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Transmission Eigenvalues for Non-regular Cases

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Abstract

We prove the existence of transmission eigenvalues in the case when the perturbation of the index of refraction may have singularity or degeneration on the boundary of its support. This singularity or degeneration is measured in terms of the distance to the boundary.

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1 Introduction

The scattering of a time-harmonic plane wave u_0 in an inhomogeneous medium can be modeled by the scattering problem for the Helmholtz equation. The total wave u

$$u(x) = u_0(x) + u_{sc}(x) \tag{1.1}$$

satisfies the Helmholtz equation

$$(\Delta + k^2(1 + m(x))u(x) = 0, \quad x \in \mathbb{R}^n, \quad n \geq 2, \tag{1.2}$$

where $k > 0$ fixed and function $m(x)$ denotes the perturbation of the index of refraction. We assume that $m(x)$ is compactly supported in some bounded domain $D \subset \mathbb{R}^n$ and belongs to $L^p(D)$ for some $\frac{n}{2} < p \leq \infty$.

Under $u_0(x)$ we understand the solution of the free Helmholtz equation

$$(\Delta + k^2)u_0(x) = 0, \quad x \in \mathbb{R}^n,$$

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in the form of Herglotz function, i.e.

$$u_0(x) = \int_{\mathbb{S}^{n-1}} e^{ik(x,\vartheta)} g_0(\vartheta)\, d\vartheta \tag{1.3}$$

with some function $g_0(\vartheta) \in L^2(\mathbb{S}^{n-1})$. Here $\mathbb{S}^{n-1}$ is the unit sphere in $\mathbb{R}^n$. The justification of such choice (1.3) of the set of solutions of the free Helmholtz equation can be found out in [15], [16], [24], [2]. The set of all such solutions we denote by U_0. This are incident waves or free waves.

We introduce the Sommerfeld radiation condition at the infinity

$$\lim_{r\to\infty} r^{\frac{n-1}{2}} \left(\frac{\partial f(x)}{\partial r} - ikf(x) \right) = 0, \quad r = |x|. \tag{1.4}$$

By U_{sc} we denote the set of all solutions of the non-homogeneous Helmholtz equation

$$(\Delta + k^2(1 + m(x))u(x) = f(x) \tag{1.5}$$

with compactly supported function f which belongs to the space $L^{\frac{2p}{p+1}}$, where p is the same as for function m. And this solution must satisfy the Sommerfeld radiation condition (1.4). This are outgoing solutions or scattered waves.

By U_m we denote the set of all solutions of the homogeneous Helmholtz equation (1.2) in the form (1.1) such that $u_0 \in U_0$ and $u_{sc} \in U_{sc}$. It is equivalent to the fact

$$(\Delta + k^2(1 + m(x))u_{sc}(x) = -k^2 m(x)u_0(x)$$

with the right hand side from the space $L^{\frac{2p}{p+1}}(D)$.

The following result is actually proved in [23].

Theorem 1.1. *For every compactly supported* $f \in L^{\frac{2p}{p+1}}(\mathbb{R}^n)$, $\frac{n}{2} < p \le \infty$, *there exists a unique outgoing solution u to the equation*

$$(\Delta + k^2)u(x) = f(x)$$

such that u belongs to the weighted space $L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)$ *with* $\delta = 0$ *for* $\frac{n}{2} < p \le \frac{n+1}{2}$ *and with* $\delta > \frac{1}{2} - \frac{n-1}{4p}$ *for* $\frac{n+1}{2} < p \le \infty$. *Moreover there is a constant* $C > 0$ *depending on k such that*

$$||u||_{L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)} \le C||f||_{L^{\frac{2p}{p+1}}(\mathbb{R}^n)}. \tag{1.6}$$

Corollary 1.2. *For any* $u_0 \in U_0, u_0 \ne 0$, *and for* $m \in L^p(D)$, $\frac{n}{2} < p \le \infty$, *there exists a unique* $u_{sc} \in U_{sc}$ *which satisfies the equation*

$$(\Delta + k^2(1 + m(x))u_{sc}(x) = -k^2 m(x)u_0(x) \tag{1.7}$$

and such that

$$||u_{sc}||_{L^{\frac{2p}{p-1}}_{-\delta}(R^n)} \le C||m||_{L^p(D)}, \tag{1.8}$$

where constant C depends on k.

It can be easily concluded (see, for example, [24]) that any $u_0 \in U_0$ belongs to $L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)$ for any $\frac{n}{2} < p \le \infty$ and with the same δ as in Theorem 1.1.

Corollary 1.3. *For any $m \in L^p(D)$, $\frac{n}{2} < p \le \infty$, there exists a unique $u_m \in U_m$ of the form (1.1) and such that*

$$\|u_m\|_{L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)} \le C(\|m\|_{L^p(D)} + 1)\|u_0\|_{L^{\frac{2p}{p-1}}_{-\delta}(\mathbb{R}^n)}, \tag{1.9}$$

where constant C depends on k.

Remark 1.4. There are the numerous of publications concerning the scattering theory for the Schrödinger operator with the potentials from L^∞_{loc} space. But we consider the potentials from L^p_{loc} spaces. That is why we have restricted the bibliographical remarks to the works that are of interest from the viewpoint of the present article. The reason is the result of Theorem 1.1 allows us to consider the index of refraction m such that it has the degenerations or the singularities (see Corollary 1.3 and Theorem 3.3 of present paper).

Using these results and analogously to Theorem 2.2 of [24] we can prove in our case the following fact.

Theorem 1.5. *Every total wave $u_m \in U_m$ has a unique decomposition into an incident wave $u_0 \in U_0$ plus a scattered wave $u_{sc} \in U_{sc}$, and every incident wave $v_0 \in U_0$ has a unique decomposition as a total wave $v_m \in U_m$ minus a scattered wave $v_{sc} \in U_{sc}$:*

$$u_m(x) = u_0(x) + u_{sc}(x), \quad v_0(x) = v_m(x) - v_{sc}(x).$$

In present paper we will consider the interior transmission problem (the problem of existence of transmission eigenvalues). In other words we consider the positive values of parameter k for which there is a non-trivial pair (u, v) solving

$$\Delta u(x) + k^2(1 + m(x))u(x) = 0, \quad x \in D,$$

$$\Delta v(x) + k^2 v(x) = 0, \quad x \in D,$$

$$u(x) = v(x), \quad \frac{\partial u}{\partial \nu}(x) = \frac{\partial v}{\partial \nu}(x), \quad x \in \partial D.$$

This problem arises naturally in inverse scattering theory. Namely, if k is not a transmission eigenvalue then the far field pattern operator (it has basic importance in inverse scattering theory) is injective with dense range (see [8], [10]). In that case one can apply the Kirsch's characterization method and can define unknown domain D (see, for example, [17]). That's why the elimination of the values of k which are the transmission eigenvalues is very important.

The study of the interior transmission problem and transmission eigenvalues has quite long history. We restrict the bibliographical remarks to the works that are of interest from the viewpoint of the present article.

This problem was first introduced in 1988 by Colton and Monk [9] in connection with an inverse scattering problem for the reduced wave equation. The discreteness of the set of transmission eigenvalues was established by Colton, Kirsch and Päivärinta [7]. The problem of existence of transmission eigenvalues, however, has been remained unsolved long time until Päivärinta and Sylvester [24] proved the first existence result. Let us mentioned here the paper of Colton, Päivärinta and Sylvester [10] where the characterization of real transmission eigenvalues was obtained. The existence of an infinite set of transmission eigenvalues was established by Cakoni, Gintides and Haddar [4]. We also mention some results on transmission eigenvalues for Maxwell's equations and for the Helmholtz equation in presence of cavities [5], [18], [6], as well as very resent and very interesting results on transmission eigenvalues for elliptic operators of arbitrary order with constant coefficients of Hitrik, Krupchyk, Ola and Päivärinta [12], [13], [14].

The big interest to the problem of transmission eigenvalues is connected to the fact that the knowledge of the transmission eigenvalues uniquely determines a radial scatterer [20], [21], [24]. For non-radial scatterers, transmission eigenvalues have also been used to infer simple properties of the scatterer [3].

All results of the mentioned works were obtained under the hypothesis that the perturbation of the index of refraction m does not change sign and satisfies the condition $|m| \geq \delta > 0$ (in the paper [7] in three dimensional case it was allowed that function $m(x)$ has the degeneration of the type $|x-y|^{\alpha}, y \in \partial D$, with $1 \leq \alpha < 3$). It can be mentioned here that the problem when the perturbation of index of refraction m changes the sign (even it is bounded) is still open and it is under the consideration by many researchers (see, for example, [19]).

The main result of this paper is Theorem 3.3, where the existence of the transmission eigenvalues are proved for the perturbation of the index of refraction m that may have the singularities or degenerations at the boundary of the domain or at some points inside of the domain.

The approach in present work is closed to the approach which was appeared in [24].

2 The Interior Transmission Problem

In this section we assume that the perturbation of the index of refraction m has special form

$$m(x) = c_0 \rho(x)^{\beta}, \quad c_0 > 0, \quad \beta \neq 0, \quad \beta > -1, \quad x \in D, \tag{2.1}$$

where $\rho(x) = \inf_{y \in \partial D} |x-y|$ is the distance to the boundary of D. We assume that this function $m(x) > 0$ for all $x \in D$. We define the weighted space $H^2_{0,\beta}(D)$ as the closure of $C_0^{\infty}(D)$ with respect to the norm

$$\|f\|^2_{H^2_{\beta}(D)} = \int_D \left(\rho^{-\beta} \sum_{|\gamma|=2} |\partial^{\gamma} f(x)|^2 + \rho^{-\beta-2} |\nabla f(x)|^2 + \rho^{-\beta-4} |f(x)|^2 \right) dx. \tag{2.2}$$

This norm is justified by Hardy inequality (see, for example, [22] and [26]).

Lemma 2.1. *(Hardy inequality) Let us assume that $\sigma > 1$. Then there is a constant $C > 0$ such that for all $f \in C_0^{\infty}(D)$*

$$\int_D \rho^{-\sigma} |f(x)|^2 \, dx \leq C \int_D \rho^{-\sigma+2} |\nabla f(x)|^2 \, dx. \tag{2.3}$$

If $\sigma > 3$ *then in addition*

$$\int_D \rho^{-\sigma+2} |\nabla f(x)|^2 \, dx \le C \sum_{|\gamma|=2} \int_D \rho^{-\sigma+4} |\partial^\gamma f(x)|^2 \, dx. \tag{2.4}$$

These two inequalities for $\beta > -1$ imply the imbedding

$$H^2_{0,\beta}(D) \subset L^2_{\frac{\beta}{2}+2}(D).$$

Moreover, it can be easily seen that for $\beta > -1$ the following embeddings hold

$$H^2_{0,\beta}(D) \subset W^1_{2,0}(D) \subset L^2(D), \tag{2.5}$$

where the embedding to $L^2(D)$ is compact. Here and later on by the symbol $W^t_{s,0}(D)$ for positive integer t and $s \ge 1$ we denote the closure of $C_0^\infty(D)$ with respect to the norm of Sobolev space $W^t_s(D)$.

Definition 2.2. We say that a wave number $k > 0$ is a transmission eigenvalue of $m(x) = c_0 \rho(x)^\beta \in L^p(D)$, $\frac{n}{2} < p \le \infty$ if any of the conditions below are satisfied.

1) There exist $u_0 \in U_0, u_0 \ne 0$, and $u_m \in U_m, u_m \ne 0$, such that

$$u_m - u_0 \in W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D). \tag{2.6}$$

2) There exists $u_m \in U_m, u_m \ne 0$, such that the unique outgoing solution u_{sc} to the equation

$$(\Delta + k^2) u_{sc} = -k^2 m u_m \tag{2.7}$$

belongs to $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$.

3) There exist $u_m \in U_m, u_m \ne 0$, and $v \in W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$ such that

$$(\Delta + k^2) v = -k^2 m u_m. \tag{2.8}$$

4) There exists $u_0 \in U_0, u_0 \ne 0$, such that the unique outgoing solution u_{sc} to the equation

$$(\Delta + k^2(1+m)) u_{sc} = -k^2 m u_0 \tag{2.9}$$

belongs to $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$.

5) There exist $u_0 \in U_0, u_0 \ne 0$, and $v \in W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$ such that

$$(\Delta + k^2(1+m)) v = -k^2 m u_0. \tag{2.10}$$

Theorem 2.3. *These 5 conditions (2.6)-(2.10) are equivalent.*

Proof. Obviously 2) implies 3). Any $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$-solution v to the equation

$$(\Delta + k^2) v = -k^2 m u_m$$

extended to be zero in $\mathbb{R}^n \setminus D$, is outgoing. But since outgoing solution is unique (see Theorem 1.5 of present article) then v is this solution. Thus, 3) implies 2). That is 2) and 3) are equivalent.

Obviously 4) implies 5). But uniqueness of the outgoing solution to the equation

$$(\Delta + k^2(1+m))v = -k^2 m u_0$$

implies that any $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$-solution of the latter equation, extended by zero in $\mathbb{R}^n \setminus D$, must be u_{sc}, so 5) implies 4). Thus 4) and 5) are also equivalent.

Due to the unique decomposition (see Theorem 1.5 of present article) the unique outgoing solution to the equation

$$(\Delta + k^2)u_{sc} = -k^2 m u_m$$

is also the unique outgoing solution to the equation

$$(\Delta + k^2(1+m))u_{sc} = -k^2 m u_0.$$

It means that 4) and 2) are equivalent. The last step is: Theorem 1.5 gives that

$$u_{sc} = u_m - u_0.$$

This equality shows that the left hand side is in $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$ if and only if the right hand side is. Hence, 1) and 2) are equivalent. Therefore, Theorem 2.3 is completely proved. □

Remark 2.4. If $\beta > 0$ the function $m = c_0\rho^\beta$ belongs to $L^\infty(D)$ and in this case we will consider $H^2_{0,\beta}(D)$ instead of $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$ since the following equality holds in that case (see (2.2))

$$W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D) = W^2_{2,0}(D) \cap H^2_{0,\beta}(D) = H^2_{0,\beta}(D).$$

If $-1 < \beta < 0$ the function $m = c_0\rho^\beta$ belongs to $L^p(D)$ for any p from the interval

$$\frac{n}{2} < p < -\frac{n-1}{\beta}, \quad n \geq 2. \tag{2.11}$$

In this case we need to consider $W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D)$ for these values of p.

Theorem 2.5. *(Characterization) $k > 0$ is a transmission eigenvalue of the function m (2.1) if and only if there is a function $u \in H^2_{0,\beta}(D), u \neq 0$, such that the following equality*

$$\int_D \frac{1}{m}(\Delta + k^2(1+m))u(x)(\Delta + k^2)\varphi(x)\,dx = 0 \tag{2.12}$$

holds for any $\varphi \in H^2_{0,\beta}(D)$.

Proof. 1) Let $k > 0$ is a transmission eigenvalue of the function m. Then due to Theorem 2.3 there exists $u_m \in U_m$ such that the unique outgoing solution to the equation

$$(\Delta + k^2)u = -k^2 m u_m$$

belongs to

$$W^2_{\frac{2p}{p-1},0}(D) \cap H^2_{0,\beta}(D) \subset H^2_{0,\beta}(D).$$

Let us prove that this solution u satisfies (2.12). Indeed, it is easy to see that (2.12) converges for this u and $\varphi \in H^2_{0,\beta}(D)$. Moreover we can integrate by parts and get

$$\int_D \frac{1}{m}(\Delta + k^2(1+m))u(x)(\Delta + k^2)\varphi(x)\,dx = \int_D u(\Delta + k^2)\left(\frac{1}{m}(\Delta + k^2(1+m))\varphi(x)\right)dx$$

$$= \int_D (\Delta + k^2)u\frac{1}{m}(\Delta + k^2(1+m))\varphi(x)\,dx = -k^2\int_D u_m(\Delta + k^2(1+m))\varphi(x)\,dx$$

$$= -k^2\int_D (\Delta + k^2(1+m))u_m(x)\varphi(x)\,dx = 0.$$

2) Let there is $u \in H^2_{0,\beta}(D)$ such that (2.12) holds for any $\varphi \in H^2_{0,\beta}(D)$. Integration by parts in (2.12) shows that

$$(\Delta + k^2)\left(\frac{1}{m}(\Delta + k^2(1+m))u(x)\right) = 0$$

in the sense of distributions (the left hand side is an element of $H^{-2}_{\beta}(D) := (H^2_{0,\beta}(D))^*$). It means that there is $u_0 \in U_0, u_0 \neq 0$, such that

$$(\Delta + k^2(1+m))u = -k^2 m u_0.$$

This equality implies that $u \in L^{\frac{2p}{p-1}}(D)$ (see Theorem 1.1 in present article). If we rewrite the latter equality as

$$(\Delta + k^2)u = -k^2 m u - k^2 m u_0$$

then in order to finish the proof of this theorem it is enough to establish that mu belongs to $L^{\frac{2p}{p+1}}(D)$ since the regularity arguments for the operator $\Delta + k^2$ provides the needed result. It is not so difficult to check (see Hardy inequality (2.3)-(2.4)) that $u \in H^2_{0,\beta}(D)$ is equivalent to $\rho^{-\frac{\beta}{2}}u \in W^2_{2,0}$. This implies (using Sobolev embedding) that

$$\rho^{-\frac{\beta}{2}}u \in L^\infty(D), \quad n = 2,3, \quad \rho^{-\frac{\beta}{2}}u \in L^r(D), \quad n \geq 4, \tag{2.13}$$

where $r < \infty$ for $n = 4$ and $r = \frac{2n}{n-4}$ for $n > 4$. Next, we can represent mu as

$$mu = c_0\rho^{\frac{3\beta}{2}}(\rho^{-\frac{\beta}{2}}u).$$

This representation and embeddings (2.13) allow us to conclude that in the case $\beta > 0$ the function $mu \in L^2(D)$. In the case $\beta < 0$ we need to assume in addition to the conditions (2.11) that in two-dimensional case $\beta > -\frac{2}{3}$. Then using Hölder inequality we may conclude that the function $mu \in L^{\frac{2p}{p+1}}(D)$ for some $\frac{n}{2} < p < \infty$. Therefore, this theorem is completely proved. □

Remark 2.6. All results that were proved above will be also true for the case

$$m(x) = c_0|x - x_0|^\beta,$$

where x_0 is an arbitrary fixed point from domain D.

3 Existence of Transmission Eigenvalues

In this section we assume that the function m satisfies either the conditions (2.1) or the conditions of Remark 2.6. Theorem 2.2 tells us that $k > 0$ is a transmission eigenvalue whenever the operator (which is understood in the sense of quadratic forms)

$$(\Delta + k^2)\left(\frac{1}{m}(\Delta + k^2(1+m))\right) = (\Delta + k^2(1+m))\left(\frac{1}{m}(\Delta + k^2)\right) \tag{3.1}$$

has a non-trivial kernel in $H^2_{0,\beta}(D)$. We will investigate the existence of this kernel by examining the spectrum of the operator as k^2 changes. We denote k^2 by $\tau, k^2 = \tau$.

The following theorem asserts that this operator (3.1), with the appropriate domain, defines a semi-bounded self-adjoint operator.

Theorem 3.1. *The quadratic form Q_τ, defined by*

$$Q_\tau(u) = \frac{1}{c_0}\int_D \rho^{-\beta}|\Delta u|^2\,dx + \tau\int_D \left(\frac{1}{c_0}\rho^{-\beta}(u\Delta\overline{u} + \overline{u}\Delta u) + \overline{u}\Delta u\right)dx +$$

$$+\tau^2\int_D (\frac{1}{c_0}\rho^{-\beta}+1)|u|^2\,dx \tag{3.2}$$

with form domain $H^2_{0,\beta}(D)$, is densely defined, closed semi-bounded quadratic form on $L^2_{\frac{\beta}{2}+2}(D)$ with the norm

$$||f||^2_{L^2_{\frac{\beta}{2}+2}(D)} = \int_D \rho^{-\beta-4}|f|^2\,dx.$$

The unique self-adjoint operator associated with this norm Q_τ is equal to

$$L = (\Delta + k^2)\left(\frac{1}{m}(\Delta + k^2(1+m))\right) = (\Delta + k^2(1+m))\left(\frac{1}{m}(\Delta + k^2)\right) \tag{3.3}$$

on the domain

$$D(L) = \{f \in H^2_{0,\beta}(D) : Lf \in L^2(D)\}.$$

In addition, the spectrum of this self-adjoint operator is pure discrete of finite multiplicity having only one accumulation point at infinity.

Proof. Let us first prove so-called Gårding's inequality. Using inequality $ab \le \varepsilon a^2 + \frac{1}{4\varepsilon}b^2$ we obtain from (3.2)

$$Q_\tau(u) \ge \frac{1}{c_0}(1 - 2\varepsilon\tau)\int_D \rho^{-\beta}|\Delta u|^2\,dx + \frac{1}{c_0}(\tau^2 - \frac{\tau}{\varepsilon})\int_D \rho^{-\beta}|u|^2\,dx -$$

$$-c_0\frac{\tau}{4\varepsilon}\int\limits_D \rho^{\beta}|u|^2\,dx+\int\limits_D |u|^2\,dx. \tag{3.4}$$

Choosing $\varepsilon=\frac{1}{4\tau}$ and using the condition $\beta>-2$, we obtain from (3.4)

$$Q_\tau(u)\geq\frac{1}{2c_0}\int\limits_D \rho^{-\beta}|\Delta u|^2\,dx-\frac{3\tau^2}{c_0}d^4\int\limits_D \rho^{-\beta-4}|u|^2\,dx-$$

$$-c_0\tau^2 d^{2\beta+4}\int\limits_D \rho^{-\beta-4}|u|^2\,dx,$$

where $d=diamD$. Now Hardy inequality (2.3)-(2.4) implies that there is a constant $C>1$ such that Gårding's inequality holds

$$Q_\tau(u)\geq\frac{1}{2c_0C}||u||^2_{H^2_\beta(D)}-\left(\frac{3\tau^2}{c_0}d^4+c_0\tau^2 d^{2\beta+4}\right)||u||^2_{L^2_{\frac{\beta}{2}+2}(D)}. \tag{3.5}$$

This inequality implies that for some $\mu>0$

$$Q_\tau(u)+\mu||u||^2_{L^2_{\frac{\beta}{2}+2}(D)}\geq\frac{1}{2c_0C}||u||^2_{H^2_\beta(D)}.$$

It means that there is a self-adjoint operator $L_\mu:=L+\mu\rho^{-\beta-4}I$ for which

$$(L_\mu u,u)_{L^2(D)}\geq\frac{1}{2c_0C}||u||^2_{H^2_\beta(D)}.$$

Since $H^2_{0,\beta}(D)\subset L^2(D)$ and this embedding is compact (see (2.5)), we may conclude from this inequality that there is a unique self-adjoint operator

$$L_\mu^{-1}:L^2(D)\to L^2(D)$$

with pure discrete spectrum of finite multiplicity. Thus, we can obtain self-adjoint operator L (3.3) as follows:

$$L:=(L_\mu^{-1})^{-1}-\mu\rho^{-\beta-4}I.$$

Due to the compact imbedding (2.5) the spectrum of the operator $(L_\mu^{-1})^{-1}$ is pure discrete of finite multiplicity having only one accumulation point at infinity. But since the operator

$$L_\mu^{-1}\rho^{-\beta-4}$$

is also compact we have the same fact for the spectrum of the operator L. Thus, Theorem 3.1 is proved. □

The next theorem can be considered as a particular case of the previous one but it is actually what we need for the existence (and non-existence) transmission eigenvalues.

Theorem 3.2. *The quadratic form Q_0, defined by*

$$Q_0(u) = \frac{1}{c_0} \int_D \rho^{-\beta} |\Delta u|^2 \, dx \tag{3.6}$$

with form domain $H^2_{0,\beta}(D)$, is densely defined, closed positive quadratic form on $L^2(D)$. The unique self-adjoint operator associated with this norm Q_0 is equal to

$$L_0 = \Delta\left(\frac{1}{m}\Delta\right) \tag{3.7}$$

on the domain

$$D(L_0) = \{f \in H^2_{0,\beta}(D) : L_0 f \in L^2(D)\}.$$

Moreover, this self-adjoint operator has pure discrete nonnegative spectrum $\mu_s \geq 0, s = 0, 1, 2, \ldots$ of finite multiplicity having only one accumulation point at infinity.

We need some notations. First, we recall the min-max characterization of the eigenvalues μ_s of a self-adjoint operator L_0 defined by a quadratic form (3.6) (see [25], p. 71)

$$\mu_s = \max_{V \subset V_s} \min_{u \in V, \|u\|_{L^2(D)}=1} Q_0(u), \tag{3.8}$$

where V_s denotes the co-dimension s subspaces of the form domain $H^2_{0,\beta}(D)$. Let us denote by S_β the following value

$$S^{\pm}_{\beta} = \max_{u \in H^2_{0,\beta}(D), \|u\|_{L^2(D)}=1} \int_D m^{\pm 1}(x) |u(x)|^2 \, dx. \tag{3.9}$$

And finally, by λ_0 we denote the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions. The Rayleigh-Ritz characterization of the first Dirichlet eigenvalue and embedding (2.5) imply

$$\lambda_0 = \inf_{u \in H^1_0(D), \|u\|_{L^2(D)}=1} \int_D |\nabla u(x)|^2 \, dx \leq$$

$$\leq \inf_{u \in H^2_{0,\beta}(D), \|u\|_{L^2(D)}=1} \int_D |\nabla u(x)|^2 \, dx. \tag{3.10}$$

Theorem 3.3. *Suppose that function m satisfies all conditions of Remark 2.4 or Remark 2.6. If*

$$k^2 < \frac{\lambda_0}{1 + S^+_\beta} \tag{3.11}$$

where S^+_β is as in (3.9), then $k > 0$ is not a transmission eigenvalue. If $k^2 \geq \frac{\lambda_0}{1+S^+_\beta}$ and

$$\lambda_0 \geq 2\sqrt{\mu_s}\left(\sqrt{1 + S^-_\beta} + \sqrt{S^-_\beta}\right), \tag{3.12}$$

where μ_s and S_β^- are as in (3.8) and (3.9), respectively, then there exist $s+1$ transmission eigenvalues k with

$$\frac{\lambda_0 - 2\sqrt{\mu_s}\sqrt{S_\beta^-} - \sqrt{\lambda_0^2 - 4\lambda_0\sqrt{\mu_s}\sqrt{S_\beta^-} - 4\mu_s}}{2(1+S_\beta^-)} \le k^2 \le$$

$$\le \frac{\lambda_0 - 2\sqrt{\mu_s}\sqrt{S_\beta^-} + \sqrt{\lambda_0^2 - 4\lambda_0\sqrt{\mu_s}\sqrt{S_\beta^-} - 4\mu_s}}{2(1+S_\beta^-)}. \tag{3.13}$$

Proof. The quadratic form Q_τ (3.2) can be rewritten as

$$Q_\tau(u) = \int_D (\Delta + k^2(1+m))\left(\frac{1}{m}(\Delta + k^2(1+m))\right)u\cdot\overline{u}\,dx -$$

$$-k^2\int_D (\Delta + k^2(1+m))u\cdot\overline{u}\,dx.$$

Integration by parts in both integrals yields

$$Q_\tau(u) = \int_D \frac{1}{m}|(\Delta + k^2(1+m))u|^2\,dx + k^2\int_D |\nabla u|^2\,dx - k^4\int_D (1+m)|u|^2\,dx.$$

Thus, using (3.9) and (3.10) we obtain

$$Q_\tau(u) \ge \tau\lambda_0 - \tau^2(1+S_\beta^+).$$

This inequality implies that if τ satisfies (3.11) then the quadratic form (3.2) is strictly positive and therefore such $k>0$ is not a transmission eigenvalue.

To prove the second part of the theorem we estimate the quadratic form (3.2) from above as follows (using also integration by parts in the second integral):

$$Q_\tau(u) \le \frac{1}{c_0}\int_D \rho^{-\beta}|\Delta u|^2\,dx + 2\tau\int_D \frac{1}{c_0}\rho^{-\beta}|u||\Delta u|\,dx - \tau\int_D |\nabla u|^2\,dx +$$

$$+\tau^2\int_D \left(\frac{1}{c_0}\rho^{-\beta}+1\right)|u|^2\,dx. \tag{3.14}$$

Restricting to $||u||_{L^2(D)} = 1$ and using the Cauchy-Schwartz inequality we obtain from (3.10) and (3.14)

$$Q_\tau(u) \le \frac{1}{c_0}\int_D \rho^{-\beta}|\Delta u|^2\,dx + 2\tau\left(\int_D \frac{1}{c_0}\rho^{-\beta}|\Delta u|^2\,dx\right)^{\frac{1}{2}}\left(\int_D \frac{1}{c_0}\rho^{-\beta}|u|^2\,dx\right)^{\frac{1}{2}} -$$

$$-\tau\lambda_0 + \tau^2 + \tau^2\int_D \frac{1}{c_0}\rho^{-\beta}|u|^2\,dx.$$

Restricting now to V_s (see (3.8)) and using (3.9) we obtain from the latter inequality

$$Q_\tau(u) \le \tau^2(1+S_\beta^-) - \tau\left(\lambda_0 - 2\sqrt{\mu_s}\sqrt{S_\beta^-}\right) + \mu_s.$$

We minimize the right hand-side of this inequality by choosing $\tau^* = \frac{\lambda_0 - 2\sqrt{\mu_s}\sqrt{S_\beta^-}}{2(1+S_\beta^-)}$ to obtain

$$Q_{\tau^*}(u) \le -\frac{\left(\lambda_0 - 2\sqrt{\mu_s}\sqrt{S_\beta^-}\right)^2}{4(1+S_\beta^-)} + \mu_s. \tag{3.15}$$

This inequality shows that Q_{τ^*} restricted to V_s is non-positive if the right hand-side of (3.15) is non-positive. But it is equivalent to (3.12). Using the continuity arguments of Q_τ with respect to τ (see, for example, Lemmas 5.3-5.5 of [24]) we may conclude now that if the conditions (3.12) and (3.13) are satisfied then there are at least $s+1$ transmission eigenvalues. Thus Theorems 3.3 is completely proved. □

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