Communications in **M**athematical **A**nalysis
Volume 14, Number 2, pp. 113–128 (2013) www.math-res-pub.org/cma
ISSN 1938-9787

Poly-Bergman Type Spaces on the Siegel Domain

Josué Ramírez Ortega[*]
Facultad de Matemáticas, Universidad Veracruzana,
Lomas del Estadio s/n, Zona Universitaria Xalapa, Veracruz, México

Armando Sánchez Nungaray[†]
Facultad de Matemáticas, Universidad Veracruzana,
Lomas del Estadio s/n, Zona Universitaria Xalapa, Veracruz, México

(Communicated by Vladimir Rabinovich)

Abstract

We introduce poly-Bergman type spaces on the Siegel domain $D_n \subset \mathbb{C}^n$, and prove that they are isomorphic to tensor products of one-dimensional spaces generated by orthogonal polynomials of two kinds: Laguerre and Hermite polynomials. The linear span of all poly-Bergman type spaces is dense in the Hilbert space $L^2(D_n, d\mu_\lambda)$, where $d\mu_\lambda = (\operatorname{Im} z_n - |z_1|^2 - \cdots - |z_{n-1}|^2)^\lambda dx_1 dy_1 \cdots dx_n dy_n$ and $\lambda > -1$.

AMS Subject Classification: Primary 32A36; Secondary 30H20.

Keywords: Bergman spaces, Siegel domain, Polyanalytic functions.

1 Introduction

In this paper we generalize the concept of the polyanalytic function for the Siegel domain $D_n \subset \mathbb{C}^n$, which is the unbounded realisation of the unit ball $\mathbb{B}^n \subset \mathbb{C}^n$.

The spaces of polyanalytic functions on the unit disc $\mathbb{D}$, or on the upper half-plane as its unbounded realisation, were introduced and studied in [1, 2, 5, 6]). Recall some known facts. Let $\Pi \subset \mathbb{C}$ be the upper half-plane and let $l \in \mathbb{N}$. We denote by $\mathcal{A}_l^2(\Pi)$ $[\tilde{\mathcal{A}}_l^2(\Pi)]$ the subspace of $L^2(\Pi)$ consisting of all l-analytic functions [l-anti-analytic functions], i.e., the functions satisfying the equation $(\partial/\partial\overline{z})^l\varphi = 0$ $[(\partial/\partial z)^l\varphi = 0]$. The function space $\mathcal{A}_l^2(\Pi)$ is called poly-Bergman space of Π. Let $\mathcal{A}_{(l)}^2(\Pi) = \mathcal{A}_l^2(\Pi) \ominus \mathcal{A}_{l-1}^2(\Pi)$ and $\tilde{\mathcal{A}}_{(l)}^2(\Pi) = \tilde{\mathcal{A}}_l^2(\Pi) \ominus \tilde{\mathcal{A}}_{l-1}^2(\Pi)$ be the spaces of true-l-analytic functions and true-l-anti-analytic functions, respectively.

[*]E-mail address: jro3001@gmail.com
[†]E-mail address: sancheznungaray@gmail.com

Let $\chi_{\pm}$ stand for the characteristic function of $\mathbb{R}_{\pm} = \{x \in \mathbb{R} : \pm x \geq 0\}$. The main result of [10] says that the space $L^2(\Pi)$ admits the decomposition

$$L^2(\Pi) = \bigoplus_{l=1}^{\infty} \mathcal{A}^2_{(l)}(\Pi) \oplus \bigoplus_{l=1}^{\infty} \tilde{\mathcal{A}}^2_{(l)}(\Pi),$$

and that there exists an unitary operator $W : L^2(\Pi) \to L^2(\Pi)$ such that the restriction mappings

$$W : \mathcal{A}^2_{(l)}(\Pi) \to L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l-1},$$

$$W : \tilde{\mathcal{A}}^2_{(l)}(\Pi) \to L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l-1},$$

are isometric isomorphisms, where $\mathcal{L}_l$ is the one-dimensional space generated by the Laguerre function of degree l and order $\lambda > -1$. Note that the above restriction mappings from poly-Bergman spaces and anti-poly-Bergman spaces are the analogue of the Bargmann type transform.

For the Bergman space $\mathcal{A}^2_\lambda(D_n)$ of the Siegel domain D_n, the analogues of the classical Bargmann transform and its inverse for five different types of commutative subgroups of biholomorphisms of D_n were constructed in [8]. In particular, for the nilpotent case, an isometric isomorphisms

$$U : \mathcal{A}^2_\lambda(D_n) \to L^2(\mathbb{R}^{n-1} \times \mathbb{R}_+)$$

was explicitly described.

In this work the polyanalytic function spaces are defined via the complex structure of $\mathbb{C}^n$ induced by the tangential Cauchy-Riemann equations, which were given for the Heisenberg group in [3]. Let $L = (l_1, \ldots, l_n) \in \mathbb{N}^n$. The poly-Bergman type space $\mathcal{A}^2_{\lambda L}(D_n)$, or simply denoted by $\mathcal{A}^2_{\lambda L}$, is the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all L-analytic functions, i.e., functions that satisfy the equations

$$\begin{aligned} \left(\frac{\partial}{\partial \overline{z_k}} - 2iz_k \frac{\partial}{\partial \overline{z_n}}\right)^{l_k} f &= 0, \quad 1 \leq k \leq n-1, \\ \left(\frac{\partial}{\partial \overline{z_n}}\right)^{l_n} f &= 0, \end{aligned}$$

where, as usual, $\frac{\partial}{\partial \overline{z}_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} - \frac{1}{i}\frac{\partial}{\partial y_k}\right)$ and $\frac{\partial}{\partial z_k} = \frac{1}{2}\left(\frac{\partial}{\partial x_k} + \frac{1}{i}\frac{\partial}{\partial y_k}\right)$. In particular, a function f is analytic in the Siegel domain if it satisfies

$$\begin{aligned} \frac{\partial f}{\partial \overline{z_k}} - 2iz_k \frac{\partial f}{\partial \overline{z_n}} &= 0, \quad 1 \leq k \leq n-1, \\ \frac{\partial f}{\partial \overline{z_n}} &= 0. \end{aligned}$$

Functions in $\mathcal{A}^2_{\lambda L}$ will be also called polyanalytic functions.

Anti-polyanalytic functions are just the complex conjugation of polyanalytic functions, but the spaces of polyanalytic and anti-polyanalytic functions are mutually orthogonal. For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the anti-poly-Bergman type space $\tilde{\mathcal{A}}^2_{\lambda L}(D_n)$ (or simply $\tilde{\mathcal{A}}^2_{\lambda L}$)

as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all L-anti-analytic functions, i.e., functions satisfying the equations

$$\begin{aligned}\left(\frac{\partial}{\partial z_k} + 2i\overline{z_k}\frac{\partial}{\partial z_n}\right)^{l_k} f &= 0, \quad k = 1, ..., n-1, \\ \left(\frac{\partial}{\partial z_n}\right)^{l_n} f &= 0.\end{aligned}$$

We define the spaces of true-L-analytic and true-L-anti-analytic functions as

$$\mathcal{A}^2_{\lambda(L)} = \mathcal{A}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \mathcal{A}^2_{\lambda, L-e_j}\right),$$

$$\tilde{\mathcal{A}}^2_{\lambda(L)} = \tilde{\mathcal{A}}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \tilde{\mathcal{A}}^2_{\lambda, L-e_j}\right),$$

where $\mathcal{A}^2_{\lambda S} = \tilde{\mathcal{A}}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$, and $\{e_k\}_{k=1}^n$ stand for the canonical basis of $\mathbb{R}^n$.

The main results obtained in this work are as follows:

1. The space $L^2(D_n, d\mu_\lambda)$ admits the decomposition

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}^2_{\lambda(L)}\right) \bigoplus \left(\bigoplus_{L\in\mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)}\right).$$

2. There exists an unitary operator

$$W : L^2(D_n, d\mu_\lambda) \longrightarrow L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, y^\lambda dy)$$

such that for each $L \in \mathbb{N}^n$ the restricted mappings

$$W : \mathcal{A}^2_{\lambda(L)} \to L^2(\mathbb{R}^{n-1}) \otimes H_{L-e} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l_n-1},$$

$$W : \tilde{\mathcal{A}}^2_{\lambda(L)} \to H_{L-e} \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l_n-1},$$

are isometric isomorphisms, where H_{L-e} is the one-dimensional space generated by the product $h_{l_1-1}(y_1)\cdots h_{l_{n-1}-1}(y_{n-1})$ and $\{h_j(y)\}_{j=0}^\infty$ is the orthonormal basis for $L^2(\mathbb{R}, dy)$ consisting of the Hermite functions.

Let $\sigma \in \{\pm 1\}^n$ and $L \in \mathbb{N}^n$. The subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all (L,σ)-analytic functions is defined in Section 7. Such subspace is denoted by $\mathcal{A}^2_{\lambda L\sigma}$, and is called mixed poly-Bergman type space or σ-poly-Bergman type space. In particular, if $\sigma = L = (1, \ldots, 1)$, then $\mathcal{A}^2_{\lambda L\sigma}$ is just the usual weighted Bergman space of D_n. We define the space of true-(L,σ)-analytic functions as

$$\mathcal{A}^2_{\lambda(L)\sigma} = \mathcal{A}^2_{\lambda L\sigma} \ominus \left(\sum_{k=1}^{n} \mathcal{A}^2_{\lambda, L-e_k,\sigma}\right),$$

where $\mathcal{A}^2_{\lambda S\sigma} = \{0\}$ if $S \notin \mathbb{N}^n$. We prove that $L^2(D_n, d\mu_\lambda)$ admits the decomposition

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}^2_{\lambda(L)\sigma}\right) \bigoplus \left(\bigoplus_{L\in\mathbb{N}^n} \mathcal{A}^2_{\lambda(L),-\sigma}\right).$$

We also establish the relationship between the poly-Bergman type spaces and the σ-poly-Bergman type spaces.

2 CR Manifolds

For a smooth submanifold M in $\mathbb{C}^n$, recall that $T_p(M)$ is the real tangent space of M at the point p. In general, $T_p(M)$ is not invariant under the complex structure map J for $T_p(\mathbb{C}^n)$. For a point $p \in M$, the complex tangent space of M at p is the vector space

$$H_p(M) = T_p(M) \cap J\{T_p(M)\}.$$

This space is called the holomorphic tangent space. Using the Euclidian inner product on $T_p(\mathbb{R}^{2n})$, denote by $X_p(M)$ the totally real part of the tangent space of M which is the orthogonal complement of $H_p(M)$ in $T_p(M)$. We have that $T_p(M) = H_p(M) \oplus X_p(M)$ and $J(X_p(M))$ is trasversal to $T_p(M)$. A submanifold M of $\mathbb{C}^n$ is called a CR submanifold of $\mathbb{C}^n$ if $\dim_{\mathbb{R}} H_p(M)$ is independient of $p \in M$. The complexifications of $T_p(M)$, $H_p(M)$ and $X_p(M)$ are denoted by $T_p(M)\otimes\mathbb{C}$, $H_p(M)\otimes\mathbb{C}$ and $X_p(M)\otimes\mathbb{C}$, respectively. Since the space $H_p(M)$ is J-invariant, the complex structure map J on $T_p(\mathbb{R}^{2n})\otimes\mathbb{C}$ induce a complex structure map on $H_p(M)\otimes\mathbb{C}$ by restriction. Moreover $H_p(M)\otimes\mathbb{C}$ is the direct sum of the $+i$ and $-i$ eigenspace of J which are denoted by $H_p^{1,0}(M)$ and $H_p^{0,1}(M)$, respectively.

The following result establishes the form of the basis of $H_p(M)$. It also provides an expression for the generators of $H_p(M)$. We refer to [3] for its proof.

Theorem 2.1. *Suppose $M = \{(x+iy, w) \in \mathbb{C}^d \times \mathbb{C}^{n-d} : \ \ y = h(x,w)\}$, where $h : \mathbb{R}^d \times \mathbb{C}^{n-d} \to \mathbb{R}^d$ is of class C^m $(m \geq 2)$ with $h(0)$ and $Dh(0) = 0$. A basis for $H_p^{1,0}(M)$ near the origin is given by*

$$\Lambda_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^{d} \left(\sum_{m=1}^{d} \mu_{lm} \frac{\partial h_m}{\partial w_k} \frac{\partial}{\partial z_l} \right), \quad 1 \leq k \leq n-d,$$

where μ_{lm} is the (l,m)-th element of the $d \times d$ matrix

$$\left(I - i\frac{\partial h}{\partial x}\right)^{-1}.$$

A basis for $H_p^{0,1}$ near the origin is given by $\overline{\Lambda_1}, \ldots, \overline{\Lambda_{n-d}}$.

If the function h is independient of the variable x, then the local basis of $H_p^{1,0}(M)$ has the following more simple form

$$\Lambda_k = \frac{\partial}{\partial w_k} + 2i \sum_{l=1}^{d} \frac{\partial h_l}{\partial w_k} \frac{\partial}{\partial z_l}, \quad 1 \leq k \leq n-d. \tag{2.1}$$

We refer to Example 7.3-1 of [3] for the details on the following construction of the Heisenberg group, which use the equation (2.1). For the real hypersurface in $\mathbb{C}^n$ defined by

$$M = \{(z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \ \mathrm{Im}\, z_n = |z'|^2\},$$

the generators for $H^{1,0}(M)$ are given by

$$\Lambda_k = \Lambda_{k-}^{-} = \frac{\partial}{\partial z_k} + 2i\overline{z_k}\frac{\partial}{\partial z_n}, \quad 1 \le k \le n-1, \tag{2.2}$$

and the generators for $H^{0,1}(M)$ are given by

$$\overline{\Lambda_k} = \Lambda_{k+}^{+} = \frac{\partial}{\partial \overline{z_k}} - 2iz_k\frac{\partial}{\partial \overline{z_n}}, \quad 1 \le k \le n-1. \tag{2.3}$$

3 Cauchy-Riemann Equations for the Siegel Domain

Let $d\mu(z) = dx_1dy_1 \ldots dx_ndy_n$ stand for the standard Lebesgue measure in $\mathbb{C}^n$, where $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ and $z_k = x_k + iy_k$. We often rewrite z as (z', z_n), where $z' = (z_1, \ldots, z_{n-1})$. The standard norm in $\mathbb{C}^n$ is denoted by $|\cdot|$. In the Siegel domain

$$D_n = \{z = (z', z_n) \in \mathbb{C}^{n-1} \times \mathbb{C} : \ \mathrm{Im}\, z_n - |z'|^2 > 0\}$$

we consider the weighted Lebesgue measure

$$d\mu_\lambda(z) = (\mathrm{Im}\, z_n - |z'|^2)^\lambda d\mu(z), \quad \lambda > -1.$$

Let $\mathcal{A}^2_\lambda(D_n)$ be the weighted Bergman space, defined as the space of all holomorphic functions in $L^2(D_n, d\mu_\lambda)$. Thus, for $f \in \mathcal{A}^2_\lambda(D_n)$,

$$\frac{\partial f}{\partial \overline{z}_k} = 0, \quad k = 1, ..., n,$$

equivalently,

$$\begin{aligned} \overline{\Lambda}_k f &= 0, \quad k = 1, ..., n-1, \\ \frac{\partial}{\partial \overline{z}_n} f &= 0. \end{aligned}$$

We use all the powers of the $\overline{\Lambda}_k$'s operators to define the first class of poly-Bergman type spaces in the Siegel domain, i.e., we define a certain class of polyanalytic function spaces. Fortunately, such spaces densely fill the space $L^2(D_n, d\mu_\lambda)$, and are isomorphic to tensor products of certain L^2-spaces.

Let $\mathcal{D} = \mathbb{C}^{n-1} \times \Pi$, where $\Pi = \mathbb{R} \times \mathbb{R}_+ \subset \mathbb{C}$. We realize the poly-Bergman type spaces as subspaces of $L^2(\mathcal{D}, d\eta_\lambda)$ in order to apply Fourier transform techniques for their study. Consider the following mapping from $\mathcal{D}$ to D_n:

$$\kappa : w = (w', u_n + iv_n) \longmapsto z = (w', u_n + iv_n + i|w'|^2), \quad \text{(i.e. } z' = w').$$

Consider also the unitary operator $U_0 : L^2(D_n, d\mu_\lambda) \to L^2(\mathcal{D}, d\eta_\lambda)$ given by

$$(U_0 f)(w) = f(\kappa(w)),$$

where

$$d\eta_\lambda(w) = v_n^\lambda d\mu(w).$$

In [8] the authors showed that the space $\mathcal{A}_0(\mathcal{D}) = U_0(\mathcal{A}^2_\lambda(D_n))$ consists of all functions $\varphi(w', w_n) = (U_0 f)(w)$ satisfying the equations

$$\begin{aligned} U_0 \tfrac{\partial}{\partial \overline{z}_k} U_0^{-1} \varphi &= \left(\tfrac{\partial}{\partial \overline{w}_k} - w_k \tfrac{\partial}{\partial v_n}\right)\varphi &= 0, \quad 0 \le k \le n-1, \\ U_0 \tfrac{\partial}{\partial \overline{z}_n} U_0^{-1} \varphi &= \tfrac{\partial}{\partial \overline{w}_n} \varphi &= 0, \end{aligned} \tag{3.1}$$

where $\frac{\partial}{\partial \overline{w}_n} = \frac{1}{2}\left(\frac{\partial}{\partial u_n} + i \frac{\partial}{\partial v_n}\right)$. For functions satisfying this last equation, the first type equation in (3.1) can be rewritten as

$$U_0 \frac{\partial}{\partial \overline{z}_k} U_0^{-1} \varphi = \left(\frac{\partial}{\partial \overline{w}_k} - i w_k \frac{\partial}{\partial u_n}\right)\varphi = 0, \quad k = 1, ..., n-1. \tag{3.2}$$

This equation justify why we are using the Δ_k's operators because

$$U_0 \overline{\Lambda}_k U_0^{-1} = \frac{\partial}{\partial \overline{w}_k} - i w_k \frac{\partial}{\partial u_n}, \quad k = 1, ..., n-1.$$

On the other hand, the differential operators $\partial/\partial z_k$ $(k = 1, ..., n-1)$ are used to define the anti-analytic function space, but they can be replaced by the operators given in (2.2). In particular,

$$U_0 \Lambda_k U_0^{-1} = \frac{\partial}{\partial w_k} + i \overline{w}_k \frac{\partial}{\partial u_n}, \quad k = 1, ..., n-1.$$

In addition we have

$$U_0 \frac{\partial}{\partial z_n} U_0^{-1} = \frac{\partial}{\partial w_n} = \frac{1}{2}\left(\frac{\partial}{\partial u_n} - i \frac{\partial}{\partial v_n}\right).$$

As expected, we use the Λ_k's operators to define anti-polyanalytic function spaces.

To define mixed poly-Bergman type spaces we additionally use the differential operators

$$\Lambda^+_{k-} = \frac{\partial}{\partial z_k} - 2i \overline{z_k} \frac{\partial}{\partial \overline{z_n}}, \quad 1 \le k \le n-1, \tag{3.3}$$

$$\Lambda^-_{k+} = \overline{\Lambda^+_{k-}} = \frac{\partial}{\partial \overline{z_k}} + 2i z_k \frac{\partial}{\partial z_n}, \quad 1 \le k \le n-1. \tag{3.4}$$

We have

$$U_0 \Lambda^+_{k-} U_0^{-1} = \frac{\partial}{\partial w_k} - i \overline{w}_k \frac{\partial}{\partial u_n}, \quad k = 1, ..., n-1,$$

$$U_0 \Lambda^-_{k+} U_0^{-1} = \frac{\partial}{\partial \overline{w}_k} + i w_k \frac{\partial}{\partial u_n}, \quad k = 1, ..., n-1.$$

4 Orthogonal Polynomials

In this section we introduce Laguerre and Hermite polynomials, which will be used to describe poly-Bergman type spaces. As usual, the Laguerre polynomials of order λ are defined by

$$L_j^\lambda(y) := e^y \frac{y^{-\lambda}}{j!} \frac{d^j}{dy^j}(e^{-y} y^{j+\lambda}), \quad j = 0, 1, 2, \ldots$$

Laguerre polynomials constitute an orthogonal basis for the space $L^2(\mathbb{R}_+, y^\lambda e^{-y} dy)$, thus the set of Laguerre functions

$$\ell_j^\lambda(y) = (-1)^j c_j L_j^\lambda(y) e^{-y/2}, \quad j = 0, 1, 2, \ldots$$

is an orthonormal basis of $L^2(\mathbb{R}_+, y^\lambda dy)$, where $c_j = \sqrt{j!/\Gamma(j+\lambda+1)}$ and Γ is the standard gamma function. The second type of polynomials we are interested in is the set of Hermite polynomials:

$$H_j(y) := (-1)^j e^{y^2} \frac{d^j}{dy^j} e^{-y^2}, \quad j = 0, 1, 2, \ldots$$

Hermite polynomials constitute an orthonormal basis for the space $L^2(\mathbb{R}, e^{-y^2} dy)$, thus the set of Hermite functions

$$h_j(y) = \frac{(-1)^j}{(2^n \sqrt{\pi} n!)^{1/2}} H_j(y) e^{-y^2/2}, \quad j = 0, 1, 2, \ldots$$

is an orthonormal basis of $L^2(\mathbb{R})$. Therefore, the set of functions

$$h_{J'}(y_1, \ldots, y_{n-1}) = \prod_{k=1}^{n-1} h_{j_k}(y_k), \quad J' = (j_1, \ldots, j_{n-1}) \in \mathbb{Z}_+^{n-1} \tag{4.1}$$

is an orthonormal basis of $L^2(\mathbb{R}^{n-1})$. Here $\mathbb{Z}_+ = \{0\} \cup \mathbb{N}$ and $\mathbb{Z}_- = \mathbb{Z} \setminus \mathbb{N}$. For $J', L' \in \mathbb{Z}_+^{n-1}$ we say that $J' \leq L'$ if $j_k \leq l_k$ with $k = 1, \ldots, n-1$.

5 Poly-Bergman Type Spaces

For $L = (l_1, \ldots, l_n) \in \mathbb{N}^n$, we define the poly-Bergman type space $\mathcal{A}^2_{\lambda L}$ as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all functions f satisfying the equations

$$\begin{aligned} \left(\frac{\partial}{\partial \bar{z}_k} - 2i z_k \frac{\partial}{\partial \bar{z}_n} \right)^{l_k} f &= 0, \quad k = 1, \ldots, n-1, \\ \left(\frac{\partial}{\partial \bar{z}_n} \right)^{l_n} f &= 0. \end{aligned}$$

Let $\{e_j\}_{j=1}^n$ be the canonical basis of $\mathbb{R}^n$. We define the space of true-L-analytic functions as

$$\mathcal{A}^2_{\lambda(L)} = \mathcal{A}^2_{\lambda L} \ominus \left(\sum_{j=1}^n \mathcal{A}^2_{\lambda, L - e_j} \right),$$

where $\mathcal{A}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$.

It is much more convenient to deal with $\mathcal{A}_{0,\lambda L}(\mathcal{D}) = U_0(\mathcal{A}^2_{\lambda L}) \subset L^2(\mathcal{D}, d\eta_\lambda)$ in order to apply Fourier techniques for the study of the poly-Bergman type space. For a function $\varphi(w) = (U_0 f)(w) \in \mathcal{A}_{0,\lambda L}(\mathcal{D})$ we have

$$\begin{array}{rcccl} U_0\left(\overline{\Lambda}_k\right)^{l_k} U_0^{-1}\varphi &=& \left(\frac{\partial}{\partial \overline{w}_k} - i w_k \frac{\partial}{\partial u_n}\right)^{l_k}\varphi &=& 0, \quad k = 1, ..., n-1, \\ U_0\left(\frac{\partial}{\partial \overline{z}_n}\right)^{l_n} U_0^{-1}\varphi &=& \frac{1}{2^{l_n}}\left(\frac{\partial}{\partial u_n} + i\frac{\partial}{\partial v_n}\right)^{l_n}\varphi &=& 0. \end{array}$$

Consider now the tensor decomposition

$$L^2(\mathcal{D}, d\eta_\lambda) = L^2(\mathbb{C}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v_n^\lambda dv_n).$$

Take $w = (w', w_n) \in \mathbb{C}^{n-1} \times \Pi$, where $w' = (w_1, ..., w_{n-1})$ and $w_k = u_k + iv_k$. We write $w' = u' + iv'$, where $u' = (u_1, ..., u_{n-1})$ and $v' = (v_1, ..., v_{n-1})$, and we identify $w = (w', u_n + iv_n)$ with (u', v', u_n, v_n). Then

$$L^2(\mathcal{D}, d\eta_\lambda) = L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, v_n^\lambda dv_n), \tag{5.1}$$

where the first (second) tensor factor space consists of functions in the real (imaginary) part of the complex vector w'. Let F denote the Fourier transform on $L^2(\mathbb{R})$:

$$(F f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(u) e^{-i\xi u} du.$$

Let $F_{(n-1)}$ be the tensor product of F with itself taken $n-1$ times. Now, according to the decomposition (5.1) we introduce the unitary operators

$$U_1 = I \otimes I \otimes F \otimes I,$$

$$U_2 = F_{(n-1)} \otimes I \otimes I \otimes I.$$

Of course, $U_2 U_1$ is just the Fourier transform with respect to the variable $u = \operatorname{Re} w$.

Consider the change of variable $\zeta = (\zeta_1, ..., \zeta_n) \mapsto z = (z_1, ..., z_n)$, where $\zeta_k = \xi_k + iv_k$ and $z_k = x_k + iy_k$ are related by

$$\begin{pmatrix} \xi_k \\ v_k \end{pmatrix} = \begin{pmatrix} \sqrt{|x_n|} & \sqrt{|x_n|} \\ \frac{-1}{2\sqrt{|x_n|}} & \frac{1}{2\sqrt{|x_n|}} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad k = 1, \dots, n-1, \tag{5.2}$$

and

$$\xi_n = x_n, \quad v_n = \frac{y_n}{2|x_n|}.$$

Let $\zeta = (\zeta', \zeta_n)$, where $\zeta' = \xi' + iv'$ and $\xi' = (\xi_1, \dots, \xi_{n-1})$. According to the tensor product (5.1), consider the following unitary operators on $L^2(\mathcal{D}, d\eta_\lambda)$:

$$V_1 : \psi(\zeta', \xi_n + iv_n) \longmapsto \Psi(\zeta', x_n + iy_n) = \frac{1}{(2|x_n|)^{(\lambda+1)/2}} \psi(\zeta', x_n + i\frac{y_n}{2|x_n|}),$$

$$V_2 : \Psi(\zeta', x_n + iy_n) \longmapsto \Phi(z', x_n + iy_n) = \Psi(\sqrt{|x_n|}(x' + y') + i\frac{1}{2\sqrt{|x_n|}}(-x' + y'), x_n + iy_n).$$

Theorem 5.1. *The unitary operator $W = V_2V_1U_2U_1U_0$ maps $L^2(D_n, d\mu_\lambda)$ onto*

$$\mathcal{H} = L^2(\mathcal{D}, d\eta_\lambda) = L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}) \otimes L^2(\mathbb{R}_+, y_n^\lambda dy_n).$$

The poly-Bergman type space $\mathcal{A}^2_{\lambda L}$ is mapped by W to the subspace

$$\mathcal{H}_L^+ = L^2(\mathbb{R}^{n-1}) \otimes \left(\bigoplus_{0\le J'\le L'-e} H_{J'} \right) \otimes L^2(\mathbb{R}_+) \otimes \left(\bigoplus_{j_n=0}^{l_n-1} \mathcal{L}_{j_n} \right),$$

where $e = (1, ..., 1) \in \mathbb{Z}_+^{n-1}$, and

$$\mathcal{L}_{j_n} = gen\{\ell^\lambda_{j_n}(y_n)\} \subset L^2(\mathbb{R}_+, y_n^\lambda dy_n),$$

$$H_{J'} = gen\{h_{J'}(y')\} \subset L^2(\mathbb{R}^{n-1}, dy').$$

Corollary 5.2. *The restriction of W to the space $\mathcal{A}^2_{\lambda(L)}$ given by*

$$W : \mathcal{A}^2_{\lambda(L)} \longrightarrow \mathcal{H}^+_{(L)} = L^2(\mathbb{R}^{n-1}) \otimes H_{L'-e} \otimes L^2(\mathbb{R}_+) \otimes \mathcal{L}_{l_n-1}$$

is an isomorphisms.

Proof of Theorem 5.1. Let $\mathcal{A}_{1,\lambda L} = U_1(\mathcal{A}_{0,\lambda L}(\mathcal{D}))$. The operator U_1 is the Fourier transform with respect to the variable $u_n = \operatorname{Re} w_n$. Then $\phi(w', \xi_n + iv_n) = (U_1\varphi)(w', \xi_n + iv_n)$ belongs to $\mathcal{A}_{1,\lambda L}$ if and only if

$$\begin{aligned} \left(\frac{\partial}{\partial \overline{w}_k} + \xi_n w_k \right)^{l_k} \phi &= 0, \quad k = 1, ..., n-1, \\ \frac{i^{l_n}}{2^{l_n}} \left(\xi_n + \frac{\partial}{\partial v_n} \right)^{l_n} \phi &= 0. \end{aligned}$$

We now take the Fourier transform with respect to the variables $u_k = \operatorname{Re} w_k$. Define $\mathcal{A}_{2,\lambda L} = U_2(\mathcal{A}_{1,\lambda L})$. Then $\psi(\xi' + iv', \xi_n + iv_n) = (U_2\phi)(\xi' + iv', \xi_n + iv_n)$ belongs to $\mathcal{A}_{2,\lambda L}$ if and only if

$$\begin{aligned} \left[\tfrac{i}{2} \left(\xi_k + \tfrac{\partial}{\partial v_k} \right) + i\xi_n \left(\tfrac{\partial}{\partial \xi_k} + v_k \right) \right]^{l_k} \psi &= 0, \quad k = 1, ..., n-1, \\ \tfrac{i^{l_n}}{2^{l_n}} \left(\xi_n + \tfrac{\partial}{\partial v_n} \right)^{l_n} \psi &= 0. \end{aligned} \tag{5.3}$$

Let $\mathcal{A}'_{1,\lambda L}$ denote the image space $V_1(\mathcal{A}_{2,\lambda L})$. Then $\Psi(\zeta', x_n + iy_n) = (V_1\psi)(\zeta', x_n + iy_n)$ belongs to $\mathcal{A}'_{1,\lambda L}$ if and only if

$$\begin{aligned} \left[\tfrac{i}{2} \left(\xi_k + \tfrac{\partial}{\partial v_k} \right) + ix_n \left(\tfrac{\partial}{\partial \xi_k} + v_k \right) \right]^{l_k} \Psi &= 0, \quad k = 1, ..., n-1, \\ \tfrac{i^{l_n}|x_n|^{l_n}}{2^{l_n}} \left(sign(x_n) + 2\tfrac{\partial}{\partial y_n} \right)^{l_n} \Psi &= 0. \end{aligned} \tag{5.4}$$

Take $\mathcal{A}'_{2,\lambda L} = V_2(\mathcal{A}'_{1,\lambda L})$. Then $\Phi(z', x_n + iy_n) = (V_2\Psi)(z', x_n + iy_n)$ belongs to $\mathcal{A}'_{2,\lambda L}$ if and only if

$$\begin{aligned} \left[i\sqrt{|x_n|} \left(\tfrac{1-sign(x_n)}{2} (x_k - \tfrac{\partial}{\partial x_k}) + \tfrac{1+sign(x_n)}{2} (y_k + \tfrac{\partial}{\partial y_k}) \right) \right]^{l_k} \Phi &= 0, \\ \tfrac{i^{l_n}|x_n|^{l_n}}{2^{l_n}} \left(sign(x_n) + 2\tfrac{\partial}{\partial y_n} \right)^{l_n} \Phi &= 0. \end{aligned} \tag{5.5}$$

The general solution of the last equation in (5.5) is given by

$$\Phi(z', x_n + iy_n) = \sum_{j_n=0}^{l_n-1} \phi_{j_n}(z', x_n) y_n^{j_n} e^{-(sgn\ x_n)y_n/2}.$$

Since $\Phi(z', x_n + iy_n)$ has to be in $L^2(\mathcal{D}, d\eta_\lambda)$, we must take only positive values of x_n. Morever, by rearranging polynomial terms we can express $\Phi(z', x_n + iy_n)$ as

$$\Phi(z', x_n + iy_n) = \chi_+(x_n) \sum_{j_n=0}^{l_n-1} \Phi_{j_n}(z', x_n) \ell_{j_n}^{\lambda}(y_n). \tag{5.6}$$

where $\ell_{j_n}^{\lambda}(y)$ is the Laguerre function in $L^2(\mathbb{R}_+)$ of degree j_n. Further, the function $\chi_+(x_n)\Phi_{j_n}(z', x_n)\ell_{j_n}^{\lambda}(y_n)$ belongs to $\mathcal{A}_{3,\lambda L}$ if and only if

$$\left[i\sqrt{|x_n|} \left(\frac{\partial}{\partial y_k} + y_k \right) \right]^{l_k} \Phi_{j_n}(z', x_n) = 0, \quad x_n > 0$$

for each $k = 1, ..., n-1$. Then, the general solution of this system of equations has the form

$$\Phi_{j_n}(z', x_n) = \sum_{0 \le J' \le L' - e} \tilde{\Phi}_{J', j_n}(x', x_n)(y')^{J'} e^{-|y'|^2/2}, \quad x_n > 0.$$

We rewrite the general solution as

$$\Phi_{j_n}(z', x_n) = \sum_{0 \le J' \le L' - e} \Phi_J(x', x_n) h_{J'}(y'), \quad x_n > 0, \tag{5.7}$$

where $J = (J', j_n)$, and $h_{J'}(y')$ is the Hermite function given in (4.1). Therefore

$$\Phi(z', x_n + iy_n) = \sum_{j_n=1}^{l_n-1} \left\{ \sum_{0 \le J' \le L' - e} \chi_+(x_n) \Phi_J(x', x_n) h_{J'}(y') \right\} \ell_{j_n}^{\lambda}(y_n).$$

This completes the proof.

6 Anti-Poly-Bergman Type Spaces

Anti-polyanalytic functions are just the complex conjugation of polyanalytic functions, but the spaces of polyanalytic and anti-polyanalytic functions are mutually orthogonal. For $L = (l_1, ..., l_n) \in \mathbb{N}^n$, we define the anti-poly-Bergman type space $\tilde{\mathcal{A}}_{\lambda L}^2$ as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all functions f satisfying the equations

$$\begin{aligned} \left(\frac{\partial}{\partial z_k} + 2i\overline{z_k} \frac{\partial}{\partial z_n} \right)^{l_k} f &= 0, \quad k = 1, ..., n-1, \\ \left(\frac{\partial}{\partial z_n} \right)^{l_n} f &= 0. \end{aligned}$$

We define the space of true-L-anti-analytic functions as

$$\tilde{\mathcal{A}}^2_{\lambda(L)} = \tilde{\mathcal{A}}^2_{\lambda L} \ominus \left(\sum_{j=1}^{n} \tilde{\mathcal{A}}^2_{\lambda, L-e_j} \right),$$

where $\tilde{\mathcal{A}}^2_{\lambda S} = \{0\}$ if $S \notin \mathbb{N}^n$.

Theorem 6.1. *The Hilbert space $L^2(D_n, d\mu_\lambda)$ admits the decomposition*

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L \in \mathbb{N}^n} \mathcal{A}^2_{\lambda(L)} \right) \bigoplus \left(\bigoplus_{L \in \mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)} \right).$$

Proof. We have

$$\begin{aligned}
\bigoplus_{L \in \mathbb{N}^n} \mathcal{H}^+_{(L)} &= L^2(\mathbb{R}^{n-1}) \otimes \left(\bigoplus_{L' \in \mathbb{N}^{n-1}} H_{L'-e} \right) \otimes L^2(\mathbb{R}_+) \otimes \left(\bigoplus_{l_n \in \mathbb{N}} \mathcal{L}_{l_n - 1} \right) \\
&= L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}_+) \otimes L^2(\mathbb{R}_+).
\end{aligned}$$

Similarly, we have

$$\bigoplus_{L \in \mathbb{N}^n} \mathcal{H}^-_{(L)} = L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}_-) \otimes L^2(\mathbb{R}_+).$$

It is obvious that the direct sum of all the spaces $\mathcal{H}^+_{(L)}$ and $\mathcal{H}^-_{(L)}$ is equal to $L^2(\mathcal{D}, d\eta_\lambda)$. Using the fact that W is unitary and corollaries 5.2 and 6.3 we obtain

$$\begin{aligned}
L^2(D_n, d\mu_\lambda) &= W^* \left(L^2(\mathcal{D}, d\eta_\lambda) \right) \\
&= W^* \left(\left(\bigoplus_{L \in \mathbb{N}^n} \mathcal{H}^+_{(L)} \right) \bigoplus \left(\bigoplus_{L \in \mathbb{N}^n} \mathcal{H}^-_{(L)} \right) \right) \\
&= \left(\bigoplus_{L \in \mathbb{N}^n} W^* \left(\mathcal{H}^+_{(L)} \right) \right) \bigoplus \left(\bigoplus_{L \in \mathbb{N}^n} W^* \left(\mathcal{H}^-_{(L)} \right) \right) \\
&= \left(\bigoplus_{L \in \mathbb{N}^n} \mathcal{A}^2_{\lambda(L)} \right) \bigoplus \left(\bigoplus_{L \in \mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)} \right).
\end{aligned}$$

□

Theorem 6.2. *Under the unitary operator W, the anti-poly-Bergman type space $\tilde{\mathcal{A}}^2_{\lambda L}$ is isomorphic to the subspace*

$$\mathcal{H}^-_L = \left(\bigoplus_{0 \le J' \le L'-e} H_{J'} \right) \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}_-) \otimes \left(\bigoplus_{j_n=0}^{l_n-1} \mathcal{L}_{j_n} \right),$$

where

$$H_{J'} = gen\{h_{J'}(x')\} \subset L^2(\mathbb{R}^{n-1}, dx').$$

Corollary 6.3. *The restriction of W to the space* $\tilde{\mathcal{A}}^2_{\lambda(L)}$

$$W : \tilde{\mathcal{A}}^2_{\lambda(L)} \longrightarrow \mathcal{H}^-_{(L)} = H_{L'-e} \otimes L^2(\mathbb{R}^{n-1}) \otimes L^2(\mathbb{R}_-) \otimes \mathcal{L}_{l_n-1}.$$

is an isomorphisms.

Proof of Theorem 6.2. This proof is similar to that of Theorem 5.1. Let $\tilde{\mathcal{A}}_{0,\lambda L}(\mathcal{D}) = U_0(\tilde{\mathcal{A}}^2_{\lambda L}) \subset L^2(\mathcal{D}, d\eta_\lambda)$. For $\varphi(w) = (U_0 f)(w) \in \tilde{\mathcal{A}}_{0,\lambda L}(\mathcal{D})$ we have

$$\begin{aligned} U_0 (\Lambda_k)^{l_k} U_0^{-1}\varphi &= \left(\frac{\partial}{\partial w_k} + i\overline{w_k}\frac{\partial}{\partial u_n}\right)^{l_k}\varphi = 0, \quad k = 1,...,n-1, \\ U_0\left(\frac{\partial}{\partial z_n}\right)^{l_n} U_0^{-1}\varphi &= \frac{1}{2^{l_n}}\left(\frac{\partial}{\partial u_n} - i\frac{\partial}{\partial v_n}\right)^{l_n}\varphi = 0. \end{aligned}$$

We take now the Fourier transform with respect to all the variables $u_k = \operatorname{Re} w_k$. Define $\tilde{\mathcal{A}}_{2,\lambda L} = U_2 U_1(\tilde{\mathcal{A}}_{0,\lambda L})$. Then $\psi(\xi' + iv', \xi_n + iv_n) = (U_2 U_1 \varphi)(\xi' + iv', \xi_n + iv_n)$ belongs to $\tilde{\mathcal{A}}_{2,\lambda L}$ if and only if

$$\begin{aligned} \left[\frac{i}{2}\left(\xi_k - \frac{\partial}{\partial v_k}\right) - i\xi_n\left(\frac{\partial}{\partial \xi_k} - v_k\right)\right]^{l_k}\psi &= 0, \quad k = 1,...,n-1, \\ \frac{i^{l_n}}{2^{l_n}}\left(\xi_n - \frac{\partial}{\partial v_n}\right)^{l_n}\psi &= 0. \end{aligned}$$

Take $\tilde{\mathcal{A}}_{3,\lambda L} = V_2 V_1(\tilde{\mathcal{A}}_{2,\lambda L})$. Then $\Phi(z', x_n + iy_n) = (V_2 V_1 \psi)(z', x_n + iy_n)$ belongs to $\tilde{\mathcal{A}}_{3,\lambda L}$ if and only if

$$\begin{aligned} \left[i\sqrt{|x_n|}\left(\frac{1-sign(x_n)}{2}(x_k + \frac{\partial}{\partial x_k}) + \frac{1+sign(x_n)}{2}(y_k - \frac{\partial}{\partial y_k})\right)\right]^{l_k}\Phi &= 0, \\ \frac{i^{l_n}|x_n|^{l_n}}{2^{l_n}}\left(sign(x_n) - 2\frac{\partial}{\partial y_n}\right)^{l_n}\Phi &= 0. \end{aligned} \tag{6.1}$$

The general solution of the last equation in (6.1) is given by

$$\Phi(z', x_n + iy_n) = \sum_{j_n=0}^{l_n-1} \phi_{j_n}(z', x_n) y_n^{j_n} e^{(sgn\ x_n)y_n/2}.$$

Since $\Phi(z', x_n + iy_n)$ has to be in $L^2(\mathcal{D}, d\eta_\lambda)$, we must take only negative values of x_n. Moreover, by rearranging polynomial terms we can express $\Phi(z', x_n + iy_n)$ as

$$\Phi(z', x_n + iy_n) = \chi_-(x_n) \sum_{j_n=0}^{l_n-1} \Phi_{j_n}(z', x_n)\ell^\lambda_{j_n}(y_n).$$

where $\ell^\lambda_{j_n}(y)$ is the Laguerre function in $L^2(\mathbb{R}_+)$ of degree j_n. Further, the function $\chi_-(x_n)\Phi_{j_n}(z', x_n)\ell^\lambda_{j_n}(y_n)$ belongs to $\tilde{\mathcal{A}}_{3,\lambda L}$ if and only if

$$\left[i\sqrt{|x_n|}\left(\frac{\partial}{\partial x_k} + x_k\right)\right]^{l_k}\Phi_{j_n}(z', x_n) = 0, \quad x_n < 0,$$

for each $k = 1,...,n-1$. Then, the general solution of this system of equations has the form

$$\Phi_{j_n}(z', x_n) = \sum_{0 \le J' \le L'-e} \tilde{\Phi}_{J',j_n}(y', x_n)(x')^{J'} e^{-|x'|^2/2}, \quad x_n < 0.$$

We rewrite the general solution as

$$\Phi_{j_n}(z', x_n) = \sum_{0\le J'\le L'-e} \Phi_J(y', x_n) h_{J'}(x'), \quad x_n < 0,$$

where $J = (J', j_n)$, and $h_{J'}(x')$ is the Hermite function given in (4.1). Therefore

$$\Phi(z', x_n + iy_n) = \sum_{j_n=1}^{l_n-1} \left\{ \sum_{0\le J'\le L'-e} \chi_-(x_n)\Phi_J(y', x_n) h_{J'}(x') \right\} \ell^{\lambda}_{j_n}(y_n).$$

This completes the proof.

7 Mixed Poly-Bergman Type Spaces

Let us introduce the following notation:

$$\Lambda^{\pm 1}_{k\pm 1} := \Lambda^{\pm}_{k\pm}, \quad D^{-1}_k := \partial/\partial z_k, \quad D^{+1}_k := \partial/\partial \overline{z}_k, \quad M^{-1}_k := \overline{z}_k I, \quad M^{+1}_k := z_k I.$$

Certain choices of the operators $\Lambda^{\pm}_{k\pm}$ will be taken to define mixed poly-Bergman type spaces. For each n-tuple $\sigma = (\sigma_1, ..., \sigma_n) \in \{\pm 1\}^n$, introduce the operators

$$\Lambda^{\sigma_n}_{k\sigma_k} = D^{\sigma_k}_k - 2i\sigma_n M^{\sigma_k}_k D^{\sigma_n}_n.$$

For $L = (l_1, ..., l_n) \in \mathbb{N}^n$ we define the (L,σ)-poly-Bergman type space $\mathcal{A}^2_{\lambda L\sigma}$ as the subspace of $L^2(D_n, d\mu_\lambda)$ consisting of all functions f satisfying the equations

$$\begin{aligned} (\Lambda^{\sigma_n}_{k\sigma_k})^{l_k} f &= 0, \quad k = 1, ..., n-1 \\ (D^{\sigma_n}_n)^{l_n} f &= 0. \end{aligned} \tag{7.1}$$

We will refer to $\mathcal{A}^2_{\lambda L\sigma}$ as the σ-poly-Bergman type space or the mixed poly-Bergman type space. We define the space of true-(L,σ)-analytic functions as

$$\mathcal{A}^2_{\lambda(L)\sigma} = \mathcal{A}^2_{\lambda L\sigma} \ominus \left(\sum_{j=1}^{n} \mathcal{A}^2_{\lambda, L-e_j, \sigma} \right),$$

where $\mathcal{A}^2_{\lambda S\sigma} = \{0\}$ if $S \notin \mathbb{N}^n$. Of course, for $\sigma = e = (1, ..., 1)$, $\mathcal{A}^2_{\lambda L\sigma}$ is just the poly-Bergman type space, and $\mathcal{A}^2_{\lambda L, -e}$ is the anti-poly-Bergman type space $\tilde{\mathcal{A}}_{\lambda L}$.

For $\sigma \in \{\pm 1\}^n$ consider the following bijective mappings on D_n and $\mathcal{D}$, respectively:

$$C_\sigma : (z_1, ..., z_{n-1}, z_n) \mapsto (x_1 + \sigma_1 iy_1, \ \dots, x_{n-1} + \sigma_{n-1} iy_{n-1}, \ \sigma_n x_n + iy_n),$$

$$\tilde{C}_\sigma : (w_1, ..., w_{n-1}, u_n + iv_n) \mapsto (u_1 + \sigma_1 iv_1, \ \dots, u_{n-1} + \sigma_{n-1} iv_{n-1}, \ \sigma_n u_n + iv_n),$$

i.e., we make complex conjugation in the variables $z_k = x_k + iy_k$ and $w_k = u_k + iv_k$ whenever $\sigma_k = -1$ for $k = 1, ..., n-1$.

Consider now the following unitary self-adjoint operators on $L^2(D_n, d\mu_\lambda)$ and $L^2(\mathcal{D}, d\eta_\lambda)$, respectively:

$$T_\sigma : f \mapsto f \circ C_\sigma,$$

$$\tilde{T}_\sigma : f \mapsto f \circ \tilde{C}_\sigma.$$

It is easy to see that $\tilde{T}_\sigma = U_0 T_\sigma U_0^*$. Mixed poly-Bergman type spaces can be realised as spaces of polyanalytic functions under T_σ.

Lemma 7.1. *The operator T_σ maps the σ-poly-Bergman type space onto the poly-Bergman type space:*

$$T_\sigma(\mathcal{A}_{\lambda L\sigma}) = \mathcal{A}_{\lambda L}. \tag{7.2}$$

Proof. Suppose that $\sigma_n = 1$. It is easy to see that $T_\sigma^* \overline{\Lambda_k} T_\sigma = \overline{\Lambda_k}$ if $\sigma_k = 1$, and $T_\sigma^* \overline{\Lambda_k} T_\sigma = \Lambda_{k-}^+$ if $\sigma_k = -1$. That is, $T_\sigma \Lambda_{k\sigma_k}^+ T_\sigma^* = \overline{\Lambda_k}$. Analogously, we have $T_\sigma \Lambda_{k\sigma_k}^- T_\sigma^* = \overline{\Lambda_k}$ for $\sigma_n = -1$. Therefore

$$T_\sigma \Lambda_{k\sigma_k}^{\sigma_n} T_\sigma^* = \overline{\Lambda_k}$$

no matters if $\sigma_n = 1$ or $\sigma_n = -1$. We have also $T_\sigma D_n^{\sigma_n} T_\sigma^* = D_n^{+1}$. Finally, a function $f \in L^2(D_n, d\mu_\lambda)$ satisfy equations (7.1) if and only if $T_\sigma f$ belongs to $\mathcal{A}^2_{\lambda L}$. □

The set $\{1, -1\}^n$ is a group under the multiplication $\sigma\tau := (\sigma_1\tau_1, ..., \sigma_n\tau_n)$, where $\sigma = (\sigma_1, ..., \sigma_n)$ and $\tau = (\tau_1, ..., \tau_n)$. Of course $e = (1, ..., 1)$ is the identity in this group.

Lemma 7.2. *The operator T_σ maps the $-\sigma$-poly-Bergman type space onto the anti-poly-Bergman type space:*

$$T_\sigma(\mathcal{A}_{\lambda L, -\sigma}) = \tilde{\mathcal{A}}_{\lambda L}.$$

Moreover

$$T_\sigma(\mathcal{A}_{\lambda L, \tau}) = \mathcal{A}_{\lambda L, \sigma\tau}.$$

Proof. The set of operators T_σ is a group and $T_\sigma T_\tau = T_{\sigma\tau}$. Thus

$$T_\sigma(\mathcal{A}_{\lambda L, \tau}) = T_\sigma T_\tau T_\tau(\mathcal{A}_{\lambda L, \tau}) = T_\sigma T_\tau(\mathcal{A}_{\lambda L}) = T_{\sigma\tau}(\mathcal{A}_{\lambda L}) = \mathcal{A}_{\lambda L, \sigma\tau}.$$

□

Since $\overline{\Lambda_{k\sigma_k}^{\sigma_n}} = \Lambda_{k,-\sigma_k}^{-\sigma_n}$, the mixed poly-Bergman type space $\mathcal{A}^2_{\lambda L, -\sigma}$ consists of all conjugation functions of $\mathcal{A}^2_{\lambda L, \sigma}$. We define

$$\tilde{\mathcal{A}}^2_{\lambda L\sigma} := \mathcal{A}^2_{\lambda L, -\sigma},$$

$$\tilde{\mathcal{A}}^2_{\lambda(L)\sigma} := \mathcal{A}^2_{\lambda(L), -\sigma}.$$

Theorem 7.3. *The Hilbert space $L^2(D_n, d\mu_\lambda)$ admits the decomposition*

$$L^2(D_n, d\mu_\lambda) = \left(\bigoplus_{L \in \mathbb{N}^n} \mathcal{A}^2_{\lambda(L)\sigma} \right) \bigoplus \left(\bigoplus_{L \in \mathbb{N}^n} \tilde{\mathcal{A}}^2_{\lambda(L)\sigma} \right).$$

Proof. Follows from lemmas 7.1, 7.2 and Theorem 6.1. □

Let us see how mixed poly-Bergman type spaces are mapped by the unitary operator W. Consider the unitary self-adjoint operator

$$S_\sigma = WT_\sigma W^*.$$

We have $S_\sigma = \sigma_n V_2 \tilde{T}_\sigma V_2^*$ because of $U_1\tilde{T}_\sigma = \sigma_n \tilde{T}_\sigma U_1$, $V_1\tilde{T}_\sigma = \tilde{T}_\sigma V_1$, and $U_2\tilde{T}_\sigma = \tilde{T}_\sigma U_2$. It is easy to see that

$$S_\sigma \Phi = \sigma_n \, \Phi \circ h,$$

where

$$h : \mathcal{D} \ni (z_1, \, \dots \, , z_{n-1}, x_n + iy_n) \mapsto (w_1, \, \dots \, , w_{n-1}, \sigma_n x_n + ix_n) \in \mathcal{D}$$

and

$$\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} \frac{1+\sigma_k}{2} & \frac{1-\sigma_k}{2} \\ \frac{1-\sigma_k}{2} & \frac{1+\sigma_k}{2} \end{pmatrix} \begin{pmatrix} x_k \\ y_k \end{pmatrix}, \qquad k = 1, \dots, n-1.$$

Obviously $w_k = z_k$ if $\sigma_k = 1$; otherwise this mapping interchange the real and imaginary parts of $z_k = x_k + iy_k$: $w_k = i\overline{z_k}$. On the other hand,

$$W(\mathcal{A}_{\lambda(L)\sigma}) = S_\sigma^* W T_\sigma (\mathcal{A}_{\lambda(L)\sigma}) = S_\sigma \mathcal{H}^+_{(L)}.$$

Theorem 7.4. *The true-(L,σ)-poly-Bergman type space $\mathcal{A}^2_{\lambda(L)\sigma}$ is isomorphic to the subspace*

$$\mathcal{H}^\sigma_{(L)} = \left(H^\sigma_{L-e} \otimes L^2_\sigma(\mathbb{R}^{n-1})\right) \otimes L^2(\mathbb{R}_{\sigma_n}) \otimes \mathcal{L}_{l_n-1},$$

where H^σ_{L-e} is the one-dimensional space generated by the Hermite function h_{L-e} in the variables $\operatorname{Im} h(z)'$, *and $L^2_\sigma(\mathbb{R}^{n-1})$ is the space of L^2-functions in the variable* $\operatorname{Re} h(z)'$.

Acknowledgments

The authors thanks the referees for their careful reading of the manuscript and insightful comments.

References

[1] M. B. Balk, Polyanalytic functions and their generalizations. *Itogi Nauki i Tekhniki. Ser. Sovrem. Probl. Mat. Fund. Napr.* **85** (1991), pp 187-246.

[2] M. B. Balk and M. F. Zuev, On polyanalytic functions. *Russ. Math. Surveys* **25**, no. 5 (1970), pp 201-223.

[3] A. Boggess, *CR Manifolds and the Tangencial Cauchy-Riemann Complex*. Studies in Advanced Mathematics, CRC Press, London, 1991.

[4] T. S. Chihara, *An Introduction to Orthogonal Polynomials*, Gordon and Breach Sciences Publishers, New York, London, 1978.

[5] A. Dzhuraev, Multikernel functions of a domain, kernel operators, singular integral operators. *Soviet Math. Dokl.* **32**, no. 1 (1985), pp 251-253.

[6] A. Dzhuraev, *Methods of Singular Integral Equations*, Longman Scientific & Technical. New York, 1992.

[7] N. N. Lebedev, *Special Functions and Their Applications*, Dover Publications, New York, 1972.

[8] R. Quiroga-Barranco and N. L. Vasilevski, Commutative C^*-algebras of Toeplitz operators on the unit ball, I. Bargmann-type transforms and spectral representations of Toeplitz operators. *Integr. Equ. Oper. Theory.* **59**, no. 3 (2007), pp 379-419.

[9] N. L. Vasilevski, Toeplitz operators on the Bergman spaces: Inside-the-domain effects. *Contemp. Math.* **289** (2001), pp. 79-146.

[10] N. L. Vasilevski, On the structure of Bergman and poly-Bergman spaces. *Integr. Equ. Oper. Theory.* **33**, no. 4 (1999), pp 471-488.