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ASYMPTOTIC BEHAVIOR AND STABILITY OF SOLUTIONS TO BAROTROPIC VORTICITY EQUATION ON A SPHERE

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Abstract

Orthogonal projectors on the subspace $\mathbf{H}_n$ of homogeneous spherical polynomials of degree n and on the subspace $\mathbb{P}^N$ of spherical polynomials of degree $n \leq N$ are defined for functions on the unit sphere S, and their derivatives Λ^s of real degree s are introduced by using the multiplier operators. A family of Hilbert spaces $\mathbb{H}^s$ of generalized functions having fractional derivatives of real degree s on S is introduced, and some embedding theorems for functions from $\mathbb{H}^s$ and Banach spaces $\mathbb{L}^p(S)$ and $\mathbb{L}^p(0,T;\mathbb{X})$ on S are given.

Non-stationary and stationary problems for barotropic vorticity equation (BVE) describing the vortex dynamics of viscous incompressible fluid on a rotating sphere S are considered. A theorem on the unique weak solvability of nonstationary problem and theorem on the existence of weak solution to stationary problem are given, and a condition guaranteeing the uniqueness of such steady solution is also formulated.

The asymptotic behaviour of solutions of nonstationary BVE as $t \to \infty$ is studied. Particular forms of the external vorticity source have been found which guarantee the existence of such bounded set $\mathbf{B}$ in a phase space $\mathbf{X}$ that eventually attracts all solutions to the BVE. It is shown that the asymptotic behaviour of the BVE solutions depends on both the structure and the smoothness of external vorticity source. Sufficient conditions for the global asymptotic stability of both smooth and weak solutions are also given.

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1 Introduction

Although the three-dimensional Euler and Navier-Stokes equations are the fundamental equations for the numerical simulation of dynamics of atmosphere and global climatic processes, the shallow-water equations is also widely used as a good approximation for the large-scale atmospheric motions, since the characteristic length scale of horizontal motions is much larger than that of vertical motions [21]. The shallow-water equations support both fast (gravity) waves and slow (Rossby-Haurwitz) waves [6]. The barotropic vorticity equation (BVE) is obtained from the shallow water model as a result of filtering the surface gravity waves. The unique weak solvability of the barotropic vorticity

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equation for ideal fluid on a sphere was shown by Szeptycki [33] under the conditions that the initial stream function $\psi^0(x) \in \mathbb{W}_2^2(S) \cap \mathbb{L}^\infty(S)$, the external vorticity source $F(t,x) \in \mathbb{L}^2(0,T;\mathbb{W}_2^1(S)) \cap \mathbb{L}^\infty(Q)$. Then there exists the unique solution $\psi(t,x)$ such that

$$\psi \in \mathbb{L}^\infty(0,T;\mathbb{W}_2^2(S)), \quad \Delta\psi \in \mathbb{L}^\infty(Q), \quad \psi_t \in \mathbb{L}^\infty(0,T;\mathbb{W}_2^1(S)).$$

Using the theory of integral equations, the global existence, uniqueness and regularity properties of classical solutions of the vorticity equation was proved by Chern, Colin and Kaper [12, Theorems 1-3]. The Schauder fixed-point theorem is used on a set defined by Hölder norms. Ben-Artzi [4] constructed for the Navier–Stokes flow a global-in-time solution when the initial vorticity is integrable. Unlike Giga, Miyakawa and Osada [13] his proof does not appeal to the Nash estimate, but relies on a simple property of the heat equation which makes it easy to prove the continuous dependence of the solution on the initial vorticity. Brezis [5] proves the uniqueness of a continuous solution in $\mathbb{L}^1(\mathbb{R}^2)$ for the vorticity equation for planar Navier–Stokes flow provided that the solution is locally bounded.

The space $\mathbb{L}^2(\mathbb{R}^2)$ is naturally associated with the Navier-Stokes equation, since it is the energy space, because the square of the $\mathbb{L}^2$ norm of u is the total (kinetic) energy of the fluid, which is nonincreasing with time. As it mentioned by Gallagher and Gallay [12], the first mathematical result on the Cauchy problem is due to Leray [19] who proved that, for any initial data $u_0 \in \mathbb{L}^2(\mathbb{R}^2)$, the Navier-Stokes system has a unique global solution $u \in \mathbb{C}^0([0,+\infty),\mathbb{L}^2(\mathbb{R}^2))$ such that $u(\cdot,0) = u_0$ and $\nabla u \in \mathbb{L}^2((0,+\infty),\mathbb{L}^2(\mathbb{R}^2))$. The unique solvability of two-dimensional Navier-Stokes equation with a measure as initial vorticity was considered by Gallagher and Gallay [12] in which the authors study the uniqueness of solutions to the Cauchy problem for the vorticity equation in the whole plane $\mathbb{R}^2$ when the initial vorticity is taken from $M(\mathbb{R}^2)$, the space of finite Radon measures. The existence of a global solution to the problem was established earlier (Giga, Miyakawa and Osada [13], Kato [16]), but the uniqueness for an arbitrary finite Radon measure as initial vorticity has been proved by Gallagher and Gallay [12]. To be precise, the authors prove that, any solution of the 2D Navier–Stokes equation, whose vorticity distribution is uniformly bounded in $\mathbb{L}^1(\mathbb{R}^2)$ for positive times is entirely determined by the trace of the vorticity at $t = 0$, which is a finite measure.

The analysis of classes of functions, in which there exists unique solution of the vorticity equation on a rotating sphere, is of special theoretical interest. Such analysis becomes particularly important in the study of stability of solutions. All known results on the unique solvability of 2D Navier-Stokes equations in a bounded domain of Euclidean space $\mathbb{R}^2$ differ mainly in (i) technique, (ii) spaces of generalized functions under consideration and (iii) procedure to construct approximate solutions [3, 17, 20, 30, 34].

The vorticity equation considered here takes into account the Rayleigh friction of the form $\sigma\Delta\psi$, the term $2\psi_\lambda$ describing the rotation of sphere, the external vorticity source (forcing) $F(t,x)$, and the turbulent viscosity term of common form $\nu(-\Delta)^{s+1}\psi$, where $s \geqslant 1$ is an arbitrary real number. The case $s = 1$ corresponds to the classical form used in Navier-Stokes equations (see Ladyzhenskaya [17], Temam [34, 35], Szeptycki [32, 33], Dymnikov and Filatov [7], Ilyin and Filatov [14, 15], while the case $s = 2$ was considered by Simmons, Wallace and Branstator [22], Dymnikov and Skiba [8-10], Skiba [28], etc. The turbulent term of such form for natural numbers s is applied by Lions [20] for studying the solvability of Navier-Stokes equations in a limited area by the artificial viscosity method.

Here we considered only real solutions. Note that for $s = 1$, $\psi_0 \in \mathbb{W}_2^2(S)$, $F \in \mathbb{L}^2(0,T;\mathbb{W}_2^1(S)) \cap \mathbb{L}^\infty(0,T;\mathbb{L}^2(S))$ and $\sigma = 0$, the theorem on the existence and uniqueness of solution $\psi \in \mathbb{L}^\infty(0,T;\mathbb{W}_2^2(S))$ such that $\psi_t \in \mathbb{L}^\infty(0,T;\mathbb{L}^2(S))$ and $\Delta\psi \in \mathbb{L}^2(0,T;\mathbb{W}_2^1(S))$ was proved by Szeptycki ([33], Theorem 3.1). The rotation of sphere is not considered by Szeptycki. It is also shown in [33] that if additionally $\Delta\psi_0 \in \mathbb{L}^\infty(S)$ and $F \in \mathbb{L}^\infty(Q)$ then $\Delta\psi \in \mathbb{L}^\infty(Q)$. The unique solvability of generalized problem for stream function ψ from $\mathbb{L}^2(0,T;\mathbb{W}_2^2(S)) \cap C(0,T;\mathbb{W}_2^1(S))$, $s = 1$, $\sigma = 0$, $\psi_0 \in \mathbb{W}_2^3(S)$, $F \in \mathbb{L}^2(0,T;\mathbb{W}_2^{-1}(S))$ and $F_t \in \mathbb{L}^2(0,T;\mathbb{W}_2^{-2}(S))$ was proved by Ilyin and Filatov [14] ($\mathbb{W}_2^n(S)$ are the Sobolev spaces of the functions orthogonal to a constant on a sphere; see also Dymnikov and Filatov [7] and Ilyin and Filatov [15]). The rotation of sphere was taken into account in [14]. The existence and uniqueness of BVE solution (see below (4.1)) for $s = 1$ and $s = 2$ was proved in [29]. The unique solvability of nonstationary BVE for arbitrary real number $s \geq 1$,as well as the existence of weak solution to the stationary BVE, was shown in [23]. A condition guaranteeing the uniqueness of such steady solution is also given in there. In the works [7, 14, 15], a function on the sphere is treated as the trace of the corresponding function of $\mathbb{R}^3$. Unlike this, in [23, 28, 29] and in the present work, the functional spaces are introduced directly on the sphere.

The asymptotic behaviour of solutions of nonstationary BVE as $t \to \infty$ is studied. Particular forms of the external vorticity source have been found which guarantee the existence of such bounded set **B** in a phase space **X** that eventually attracts all solutions to the BVE. It is shown that the asymptotic behaviour of the BVE solutions depends on both the structure and the smoothness of external vorticity source. Sufficient conditions for the global asymptotic stability of both smooth and weak BVE solutions are also given.

2 Orthogonal Projectors and Fractional Derivatives

Let $S = \{x \in \mathbb{R}^3 : |x| = 1\}$ be a unit sphere in the three-dimensional Euclidean space; we denote by $\mathbb{C}^\infty(S)$ the set of infinitely differentiable functions on S and by

$$\langle f,g\rangle = \int_S f(x)\overline{g}(x)dS \tag{2.1}$$

and

$$\|f\| = \langle f,f\rangle^{1/2} \tag{2.2}$$

the inner product and norm in $\mathbb{C}^\infty(S)$, respectively. Here $x = (\lambda,\mu)$ is a point on the sphere, $dS = d\lambda d\mu$ is an element of sphere surface, $\mu = \sin\phi$; $\mu \in [-1,1]$, ϕ is the latitude, $\lambda \in [0,2\pi)$ is the longitude and $\overline{g}$ is the complex conjugate of function g.

It is known that spherical harmonics

$$Y_n^m(\lambda,\mu) = \left[\frac{2n+1}{4\pi}\frac{(n-m)!}{(n+m)!}\right]^{1/2} P_n^m(\mu)\,e^{im\lambda}\ ,\ \ n \geq 0,\ \ |m| \leq n$$

form orthogonal basis in $\mathbb{C}^\infty(S)$:

$$\left\langle Y_n^m, Y_l^k \right\rangle = \delta_{mk}\delta_{nl}$$

where

$$\delta_{mk} = \begin{cases} 1, & \text{if} \quad m = k \\ 0, & \text{if} \quad m \neq k \end{cases}$$

is the Kronecker delta, and

$$P_n^m(\mu) = \frac{\left(1-\mu^2\right)^{m/2}}{2^n n!}\frac{d^{n+m}}{d\mu^{n+m}}\left(\mu^2-1\right)^n$$

is the associated Legendre function of degree n and zonal wavenumber m.

Let n and m be integer, $n \geq 0$, and $|m| \leq n$. Each spherical harmonic $Y_n^m(\lambda,\mu)$ is the eigenfunction of spectral problem

$$-\Delta Y_n^m = \chi_n Y_n^m \quad , \quad |m| \leq n$$

corresponding to the eigenvalue

$$\chi_n = n\,(n+1)$$

of multiplicity $2n+1$. Here

$$-\Delta = -\frac{\partial}{\partial\mu}\left[\left(1-\mu^2\right)\frac{\partial}{\partial\mu}\right] - \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\lambda^2} \tag{2.3}$$

is symmetric and positive definite Laplace operator on S.

For each integer $n \geq 0$, the span of $2n+1$ spherical harmonics $Y_n^m(\lambda,\mu)$ ($|m| \leq n$) forms a generalized $(2n+1)$-dimensional eigenspace

$$\mathbf{H}_n = \{\psi : -\Delta\psi = \chi_n\psi\} \tag{2.4}$$

corresponding to χ_n. The subspace $\mathbf{H}_n$ is invariant not only with respect to the Laplace operator but also to any transformation of the $SO(3)$ group of rotations of sphere about arbitrary axis through its center.

In order to simplify notation we will also use a multi-index $\alpha \equiv (m,n) \equiv (m_\alpha, n_\alpha)$ for the wavenumber (m,n):

$$\overline{\alpha} \equiv (-m,n) \equiv (-m_\alpha, n_\alpha)\,,$$

$$Y_\alpha \equiv Y_n^m, \qquad \chi_\alpha \equiv \chi_n = n(n+1),$$

$$\sum_{\alpha(k)} \equiv \sum_{n=k}^{\infty}\sum_{m=-n}^{n}. \tag{2.5}$$

We now introduce operators of projection and fractional differentiation (fractional derivatives) of functions on the sphere [23]. Since $(-\Delta)^k\psi \in \mathbb{C}^\infty(S)$ for every $\psi \in \mathbb{C}^\infty(S)$ and any natural k, then due to the formula

$$(-\Delta)^k\psi = \sum_{n=1}^{\infty}\sum_{m=-n}^{n}[n\,(n+1)]^k\ \psi_n^m Y_n^m$$

the Fourier coefficients $\psi_n^m = \langle\psi, Y_n^m\rangle$ of ψ tend to zero as $n \to \infty$ faster than the sequence $1/n^k$ for any degree k.

Definition 2.1. The completion of $\mathbb{C}^\infty(S)$ in the norm (2.2) is the Hilbert space

$$\mathbb{L}^2(S) = \oplus_{n=0}^{\infty}\mathbf{H}_n$$

being the direct orthogonal sum of subspaces $\mathbf{H}_n$ of generalized functions on S with inner product (2.1).

Definition 2.2. Let ω be an angle between two unit radius-vectors $\vec{x}_1$, $\vec{x}_2$ corresponding to the points $x_1, x_2 \in S$. Then $\vec{x}_1 \cdot \vec{x}_2 = \cos\omega$ is the scalar product of vectors $\vec{x}_1$ and $\vec{x}_2$. A function $z(\vec{x}\cdot\vec{y})$ depending only on the distance $\rho(x,y) = \arccos(\vec{x}\cdot\vec{y}) = \omega$ between two points x and y of sphere is called the zonal function. The convolution of a function $\psi \in \mathbb{L}^2(S)$ with a zonal function $Z(\vec{x}\cdot\vec{y}) \in \mathbb{L}^2(S)$ is defined by

$$(\psi * z)(x) = \frac{1}{4\pi}\int_S \psi(y)\; Z(\vec{x}\cdot\vec{y})\, dS(y)$$

(see [23]).

Let $n \geq 0$. Orthogonal projector $Y_n : \mathbb{L}^2(S) \mapsto \mathbf{H}_n$ of $\mathbb{L}^2(S)$ on the subspace $\mathbf{H}_n$ of homogeneous spherical polynomials of degree n is introduced by

$$Y_n(\psi;x) = (2n+1)(\psi * P_n)(x). \tag{2.6}$$

In order to show that (2.6) is really the projector we first prove that

$$Y_n(\psi;x) = \sum_{m=-n}^{n} \psi_n^m Y_n^m(x)$$

and hence, $Y_n(\psi) \in \mathbf{H}_n$. Indeed,

$$P_n(\vec{x}_1\cdot\vec{x}_2) = \frac{4\pi}{2n+1}\sum_{m=-n}^{n} Y_n^m(x_1)\; \overline{Y_n^m}(x_2)$$

for two radius-vectors $\vec{x}_1$ and $\vec{x}_2$ [23]. Therefore, due to Definition 2.2, and formulas (2.6) and (2.5) we have

$$Y_n(\psi;x) = \sum_{m=-n}^{n} \psi_n^m Y_n^m(x)$$

and hence, $Y_n(Y_n(\psi)) = Y_n(\psi)$ for all functions $\psi \in \mathbb{L}^2(S)$ [23]. For the sake of brevity we will sometimes write simply $Y_n(\psi)$ instead of $Y_n(\psi;x)$.

Obviously, any function from subspace $\mathbf{H}_0$ is constant:

$$Y_0(\psi) = \frac{1}{4\pi}\int_S \psi(y)\; dS(y) = Const.$$

Let $N \geqslant 0$ be an integer. We introduce finite dimensional subspaces $\mathbb{P}^N$ and $\mathbb{P}_0^N$ of spherical polynomials of degree $n \leq N$ as direct orthogonal sums of subspaces $\mathbf{H}_n$:

$$\mathbb{P}^N = \oplus_{n=0}^{N}\mathbf{H}_n, \quad \mathbb{P}_0^N = \oplus_{n=1}^{N}\mathbf{H}_n = \left\{\psi \in \mathbb{P}^N : Y_0(\psi) = 0\right\}.$$

Note that the Parseval-Steklov identities

$$\|\psi\|^2 = \sum_{\alpha(0)} |\psi_\alpha|^2 = \sum_{n=0}^{\infty} \|Y_n(\psi)\|^2 \tag{2.7}$$

$$\langle\psi,h\rangle=\sum_{\alpha(0)}\psi_\alpha\overline{h}_\alpha=\sum_{n=0}^{\infty}\langle Y_n(\psi),Y_n(h)\rangle$$

hold for any functions $\psi,h\in\mathbb{L}^2(S)$. Due to (2.7), each function $\psi(x)\in\mathbb{L}^2(S)$ is represented by its own Fourier-Laplace series

$$\psi(x)=\sum_{n=0}^{\infty}Y_n(\psi;x)\equiv\sum_{n=0}^{\infty}\sum_{m=-n}^{n}\psi_n^mY_n^m(x).$$

Definition 2.3. Let $s>0$ and $\psi(x)\in\mathbb{C}^\infty(S)$. A spherical operator $\Lambda^s=(-\Delta)^{s/2}$ of real order s is defined by means of equations

$$Y_n(\Lambda^s\psi)=\chi_n^{s/2}Y_n(\psi)=[n(n+1)]^{s/2}Y_n(\psi)$$

valid for any natural number n.

Thus, Λ^s is a multiplier operator which is completely defined by infinite set of multiplicators $\{\chi_n^{s/2}\}_{n=0}^{\infty}$. We will consider Λ^s as a derivative of real order s of functions on a sphere, besides,

$$\Lambda^s\psi(x)=\sum_{n=1}^{\infty}\chi_n^{s/2}Y_n(\psi;x)\equiv\sum_{\alpha(1)}\chi_\alpha^{s/2}\psi_\alpha Y_\alpha(x). \tag{2.8}$$

In particular, $\Lambda^{2n}=(-\Delta)^n$ for any natural n, and operator Λ can be interpreted as the square root of nonnegative and symmetric Laplace operator (2.3). It is well known that the main disadvantage of local derivatives $\partial^n/\partial\lambda^n$ and $\partial^n/\partial\mu^n$ is that they depend on the choice of coordinate system (i.e., on sphere rotation). The new derivatives Λ^s and projectors Y_n are invariant with respect to any element of the group $SO(3)$ of sphere rotations [23], and hence are free from this disadvantage.

3 Hilbert Spaces

In this section we introduce a family of Hilbert spaces $\mathbb{H}^s$ of generalized functions (distributions) on sphere S, that depends on a real parameter s, besides, a function $\psi\in\mathbb{H}^s$ for some s if its sth fractional derivative belongs to the space $\mathbb{L}^2(S)$ [23].

Definition 3.1. We denote by $\mathbb{C}_0^\infty(S)$ the space of infinitely differentiable functions which are orthogonal to any constant on the sphere:

$$\mathbb{C}_0^\infty(S)=\{\psi\in\mathbf{C}^\infty(S):Y_0(\psi)=0\}.$$

Note that operator Λ^s may be defined on functions $\mathbb{C}_0^\infty(S)$ by means of (2.8) for every real degree s.

Definition 3.2. For any real s, we introduce in $\mathbb{C}_0^\infty(S)$ the inner product $\langle\cdot,\cdot\rangle_s$ and norm $\|\cdot\|_s$ in the following way:

$$\langle\psi,h\rangle_s=\langle\Lambda^s\psi,\Lambda^sh\rangle=$$

$$=\sum_{n=1}^{\infty}\chi_n^s\langle Y_n(\psi),Y_n(h)\rangle\equiv\sum_{\alpha(1)}\chi_\alpha^s\psi_\alpha\overline{h}_\alpha,$$

$$\|\psi\|_s = \|\Lambda^s\psi\| = \langle\psi,\psi\rangle_s^{1/2} =$$

$$= \left\{ \sum_{n=1}^{\infty} \chi_n^s \|Y_n(\psi)\|^2 \right\}^{1/2} \equiv \left\{ \sum_{\alpha(1)} \chi_\alpha^s |\psi_\alpha|^2 \right\}^{1/2} . \tag{3.1}$$

Definition 3.3. Let s be a real. The Hilbert spaces obtained by closing the space $\mathbb{C}_0^\infty(S)$ in the norms (3.1) we denote as $\mathbb{H}^s$.

For the sake of brevity we will keep the symbols $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$ for the inner product and norm in $\mathbb{H}^0$ (see (2.1) and (2.2)). It is shown in [1] that

$$\mathbb{C}_0^\infty(S) \subset \mathbb{H}^r \subset \mathbb{H}^s \subset \mathbb{H}^0 \subset \mathbb{H}^{-s} \subset \mathbb{H}^{-r} \tag{3.2}$$

are continuous if $0 < s < r$, and the dual space $(\mathbb{H}^s)^*$ coincides with $\mathbb{H}^{-s}$ for all $s \geqslant 0$.

Let s and r be real numbers. Operator $\Lambda^r : \mathbb{C}_0^\infty(S) \mapsto \mathbb{C}_0^\infty(S)$ is symmetric,

$$\langle\Lambda^r\psi, h\rangle_s = \langle\psi, \Lambda^r h\rangle_s \ ,$$

and hence, closable, that is it can be extended as operator acting on the whole space $\mathbb{H}^s$.

Definition 3.4. An element $z \in \mathbb{H}^s$ is called the rth derivative $\Lambda^r\psi$ of a function $\psi \in \mathbb{H}^s$ if

$$\langle z, h\rangle_s = \langle z, \Lambda^r h\rangle_s$$

holds for all $h \in \mathbb{C}_0^\infty(S)$, where $\Lambda^r h$ is defined by (2.8).

The following assertion establishes embedding estimates for functions of the family $\mathbb{H}^s$ (see (3.2)).

Lemma 3.5. [28]. *Let s be real, $r > 0$, and $\psi \in \mathbb{H}^{s+r}$. Then $\psi \in \mathbb{H}^s$ and*

$$\|\psi\|_s \leq 2^{-r/2} \|\psi\|_{s+r} \tag{3.3}$$

$$\|\psi\|_{s+r} = \|\Lambda^r\psi\|_s .$$

Corollary 3.6. *Let s and r be real numbers. The mapping $\Lambda^r : \mathbb{H}^{s+r} \longmapsto \mathbb{H}^s$ is isometry and isomorphism. In particular, at $r = -2s$, the operator $\Lambda^{-2s} : \mathbb{H}^{-s} \longmapsto \mathbb{H}^s$ is isometric isomorphism.*

Lemma 3.7. *(Poincare inequality,* [29]*). For any $\psi \in \mathbb{H}^1$,*

$$\|\psi\| \leq 1/\sqrt{2}\,\|\nabla\psi\| = 1/\sqrt{2}\,\|\Lambda\psi\| . \tag{3.4}$$

In fact, a more general assertion than Lemma 3.7 is valid:

Lemma 3.8. [28]. *Let r, s and t be real numbers, $r < t$, and $a = \sqrt{2}$. Then for any $\psi \in \mathbb{H}^{s+t}$,*

$$\|\Lambda^r\psi\|_s \leq a^{r-t} \left\|\Lambda^t\psi\right\|_s \tag{3.5}$$

4 Barotropic Vorticity Equation

Let us consider the nonlinear nonstationary problem

$$\frac{\partial}{\partial t}\Delta\psi + J\left(\psi, \Delta\psi + 2\mu\right) = -\sigma\Delta\psi + \nu(-\Delta)^{s+1}\psi + F \tag{4.1}$$

$$\Delta\psi\left(0,x\right) = \Delta\psi_0\left(x\right) \tag{4.2}$$

for the barotropic vorticity equation (4.1) describing the behavior of relative vorticity $\Delta\psi\left(t,x\right)$ in a viscous two-dimensional incompressible rotating fluid on unit sphere S [23].

The problem takes into account a forcing (an external vorticity source) $F\left(t,x\right)$ and Rayleigh friction $\sigma\Delta\psi$ in the planetary boundary layer. Equation (4.1) is written in a non-dimensional form using geographical coordinate system (λ,μ) with the pole N being on the central axis of rotation of sphere S. Here ψ is the stream function, $\Delta\psi + 2\mu$ is the absolute vorticity,

$$J(\psi,h) = \frac{\partial\psi}{\partial\lambda}\frac{\partial h}{\partial\mu} - \frac{\partial\psi}{\partial\mu}\frac{\partial h}{\partial\lambda} = (\overrightarrow{n}\times\nabla\psi)\cdot\nabla h \tag{4.3}$$

is the Jacobian, $J\left(\psi, 2\mu\right) = 2\psi_\lambda$ is the term that takes into account the rotation of sphere, $\overrightarrow{n}$ is the outward unit normal vector at each point of sphere S, and the symbols "$\cdot$" and "$\times$" denote the scalar and vector product of vectors, respectively. Besides,

$$\nabla h = \left(\frac{1}{\sqrt{1-\mu^2}}\frac{\partial h}{\partial\lambda},\ \sqrt{1-\mu^2}\frac{\partial h}{\partial\mu}\right)$$

is the gradient of h, Δ is the spherical Laplace operator (2.3) and symbols ψ_t, ψ_λ and ψ_μ denote partial derivatives of ψ with respect to t, λ and μ, respectively. The velocity vector

$$\overrightarrow{v} = \overrightarrow{n}\times\nabla\psi$$

with components

$$u = -\sqrt{1-\mu^2}\psi_\mu\ , \quad v = \frac{1}{\sqrt{1-\mu^2}}\psi_\lambda$$

is solenoidal:

$$\nabla\cdot\overrightarrow{v} = 0.$$

We will consider the turbulent viscosity term of common form $\nu(-\Delta)^{s+1}\psi$, where $s \geqslant 1$ is arbitrary real number [23]. The case $s = 1$ corresponds to classical viscosity term in Navier-Stokes equations [7, 15, 17, 32-34], while the case $s = 2$ was considered in [8-10, 22, 28]. The turbulent term of such form for natural numbers s is applied in [20] for proving the solvability of Navier-Stokes equations in a limited area by means of the method of artificial viscosity.

The equation (4.1) is obtained by applying the operator *curl* to the equations of 2D fluid motions. Since the sphere is a smooth manifold without edges, such a transformation has resulted in the fact that if ψ is a solution of problem

(4.1), (4.2) then $\psi + const$ is also the solution for any $const$. In order to eliminate this constant the problem (4.1), (4.2) will be considered in classes of functions which are orthogonal to a constant on the sphere:

$$Y_0(\psi) = 0 \ , \quad Y_0(F) = 0.$$

5 Properties of the Jacobian

We now study some properties of the Jacobian (4.3) which will be used in the following sections. Let all functions under consideration be complex-valued. It is clear that

$$J(\psi, h) = -J(h, \psi) \ , \ \ J(\psi, \psi) = 0. \tag{5.1}$$

Let n be a natural number, and r be a real number. Since

$$\Lambda^s Y_n(\psi) = \chi_n^{s/2} Y_n(\psi),$$

we obtain

$$J(\psi, \Lambda^s \psi) = 0$$

for any homogeneous spherical polynomial ψ of degree n ($\psi \in \mathbf{H}_n$). Obviously,

$$J(\psi, h) = 0$$

for any zonal functions $\psi(x) = \psi(\overrightarrow{z} \cdot \vec{x})$ and $h(x) = h(\overrightarrow{z} \cdot \vec{x})$, where z is a pole of sphere S. Note that a smooth vector field $\overrightarrow{n} \times \nabla\psi$ is solenoidal, and due to (4.3),

$$J(\psi, h) = \nabla \cdot [h\ (\overrightarrow{n} \times \nabla\psi)]. \tag{5.2}$$

Suppose that a smooth vector field $\overrightarrow{X}$ defined on S has a compact support $K \subset S$, that is, $\overrightarrow{X}(x) = 0$ if $x \notin K$. Then

$$\int_S \nabla \cdot \overrightarrow{X}\ dS = 0.$$

Using the theorem on the partition of unity, we obtain

$$\int_S J(\psi, h)\ dS = 0. \tag{5.3}$$

It is easy to show that the relation

$$J(h\psi, g) = hJ(\psi, g) - \psi J(g, h) \tag{5.4}$$

is valid for all continuously differentiable functions ψ, g and h on S. Integrating (5.4) over S and using (5.3) and (5.1) we obtain that

$$\langle J(\psi,\, g), h\rangle = \langle J(\, g, \overline{h}), \overline{\psi}\rangle = -\langle J(\psi,\, \overline{h}), \overline{g}\rangle \tag{5.5}$$

holds for sufficiently smooth complex-valued functions ψ, g and h on S.

Let $\mathbb{C}$ be the set of complex numbers, and let $\psi \in \mathbb{C}^\infty(S)$, $\psi : S \to \mathbb{C}$. Then applying the gradient operator to the superposition $G(\psi) = G \circ \psi$ of two functions we get

$$\nabla G(\psi) = \frac{\partial}{\partial \psi} G(\psi) \nabla \psi\,,$$

and hence,

$$J(\psi, G(\psi)) = (\overrightarrow{n} \times \nabla \psi) \cdot (\frac{\partial G}{\partial \psi} \nabla \psi)$$

at each point $x \in S$. Therefore, due to (5.5),

$$\left\langle J(\psi,\, h), \overline{G(\psi)} \right\rangle = 0. \tag{5.6}$$

Lemma 5.1. *Let r be a real number, and $\psi, h \in \mathbb{C}^\infty(S)$. Then*

$$\langle J(\psi,\, h), \overline{\psi^r}\rangle = 0\ , \quad \langle J(\psi,\, \mu), \overline{\Lambda^r \psi}\rangle = 0. \tag{5.7}$$

Indeed, the first relation follows from (5.6). Further,

$$\langle J(\psi,\, \mu), \overline{\Lambda^r \psi}\rangle = \int_{-1}^{1} \left[\int_0^{2\pi} \frac{\partial \psi}{\partial \lambda} \Lambda^r \psi d\lambda \right] \, d\mu = \frac{1}{2} \int_{-1}^{1} \left[\int_0^{2\pi} \frac{\partial}{\partial \lambda} (\Lambda^{r/2} \psi)^2 d\lambda \right] \, d\mu = 0.$$

We used here the commutativity of operators Λ^r and $\frac{\partial}{\partial \lambda}$. Indeed,

$$\Lambda^r \frac{\partial}{\partial \lambda} Y_n^m = \Lambda^r (im Y_n^m) = im \chi_n^{r/2} Y_n^m = \frac{\partial}{\partial \lambda} \Lambda^r Y_n^m$$

for each basic function (spherical harmonic) Y_n^m.

Definition 5.2. We denote by $\mathbb{L}^p(S)$ the completion of continuous functions on the sphere S in the norm

$$\|\psi\|_{\mathbb{L}^p(S)} = \left(\int_S |\psi|^p \, dS \right)^{1/p}$$

if $p \neq \infty$, and in the norm

$$\|\psi\|_{\mathbb{L}^\infty(S)} = \sup_{x \in S} \ \text{vrai}\ |\psi(x)|$$

if $p = \infty$.

Applying Schwarz inequality one can obtain from (4.3) that

$$\|J(\psi,\ h)\| = \left\{ \int_S |\nabla\psi|^2 |\nabla h|^2 \ dS \right\}^{1/2} \le \|\nabla\psi\|_{\mathbb{L}^4(S)} \|\nabla h\|_{\mathbb{L}^4(S)} . \tag{5.8}$$

Lemma 5.3. [11]. *Let $1 < p < \infty$. For all functions ψ such that $\Lambda\psi \in \mathbb{L}^p(S)$, the following norms are equivalent:*

$$\|\nabla\psi\|_{\mathbb{L}^p(S)} \asymp \|\Lambda\psi\|_{\mathbb{L}^p(S)} .$$

We now give important results on continuous embeddings of functions from spaces $\mathbb{L}^p(S)$ and $\mathbb{H}^s$.

Lemma 5.4. [2]. *Let $p > 1$ and $0 < s < 2/p$. Let r be such that $s = 2(1/p - 1/r)$, and $\psi \in \mathbb{L}^p(S)$. Then $\Lambda^{-s}\psi \in \mathbb{L}^r(S)$ and*

$$\left\|\Lambda^{-s}\psi\right\|_{\mathbb{L}^r(S)} \le C \|\psi\|_{\mathbb{L}^p(S)} .$$

In the particular case, that $p = 2$ we have

Lemma 5.5. *Let $s \in (0,1)$. If number r is such that $s = 1 - 2/r$ then each function ψ of $\mathbb{H}^s$ belongs to $\mathbb{L}^r(S)$, and*

$$\|\psi\|_{\mathbb{L}^r(S)} \le C \|\psi\|_s .$$

Using Lemmas 5.4 and 5.5, and (3.1), we obtain

$$\|\nabla\psi\|_{\mathbb{L}^4(S)} \le C_1 \|\Lambda\psi\|_{\mathbb{L}^4(S)} \le C_2 \|\Lambda\psi\|_{1/2} \le C \left\|\Lambda^{3/2}\psi\right\| .$$

The last inequality and (5.8) imply the following result:

Lemma 5.6. *Let $\psi, h \in \mathbb{H}^{3/2}$. Then $J(\psi,\ h)$ belongs to $\mathbb{L}^2(S)$, and*

$$\|J(\psi,\ h)\| \le C \left\|\Lambda^{3/2}\psi\right\| \left\|\Lambda^{3/2}h\right\| .$$

By Lemma 5.6, $\langle J(\psi,h), g\rangle$ is the continuous form in $\mathbb{H}^{3/2} \times \mathbb{H}^{3/2} \times \mathbb{H}^0$. In particular,

$$\|J(\psi,\ h)\| \le M \|\Delta\psi\| \|\Delta h\| . \tag{5.9}$$

6 Solvability of Barotropic Vorticity Equation

This section is devoted to the unique solvability of a generalized nonstationary BVE problem (4.1), (4.2). Except self-interest, the analysis of classes of functions, in which there exists a solution of the problem, is particularly important in the study of stability of solutions [24, 25]. The fluid dynamics stability has been studied by many researchers (see the bibliography in [23]).

Definition 6.1. Let $p \in [1,\infty)$. We denote by $\mathbb{L}^p(0,T;\mathbb{X})$ the Banach space of measurable functions $\psi : (0,T) \to \mathbb{X}$, image of which is in $\mathbb{X}$ and such that

$$\|\psi\|_{\mathbb{L}^p(0,T;\mathbb{X})} = \left(\int_0^T \|\psi(t)\|_{\mathbb{X}}^p \, dt \right)^{1/p} < \infty.$$

If $p = \infty$ then the norm in $\mathbb{L}^\infty(0,T;\mathbb{X})$ is defined by

$$\|\psi\|_{\mathbb{L}^\infty(0,T;\mathbb{X})} = \sup_{t\in(0,T)} \text{ vrai } \|\psi(t)\|_{\mathbb{X}}.$$

Hereafter, $Q = (0,T) \times S$ and $\mathbb{L}^p(Q) = \mathbb{L}^p(0,T;\mathbb{L}^p(S))$. The following assertion was proved in [23].

Theorem 6.2. [23]. *Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. Suppose that initial field $\Delta\psi_0 \in \mathbb{L}^2(S)$, and forcing $F(t,x) \in \mathbb{L}^2(0,T;\mathbb{H}^{-s})$. Then nonstationary problem* (4.1),(4.2) *has unique weak solution $\psi(t,x) \in \mathbb{L}^\infty(0,T;\mathbb{H}^2)$ such that*

$$\psi(t,x) \in \mathbb{L}^\infty(0,T;\mathbb{H}^0) \cap \mathbb{L}^2(0,T;\mathbb{H}^s),$$

$$\Delta\psi_t \in \mathbb{L}^2(0,T;\mathbb{H}^{-s}), \quad \Delta\psi(0,x) = \Delta\psi_0(x)$$

and

$$\int_0^t \left[\langle \Delta\psi_t, h\rangle - \langle J(\psi,h), \Delta\psi + 2\mu\rangle + \sigma\langle\Delta\psi,h\rangle\right] dt - \nu \int_0^t \left\langle \Lambda^{s+2}\psi, \Lambda^s h\right\rangle dt = \int_0^t \langle F,h\rangle \, dt \tag{6.1}$$

holds for all $t \in (0,T)$ and $h \in \mathbb{L}^2(0,T;\mathbb{H}^s)$.

Remark 6.3. Here we considered only real solutions, and therefore, used (5.5) in (6.1).

We now consider the stationary BVE for incompressible viscous fluid on a rotating sphere:

$$J(\psi, \Delta\psi + 2\mu) = -\sigma\Delta\psi + \nu(-\Delta)^{s+1}\psi + F(x). \tag{6.2}$$

The existence of weak solution of equation (6.2) was proved in [23]. It was also proved in [23] that for a sufficiently large turbulent diffusion coefficient ν, equation (6.2) has unique solution. These results can be formulated as follows:

Theorem 6.4. [23]. *Let $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$. Suppose that the vorticity source $F(x) \in \mathbb{H}^{-s}$. Then there exists at least one weak solution $\psi(x) \in \mathbb{H}^{s+2}$ of equation* (6.2) *such that*

$$\nu \left\langle \Lambda^{s+2}\psi, \Lambda^s h\right\rangle - \sigma\langle\Delta\psi, h\rangle + \langle J(\psi,h), \Delta\psi + 2\mu\rangle = \langle F,h\rangle$$

holds for all $h \in \mathbb{H}^{-s}$. In addition, if

$$\nu^2 > 2^{1-s} M \|F(x)\|_{-s}$$

then solution $\psi(x)$ is unique (here M is the constant from (5.9)*).*

Remark 6.5. The case $s = 1$ and $\sigma = 0$ was considered in [14] (see also [7,15]). For $s = 1$ and $s = 2$ the Theorem 6.4 was proved in [28, 29].

7 Asymptotic Behavior of BVE Solutions

In this section we study the asymptotic behavior of BVE solutions as $t \to \infty$. Evidently, the geometric structure of the attractive set of the BVE depends on the form of external forcing [27]. We consider here such particular forms of external vorticity source that there exists a bounded set **B** in a phase space **X**, and the trajectories of all the solutions of nonstationary problem (4.1), (4.2) are eventually attracted by **B**. Some assertions are similar to well known results obtained for solutions of two-dimensional Navier-Stockes equations in the case of a bounded domain in Euclidean space $\mathbb{R}^2$ [17].

First we suppose that forcing is steady. Due to Theorem 6.2, the operators $V_t : \mathbf{X} \to \mathbf{X}$ which solve the problem (4.1), (4.2) represent the bounded and point-dissipative semigroup [18]. Indeed the following more precise assertions are valid.

Theorem 7.1. *Let $s \geq 1$ in* (4.1) *and the steady vorticity source $F(x) \in \mathbb{H}^r$, $r \geq -1$. Then every solution $\psi(t,x)$ of problem* (4.1), (4.2) *will eventually be attracted by a bounded set* **B** *of the phase space* **X**. *Moreover,*
I. *if $F(x) \in \mathbb{H}^r$ where $r \geq 0$ then $\mathbf{X} = \mathbb{H}^2$ and*

$$\mathbf{B} = \{\psi \in \mathbb{H}^2 : \|\psi\|_2 \leq C_1(r,s)\, \|F\|_r\,, \tag{7.1}$$

II. *if $F(x) \in \mathbb{H}^{-r}$ where $r \in (0,1]$ then $\mathbf{X} = \mathbb{H}^1$ and*

$$\mathbf{B} = \{\psi \in \mathbb{H}^1 : \|\psi\|_1 \leq C_2(r,s)\, \|F\|_{-r}\,. \tag{7.2}$$

The constants $C_1(r,s)$ and $C_2(r,s)$ are determined by

$$C_1(r,s) = \frac{a^{-r}}{\sigma + 2^s\nu}\,,\quad C_2(r,s) = \frac{a^{r-1}}{\sigma + 2^s\nu}$$

where $a = \sqrt{2}$ is the constant from lemma 3.8.

Proof. **I.** Suppose that $F(x) \in \mathbb{H}_0^r$ and $r \geq 0$. Taking the inner product of equation (4.1) with $\Delta\psi$ and using (5.7) we obtain

$$\langle \Delta\psi_t, \Delta\psi \rangle = -\sigma \langle \Delta\psi, \Delta\psi \rangle + \nu \left\langle (-\Delta)^{s+1}\psi, \Delta\psi \right\rangle + \langle F, \Delta\psi \rangle$$

$$= -\sigma \langle \Delta\psi, \Delta\psi \rangle - \nu \left\| \Lambda^{s+2}\psi \right\|^2 + \langle F, \Delta\psi \rangle\,. \tag{7.3}$$

Further, using (3.3) and (3.5), we estimate the terms $\langle F, \Delta\psi \rangle$ and $\nu \left\| \Lambda^{s+2}\psi \right\|^2$ as

$$|\langle F, \Delta\psi \rangle| \leq \|F\|\, \|\Delta\psi\| \leq a^{-r}\, \|F\|_r\, \|\Delta\psi\|$$

where $a = \sqrt{2}$, and

$$\nu \left\| \Lambda^{s+2}\psi \right\|^2 \geq 2^s\nu\, \|\Delta\psi\|^2\,.$$

Then (7.3) implies

$$\|\Delta\psi\| \frac{\partial}{\partial t} \|\Delta\psi\| \leq -(\sigma + 2^s\nu)\, \|\Delta\psi\|^2 + a^{-r}\, \|F\|_r\, \|\Delta\psi\|\,.$$

If we divide the last inequality by $\|\Delta\psi\|$ then we obtain

$$\frac{\partial}{\partial t} \|\Delta\psi\| \leq -\rho\, \|\Delta\psi\| + a^{-r}\, \|F\|_r \tag{7.4}$$

where

$$\rho = \sigma + 2^s \nu. \tag{7.5}$$

It follows from (7.4) that

$$\|\Delta\psi(t)\| \le \|\Delta\psi(0)\| \exp(-\rho t) + \frac{a^{-r}}{\rho} \|F\|_r \; [1 - \exp(-\rho t)] \tag{7.6}$$

and hence,

$$\|\psi(t)\|_2 \to C_1(r,s)\,\|F\|_r \quad \text{as} \;\; t \to \infty.$$

II. We now consider the case when $F(x) \in \mathbb{H}_0^{-r}$ and $r \in (0,1]$. Taking the inner product of equation (4.1) with ψ and using (5.7) we get

$$\langle \Lambda\psi_t, \Lambda\psi \rangle = -\sigma \|\Lambda\psi\|^2 - \nu \left\|\Lambda^{s+1}\psi\right\|^2 - \langle F, \psi \rangle . \tag{7.7}$$

The application of (3.3) and (3.5) to $\langle F, \psi \rangle$ and $\nu \left\|\Lambda^{s+1}\psi\right\|^2$ gives

$$|\langle F, \psi \rangle| \le \left\|\Lambda^{-r} F\right\| \|\Lambda^r \psi\| \le a^{r-1} \|F\|_{-r} \|\Lambda\psi\|$$

where $a = \sqrt{2}$, and

$$\nu \left\|\Lambda^{s+1}\psi\right\|^2 \ge \nu 2^s \|\Lambda\psi\|^2 .$$

Then (7.7) implies

$$\frac{\partial}{\partial t} \|\Lambda\psi\| \equiv \frac{\partial}{\partial t} \|\nabla\psi\| \le -\rho \|\Lambda\psi\| + a^{r-1} \|F\|_{-r}$$

where ρ is defined by (7.5), or

$$\|\Lambda\psi(t)\| \le \|\Lambda\psi(0)\| \exp(-\rho t) + \frac{a^{r-1}}{\rho} \|F\|_{-r} [1 - \exp(-\rho t)] \tag{7.8}$$

and hence,

$$\|\psi(t)\|_1 \to C_2(r,s)\,\|F\|_{-r} \quad \text{as} \;\; t \to \infty.$$

The theorem is proved. □

Remark 7.2. If solution ψ belongs to the set **B** at some moment of time t_0 then, due to (7.6) and (7.8), it will belong to **B** for all $t > t_0$. Hence all the steady and periodic solutions (if they exist) belong to the set **B**. Obviously, the set **B** contains the maximal BVE attractor [35].

Remark 7.3. Theorem 7.1 is also valid in the case when forcing $F(t,x)$ is a periodic in time function from the space $C(0,\omega;\mathbb{H}_0^{-r})$ where ω is the period. In order to prove this fact, we should only replace in (7.1) and (7.2) the norms $\|F\|_r$ and $\|F\|_{-r}$ by the norms $\max\limits_{t\in[0,\omega]} \|F\|_r$ and $\max\limits_{t\in[0,\omega]} \|F\|_{-r}$, respectively.

8 Global Asymptotic Stability of BVE Solutions

In this section, the global asymptotic stability of BVE solutions is considered. A norm related with kinetic energy and enstrophy of perturbations is introduced, and an equation describing the evolution of this norm is derived. Then two sufficient conditions for the global asymptotic stability of a BVE solution are obtained. These conditions differ in the smoothness of the basic solution and ensure that the trajectories of all other BVE solutions will exponentially tend to the trajectory of basic solution as time tends to infinity.

In a bounded domain on the plane, a condition for the global asymptotic stability were earlier obtained by Sundström [31] for the basic flow, whose stream function had continuous derivatives up to the third order inclusive. The first theorem proved here (theorem 8.4) generalizes this result to the flows on a rotating sphere, when the BVE contains the linear drag term and the term of turbulent viscosity of more general form. However, the theorems 6.2 and 6.4 on the BVE solvability (see section 6) do not guarantee the existence of the solution whose third or higher derivatives are continuous. The second theorem proved here (theorem 8.5) gives the conditions for global asymptotic stability, in which the requirements to the smoothness of basic solution is weakened and is in full accordance with the solvability theorems. Examples are given for a super-rotation flow and for the flow in the form of homogeneous spherical polynomial of degree n.

We now analyse the stability of a solution $\widetilde{\psi}(t,\lambda,\mu)$ of BVE (4.1) with initial condition $\widetilde{\psi}(0,\lambda,\mu)$. Let $\widehat{\psi}(t,\lambda,\mu)$ be another solution of (4.1) with initial condition $\widehat{\psi}(0,\lambda,\mu)$. Then

$$\frac{\partial}{\partial t}\Delta\psi + J(\psi,\Delta\widetilde{\psi}) + J(\widetilde{\psi},\Delta\psi) + 2\frac{\partial\psi}{\partial\lambda} + J(\psi,\Delta\psi) = -[\sigma + \nu\Lambda^{2s}]\Delta\psi \tag{8.1}$$

where $s \geq 1$, $\nu > 0$ and $\sigma \geq 0$, holds for the difference

$$\psi(t,\lambda,\mu) = \widehat{\psi}(t,\lambda,\mu) - \widetilde{\psi}(t,\lambda,\mu) \tag{8.2}$$

of two solutions, besides,

$$\psi(0,\lambda,\mu) = \widehat{\psi}(0,\lambda,\mu) - \widetilde{\psi}(0,\lambda,\mu)$$

at the initial moment. The function (8.2) represents a perturbation of the basic solution $\widetilde{\psi}$. Taking the inner product (2.1) of equation (8.1) successively with ψ and $\Delta\psi$ and using (5.5) and (5.7), we obtain two equations

$$\frac{\partial}{\partial t}K(t) + \langle J(\psi,\Delta\psi),\widetilde{\psi})\rangle + 2\sigma K(t) + \nu\left\|\Lambda^{s+1}\psi\right\|^2 = 0 \tag{8.3}$$

$$\frac{\partial}{\partial t}\eta(t) - \langle J(\psi,\Delta\psi),\Delta\widetilde{\psi})\rangle + 2\sigma\eta(t) + \nu\left\|\Lambda^{s+2}\psi\right\|^2 = 0 \tag{8.4}$$

for the kinetic energy

$$K(t) = \frac{1}{2}\|\nabla\psi\|^2$$

and enstrophy

$$\eta(t) = \frac{1}{2}\|\Delta\psi\|^2$$

of perturbation.

It follows from (8.3) and (8.4) that the first Jacobian in (8.1) may change the perturbation enstrophy $\eta(t)$ but does not affect the behavior of perturbation energy $K(t)$. On the contrary, the second Jacobian in (8.1) has no effect on the perturbation enstrophy $\eta(t)$ but may change the perturbation energy $K(t)$. Both the super-rotation term and the non-linear term (the last two terms in the left part of (8.1)) have no influence on the behavior of $K(t)$ and $\eta(t)$.

Evidently, the zero solution $\widetilde{\psi} = 0$ (existing if $F \equiv 0$) is globally asymptotically stable. Indeed, in this case the nonlinear terms $\langle J(\psi, \Delta\psi), \widetilde{\psi}) \rangle$ and $\langle J(\psi, \Delta\psi), \Delta\widetilde{\psi}) \rangle$ in (8.3) and (8.4) are equal to zero, and hence, the perturbation energy and enstrophy will be constant for a non-dissipative fluid ($\sigma = \mu = 0$) and will exponentially decrease otherwise.

Let now $\widetilde{\psi}$ be a solution of general form, and let p and q be non-negative real numbers, not equal to zero simultaneously. Then the functional

$$\|\psi\|_G = [G(p,q,\psi,t)]^{1/2} \tag{8.5}$$

where

$$G(p,q,\psi,t) \equiv G(t) = pK(t) + q\eta(t) = \frac{1}{2}(p\|\nabla\psi\|^2 + q\|\Delta\psi\|^2) \tag{8.6}$$

makes a norm in the space of perturbations on the sphere S.

Multiplying (8.3) and (8.4) by p and q, respectively, and combining the results, we obtain

$$\frac{\partial}{\partial t}G(t) = -2\sigma G(t) - R(t) - \nu p\left\|\Lambda^{s+1}\psi\right\|^2 - \nu q\left\|\Lambda^{s+2}\psi\right\|^2 \tag{8.7}$$

where

$$R(t) = \langle J(\psi, \Delta\psi),\ p\widetilde{\psi} - q\Delta\widetilde{\psi}\rangle\,. \tag{8.8}$$

By using (3.4) and (3.5) one can obtain

$$-\left\|\Lambda^{s+1}\psi\right\|^2 \le -2^s\|\nabla\psi\|^2, \qquad -\left\|\Lambda^{s+2}\psi\right\|^2 \le -2^s\|\Delta\psi\|^2.$$

Then the estimation of the last two terms in (8.7) leads to

$$\frac{\partial}{\partial t}G(t) \le -2\rho G(t) - R(t) \tag{8.9}$$

with

$$\rho = \sigma + 2^s\nu$$

Example 8.1. *The super-rotation basic flow*. Let the basic solution $\widetilde{\psi}$ belongs to the subspace $\mathbf{H}_1$ defined by (2.4). Such a solution represents a super-rotation flow about some axis passing through the sphere center. Then, in the geographical system of coordinates related to this axis, $\widetilde{\psi} \equiv \widetilde{\psi}(\mu) = C\mu$ where C is a constant, and $R(t)$ defined by (8.8) is equal to zero due to (5.7). Thus the super-rotation flow (about any axis of a sphere) is Liapunov stable if $\sigma = \mu = 0$, and is the global attractor (asymptotically Liapunov stable) if $\rho > 0$ [26].

Example 8.2. *The basic flow in the form of a homogeneous spherical polynomial.* Let $\widetilde{\psi}(t,\lambda,\mu)\in\mathbf{H}_n$ for some $n\geq 2$, that is,

$$\widetilde{\psi}(t,\lambda,\mu)=Y_n(\widetilde{\psi})\equiv\sum_{m=-n}^{n}\psi_n^m(t)Y_n^m(\lambda,\mu). \tag{8.10}$$

In particular, it may have the form of Legendre polynomial of degree $n:\widetilde{\psi}(\mu)=CP_n\,(\mu)$ (zonal flow). Then $J(\psi,\Delta\psi)=0$ for any initial perturbation of the subspace $\mathbf{H}_n$ ($\psi(0,\lambda,\mu)\in\mathbf{H}_n$), and due to (8.8), $R(t)\equiv 0$. Thus, due to (8.1), any perturbation of $\mathbf{H}_n$ will never leave $\mathbf{H}_n$, and hence, $\mathbf{H}_n$ is invariant set of perturbations to a polynomial flow (8.10). It follows from (8.9) that

$$G(p,q,\psi,t)\leq G(p,q,\psi,0)\ \exp(-2\rho t)$$

and any initial perturbation $\psi(0,\lambda,\mu)$ of $\mathbf{H}_n$ will exponentially tend to zero with time not leaving $\mathbf{H}_n$. In other words, the set $\mathbf{H}_n$ belongs to to the basin of attraction of solution (8.10).

Remark 8.3. The same result is valid in the case when basic flow $\widetilde{\psi}(t,\lambda,\mu)$ is a linear combination of the flows considered in the examples 8.1 and 8.2. In particular, $\widetilde{\psi}(t,\lambda,\mu)$ may be a Rossby-Haurwitz wave [25].

We now obtain sufficient conditions for the global asymptotic stability of rather smooth BVE solution on the sphere S, when any solution perturbation tends to zero as $t\to\infty$ (see below theorems 8.4 and 8.5). In the particular case when $s=1$ and $\sigma=0$, theorem 8.4 is analogous to the assertion proved by Sundström [31] for flows in a limited domain on the plane.

First, assume that the basic solution $\widetilde{\psi}(t,\lambda,\mu)$ of equation (4.1) is rather smooth, such that two values

$$p=\sup_{t\,\geq\,0}\max_{(\lambda,\mu)\in S}|\nabla\Delta\widetilde{\psi}(t,\lambda,\mu)|\quad\text{and}\quad q=\sup_{t\,\geq\,0}\max_{(\lambda,\mu)\in S}|\nabla\widetilde{\psi}(t,\lambda,\mu)| \tag{8.11}$$

are finite. Let us estimate the inner product (8.8) by means of functional (8.6) with p and q defined by (8.11):

$$|R(t)|=|\langle J(p\widetilde{\psi}-q\Delta\widetilde{\psi},\psi),\Delta\psi\rangle|\leq 2pq\,\|\nabla\psi\|\,\|\Delta\psi\|\leq 2\sqrt{pq}G(t).$$

Substitution of this inequality in equation (8.9) leads to

Theorem 8.4. *Let $s\geq 1$, $\nu>0$ and $\sigma\geq 0$. If the smooth solution $\widetilde{\psi}(t,\lambda,\mu)$ of equation* (4.1) *is such that the numbers p and q defined by* (8.11) *are finite, and*

$$\sigma+2^s\nu>\sqrt{pq}$$

then any perturbation of $\widetilde{\psi}(t,\lambda,\mu)$ will exponentially decrease with time in the norm (8.5).

Note that both Theorem 8.4 and asymptotic-stability condition by Sundström [31] demand the uniform boundedness of $|\nabla\Delta\widetilde{\psi}(t,\lambda,\mu)|$ and $|\nabla\widetilde{\psi}(t,\lambda,\mu)|$. However, as it was mentioned earlier, the existence of BVE solutions has been proved only in the classes of twice continuously differentiable streamfunctions. We now show that the restriction (8.11) on the smoothness of basic solution can be weakened so as to agree with the requirements of the solvability theorems 6.2 and 6.4.

Indeed, let us consider a BVE solution $\widetilde{\psi}(t,\lambda,\mu)$ such that

$$p=\sup_{t\,\geq\,0}\max_{(\lambda,\mu)\in S}|\Delta\widetilde{\psi}(t,\lambda,\mu)|\quad\text{and}\quad q=\sup_{t\,\geq\,0}\max_{(\lambda,\mu)\in S}|\widetilde{\psi}(t,\lambda,\mu)| \tag{8.12}$$

are finite values. Let us estimate $|R(t)|$ using (3.4), (3.5) and ε-inequality

$$ab\leq a^2\varepsilon^2+\frac{b^2}{4\varepsilon^2}$$

where $\varepsilon > 0$:

$$|R(t)| \leq 2pq\,\|\nabla\psi\|\,\|\nabla\Delta\psi\| = 2pq\,\|\nabla\psi\|\,\left\|\Lambda^3\psi\right\| \leq 2pq\,\|\nabla\psi\|\,\|\psi\|_3 =$$

$$= (\sqrt{pq}\,\|\psi\|_1)\,(2\sqrt{pq}\,\|\psi\|_3) \leq 2q\varepsilon^2 G(t) + \frac{pq}{\varepsilon^2}\,\|\psi\|_3^2. \qquad (8.13)$$

Further, it follows from (8.6), (8.7) and Lemma 3.5 that

$$\frac{\partial}{\partial t}G(t) \leq -2\sigma G(t) - R(t) - \nu p\,\|\psi\|_{s+1}^2 - \nu q\,\|\psi\|_{s+2}^2$$

$$\leq -2\sigma G(t) - 2^{s+1}\nu G(t) - R(t) - \nu q\,\|\psi\|_{s+2}^2$$

$$\leq -2\rho G(t) - R(t) - \nu q a^{1-s}\,\|\psi\|_3^2 \qquad (8.14)$$

where $a = \sqrt{2}$. Combining (8.14) with (8.13) and putting $\varepsilon^2 = pa^{s-1}/\nu$ we eliminate the two terms containing $\|\psi\|_3^2$. The resulting inequality leads to

Theorem 8.5. *Let* $s \geq 1$, $\nu > 0$ *and* $\sigma \geq 0$. *If a solution* $\widetilde{\psi}(t,\lambda,\mu)$ *of equation* (4.1) *is such that the numbers* p *and* q *defined by* (8.12) *are finite, and*

$$\nu(\sigma + 2^s\nu) > 2^{(s-1)/2}pq \qquad (8.15)$$

then any perturbation of $\widetilde{\psi}(t,\lambda,\mu)$ *will exponentially decrease with time in the norm* (8.5).

Remark 8.6. Evidently that in the case of a steady solution $\widetilde{\psi}(\lambda,\mu)$, p and q in (8.15) are defined as

$$p = \max_{(\lambda,\mu)\in S}|\Delta\widetilde{\psi}(\lambda,\mu)| \qquad \text{and} \qquad q = \max_{(\lambda,\mu)\in S}|\widetilde{\psi}(\lambda,\mu)|\,.$$

According to conditions (8.12), only the first two derivatives of the basic solution $\widetilde{\psi}(t,\lambda,\mu)$ should be continuous functions, and hence, these conditions can be applied to a wider class of solutions to equation (4.1).

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