systems (2.8) with analytic function $g(z)$.

Example 4.3. Let us consider the nonlinear control system

$$\dot z_1 = z_2, \quad \dot z_2 = z_3, \quad \dot z_3 = u + z_3^2$$

and find an n-bang-bang control transferring the point $z^0 = (0, 0, 0.1)$ to the origin. Using the proposed algorithm, we find k, θ_k and a control $u_k(t)$ such that $\|z^{(k)}(\theta)\| < 10^{-10}$. On the first step, solving the min-problem of the form

$$-2z^0 = \int_0^{\theta} \tau^2 u(\tau)\, d\tau,\ z_2^0 = \int_0^{\theta} \tau u(\tau)\, d\tau,\ -z_3^0 = \int_0^{\theta} u(\tau)\, d\tau,$$

we obtain that $\theta_1 = 0.4390312689049587$, the control u_1 has two switchings $t_1^{(1)} = 0.1995057194364092$, $t_2^{(1)} = 0.36902135388888835$ and $u_1(\theta - 0) = -1$. Solving the Cauchy problem

$$\dot z_1^{(1)} = z_2^{(1)}, \quad \dot z_2^{(1)} = z_3^{(1)}, \quad \dot z_3^{(1)} = u_1(t) + (z_3^{(1)})^2, \quad z^{(1)}(0) = z^0,$$

we have $\|z^{(1)}(\theta_1)\| = 0.0012586396145635088$,

$$z^{(1)}(\theta_1) = \begin{pmatrix} 0.000048262522943310304 \\ 0.00030995420364153217 \\ 0.0012189228030817488 \end{pmatrix}, \quad z^0 + e^{-A_0\theta_1} z^{(1)}(\theta_1) = \begin{pmatrix} 0.000029655679191781402 \\ -0.00022519102129263703 \\ 0.10121892280308176 \end{pmatrix}.$$

On the second step we solve min-problem (4.6) and obtain that $\theta_2 = 0.4402879856552574$, the control $u_2(t)$ has two switchings $t_1^{(2)} = 0.2004299838427155$, $t_2^{(2)} = 0.3699645152688034$ and $u_2(\theta - 0) = -1$. Solving the Cauchy problem

$$\dot z_1^{(2)} = z_2^{(2)}, \quad \dot z_2^{(2)} = z_3^{(2)}, \quad \dot z_3^{(2)} = u_2(t) + (z_3^{(2)})^2, \quad z^{(2)}(0) = z^0,$$

we have $\|z^{(2)}(\theta_2)\| = 0.0001343033798545341$,

$$z^{(2)}(\theta_2) = \begin{pmatrix} 5.127351445259204 \times 10^{-6} \\ 0.00013420467173401538 \\ -4.6280911102543736 \times 10^{-7} \end{pmatrix}, \quad z^0 + \sum_{i=1}^{2} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} -0.000024350532531642497 \\ -0.00009078258026738536 \\ 0.10121845999397074 \end{pmatrix}.$$

Let us give the results for the next steps.

Step 3: $\theta_3 = 0.4384037935557613$, $t_1^{(3)} = 0.20043776078740605$, $t_2^{(3)} = 0.3690304275683015$, $u_3(\theta - 0) = -1$; $\|z^{(3)}(\theta_3)\| = 0.0009325647933427833$,

$$z^{(3)}(\theta_3) = \begin{pmatrix} -9.792500010265548 \times 10^{-8} \\ -1.5397800728669132 \times 10^{-6} \\ 0.0009325635170166639 \end{pmatrix}, \quad z^0 + \sum_{i=1}^{3} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} 0.00006584495625426516 \\ -0.0005011617439320605 \\ 0.1021510235109874 \end{pmatrix}.$$

Step 4: $\theta_4 = 0.4393448985461692$, $t_1^{(4)} = 0.20043626488760394$, $t_2^{(4)} = 0.36903320240519494$, $u_4(\theta-0) = -1$; $\|z^{(4)}(\theta_4)\| = 4.263169161312102\times10^{-6}$,

$$z^{(4)}(\theta_4) = \begin{pmatrix} -7.2590783684969885\times10^{-9} \\ -2.1097266220377417\times10^{-7} \\ -4.257939541556036\times10^{-6} \end{pmatrix}, \quad z^0 + \sum_{i=1}^{4} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} 0.0000655194448057307 \\ -0.0004995020125783636 \\ 0.10214676557144584 \end{pmatrix}.$$

Step 5: $\theta_5 = 0.4393436599430583$, $t_1^{(5)} = 0.20043626880368812$, $t_2^{(5)} = 0.3690347159894944$, $u_6(\theta-0) = -1$; $\|z^{(5)}(\theta_5)\| = 1.4946359520492283\times10^{-6}$,

$$z^{(5)}(\theta_5) = \begin{pmatrix} 4.013987086775414\times10^{-11} \\ 1.2925790243183083\times10^{-9} \\ -1.4946353925912367\times10^{-6} \end{pmatrix}, \quad z^0 + \sum_{i=1}^{5} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} 0.00006537466766646282 \\ -0.0004988440614156778 \\ 0.10214527093605325 \end{pmatrix}.$$

Step 6: $\theta_6 = 0.4393421462712416$, $t_1^{(6)} = 0.20043626867110573$, $t_2^{(6)} = 0.3690347063387003$, $u_6(\theta-0) = -1$; $\|z^{(6)}(\theta_6)\| = 9.421311485926116\times10^{-9}$,

$$z^{(6)}(\theta_6) = \begin{pmatrix} -7.830901194625525\times10^{-13} \\ -2.3550444572138706\times10^{-11} \\ 9.421282018821732\times10^{-9} \end{pmatrix}, \quad z^0 + \sum_{i=1}^{6} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} 0.00006537558648517041 \\ -0.0004988482241323852 \\ 0.10214528035733526 \end{pmatrix}.$$

Step 7: $\theta_7 = 0.4393421560328203$, $t_1^{(7)} = 0.20043626867221803$, $t_2^{(7)} = 0.3690347065099603$, $u_7(\theta-0) = -1$; $\|z^{(7)}(\theta_7)\| = 1.7028474502025607\times10^{-10}$,

$$z^{(7)}(\theta_6) = \begin{pmatrix} 2.3586904919979293\times10^{-12} \\ 1.3904432191117115\times10^{-11} \\ -1.696997281412649\times10^{-10} \end{pmatrix}, \quad z^0 + \sum_{i=1}^{7} e^{-A_0\theta_i} z^{(i)}(\theta_i) = \begin{pmatrix} 0.0000653755663572071 \\ -0.0004988481356717085 \\ 0.10214528018763554 \end{pmatrix}.$$

Step 8: $\theta_8 = 0.43934215588800285$, $t_1^{(8)} = 0.20043626871864134$, $t_2^{(8)} = 0.3690347065688252$, $u_8(\theta-0) = -1$;

$$z^{(8)}(\theta_8) = \begin{pmatrix} -2.1234430410059927\times10^{-12} \\ -7.045249222221441\times10^{-12} \\ -1.0134017695463914\times10^{-11} \end{pmatrix} \quad \text{and} \quad \|z^{(8)}(\theta_8)\| = 1.2523692011686662\times10^{-11}.$$

Graphics of the components of the trajectory $z^{(8)}(t)$ are presented on Fig. 1– 6.

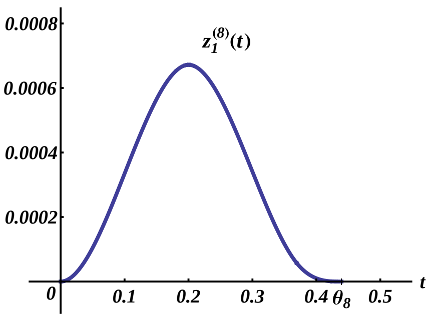


Figure 1. Graphic of the $z_1^{(8)}(t)$ on the segment $[0, \theta_8]$.

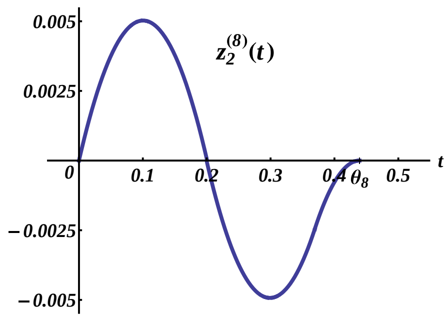


Figure 2. Graphic of the $z_2^{(8)}(t)$ on the segment $[0, \theta_8]$.

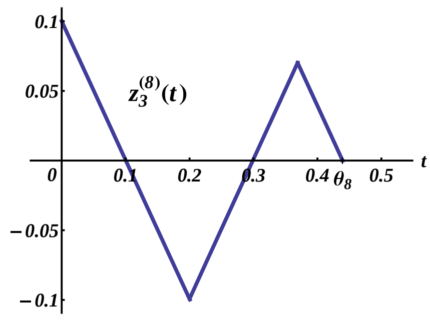
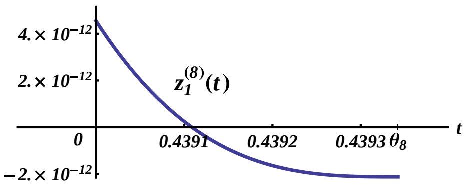
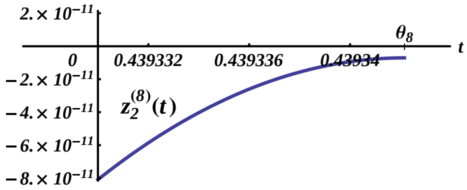


Figure 3. Graphic of the $z_3^{(8)}(t)$ on the segment $[0, \theta_8]$.





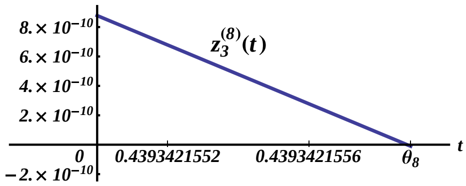


Figure 4. Graphic of the $z_1^{(8)}(t)$ on the segment $[0.439, \theta_8]$.

Figure 5. Graphic of the $z_2^{(8)}(t)$ on the segment $[0.43933, \theta_8]$.

Figure 6. Graphic of the $z_3^{(8)}(t)$ on the segment $[0.439342155, \theta_8]$.

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