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Time-frequency Integrals and the Stationary Phase Method in Problems of Acoustic Waves Propagation from Moving Sources

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Abstract

The paper is devoted to the applications of the time-frequency integrals and the two-dimensional stationary phase method for the problems of waves propagation from moving sources in acoustic dispersive media. Applying the stationary phase method we obtain the effective formula for the acoustic fields in the dispersive media generated by non-uniformly moving modulated source.

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1 Introduction

The paper is devoted to the applications of the time-frequency integrals and the two-dimensional stationary phase method for the problems of waves propagation from moving sources in

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acoustic dispersive media.

We consider the acoustic fields generated by moving in dispersive media modulated sources of the form

$$F(t,\omega) = a(t)e^{-i\omega_0 t}\delta(\mathbf{x} - \mathbf{x}_0(t)), t \in \mathbb{R}, \mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3,$$

where ω_0 is an eigenfrequency of the source, $a(t)$ is a slowly varying amplitude, $\mathbf{x}_0(t) = (x_{01}(t), x_{02}(t), x_{03}(t))$ is a vector-function defining the motion of the source.

Some assumptions with respect to the source allow us to introduce a large dimensionless parameter $\lambda > 0$ which characterizes simultaneously the slowness of the variations of: the source amplitude, the velocity of the source, and a large distance between sources and receivers. We obtain a representation of the fields as double oscillating integrals depending on the parameter $\lambda > 0$

$$\Phi_\lambda(t,\mathbf{x}) = \int_{\mathbb{R}\times\mathbb{R}} F(t,\mathbf{x},\omega,\tau,\lambda)e^{i\lambda S(t,\mathbf{x},\omega,\tau)}d\omega d\tau, \tag{1.1}$$

where F is the complex valued function and S is the real-valued function for $|\omega|$ large enough. Generally speaking integral (1.1) is divergent and we consider its regularization which is called the oscillatory integral. The phase S in (1.1) is of the form

$$S(t,\mathbf{x},\omega,\tau) = \varkappa(\omega)|\mathbf{x} - \mathbf{x}_0(\tau)| - \omega(t-\tau) - \omega_0\tau$$

where $\varkappa(\omega)$ is the wave number in a dispersive medium depending on the frequency.

Applying to the integrals (1.1) the method of the stationary phase we obtain the asymptotics of the field for large $\lambda > 0$.

This approach is applied for estimates of acoustic fields generated by non uniformly moving sources in dispersive fluids and in acoustic waveguides filled by dispersive fluids.

We would like to note that the asymptotic estimates of *one-dimensional* integrals are a standard tool of the electrodynamics (see for instance [10], Chap.3,4, [11]) and go back to A. Sommerfeld [32], and L. Brillouin [8] Ch.1. But in the case of non uniformly moving sources the representation of the fields in the form of a one-dimensional integral is not effective. In turn, the representation of the field as a double time-frequency oscillating integrals with a subsequent asymptotic analysis yields effective formulas for both the fields and for the Doppler shifts.

The acoustic and electromagnetic radiation from moving sources is a classical problem of the electrodynamics, and for the isotropic *non dispersive media* the solution of this problem is given by the *Liènard-Wiechert potential* (see for instance [21], Chap. VIII, [16], Chap. 14). But the *Liènard-Wiechert potential* is not applicable for dispersive media and our representation is new and effective tool for the investigation of electromagnetic fields generated by moving sources with variable velocity.

The paper is organized as follows. In Chapter 2 we give an auxiliary material concerning the oscillatory integrals and multidimensional stationary phase method. In Chapter 3 we consider the acoustic wave propagation from moving source in dispersive fluids and layered acoustic waveguides filled with dispersive fluids. We obtain the effective asymptotic formulas for the acoustic fields, Doppler effects, and retarded time. Note some works devoted to the acoustic wave propagation from moving source [1], [3], [4], [18], [19], [20],

[24], [26], [27], [28], [30]. The representation of the acoustic fields as a sum of integrals of type (1.1) with its asymptotic estimates first was used in [19]. This method was developed in [26], [27], [28]. In this paper we show that the mentioned approach works in the case of moving sources in dispersive homogeneous fluids and stratified waveguides filled with dispersive fluids.

2 Auxiliary Material: Stationary Phase Method for the Oscillatory Integrals

1^0. We consider the integrals of the form

$$\int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x}) e^{iS(\mathbf{x})} d\mathbf{x}, \tag{2.1}$$

where $\mathbb{R}^n \ni \mathbf{x} \to \mathbf{f}(\mathbf{x}) \in \mathbb{C}^m$ is called the amplitude and the scalar function S is called the phase. We suppose that $\mathbf{f}$ and S are infinitely differentiable (in fact it is necessary a finite number of the derivatives) and satisfy the following conditions. The amplitude $\mathbf{f}$ satisfies following conditions: for every multiindex α there exists $C_\alpha > 0$ such that

$$\left|\partial^\alpha \mathbf{f}(\mathbf{x})\right| \leq C_\alpha \langle \mathbf{x} \rangle^k, \langle \mathbf{x} \rangle = (1 + |\mathbf{x}|^2)^{\frac{1}{2}}, \tag{2.2}$$

for some $k \in \mathbb{R}$ independent of α. The phase S is such that:

(i) $S(x)$ is real for $|x|$ is large enough,

(ii) for every $|\alpha| \geq 2$ there exists $C_\alpha > 0$ such that $|\partial^\alpha S(\mathbf{x})| \leq C_\alpha$,

(iii) there exists $C > 0$ and $\rho > 0$ such that

$$|\nabla S(x)| \geq C |\mathbf{x}|^\rho$$

for $|\mathbf{x}|$ large enough.

Note that if $k \geq -n$ the integral (2.1) does not exist as absolutely convergent and we need a regularization of integral (2.1). Let $\chi \in C_0^\infty(\mathbb{R}^n)$ and $\chi(\mathbf{x}) = 1$ in a small neighborhood of the origin. We set $\chi_R(\mathbf{x}) = \chi(\mathbf{x}/R)$.

Proposition 2.1. *Let estimate* (2.2) *and conditions (i)-(iii) hold. Then there exists a limit*

$$\mathbf{F} = \lim_{R\to\infty} \int_{\mathbb{R}^n} \chi_R(\mathbf{x}) \mathbf{f}(\mathbf{x}) e^{iS(\mathbf{x})} d\mathbf{x} \tag{2.3}$$

independent of the choice of the function χ.

Proof. We introduce the differential operator L

$$Lu(\mathbf{x}) = \left(1 + |\nabla S(\mathbf{x})|^2\right)^{-1} (I - i\nabla S(\mathbf{x}) \cdot \nabla) u(\mathbf{x}), \mathbf{x} \in \mathbb{R}^n. \tag{2.4}$$

One can see that

$$Le^{iS(\mathbf{x},\mathbf{y})} = e^{iS(\mathbf{x},\mathbf{y})}. \tag{2.5}$$

Let L^τ be the transpose to L differential operator. Then taking into the account (2.5) and by integration by parts we obtain

$$\mathbf{F}_R = \int_{\mathbb{R}^n} \chi_R(\mathbf{x})\mathbf{f}(\mathbf{x})e^{iS(\mathbf{x})}d\mathbf{x} = \int_{\mathbb{R}^n} (L^\tau)^j (\chi_R(\mathbf{x})\mathbf{f}(\mathbf{x}))\, e^{iS(\mathbf{x})}d\mathbf{x}. \tag{2.6}$$

Conditions (i)-(iii) yield that

$$\left|(L^\tau)^j (\chi_R(\mathbf{x})\mathbf{f}(\mathbf{x}))\right| \le C_j \langle \mathbf{x}\rangle^{k-\rho j} \tag{2.7}$$

with the constant $C_j > 0$ independent of $R > 0$. Let $j > \frac{k+n}{\rho}$. Then the integral in the right side part of (2.6) is absolutely convergent, uniformly with respect to $R > 0$, and we can go to the limit for $R \to \infty$ in (2.6). Hence the limit in (2.3) exists, independent of χ, and

$$\mathbf{F}= \lim_{R\to\infty} \mathbf{F}_R = \int_{\mathbb{R}^n} \left(\left(L^\tau\right)^j \mathbf{f}(\mathbf{x})\right) e^{iS(\mathbf{x})}d\mathbf{x}. \tag{2.8}$$

where $j > \frac{k+n}{\rho}$. □

The integrals defined by formula (2.8) are called *oscillatory.*

2^0. We consider an integral depending on the parameter $\lambda > 0$ of the form

$$\mathbf{I}_\lambda = \int_{\mathbb{R}^n} \mathbf{f}(\mathbf{x})e^{i\lambda S(\mathbf{x})}d\mathbf{x},$$

where $\mathbf{f}, S$ satisfy condition (2.2), (i)-(III), and S is a real-valued function.

We say that $\mathbf{x}_0$ is a non-degenerate stationary point of the phase S if

$$\nabla S(\mathbf{x}_0) = 0,$$

and

$$\det S''(\mathbf{x}_0) \neq 0,$$

where $S''(\mathbf{x}) = \left(\frac{\partial^2 S(\mathbf{x})}{\partial x_i \partial x_j}\right)_{i,j=1}^n$ is the Hess matrix of the phase S.

Proposition 2.2. *(see for instance [12], [7]) Let there exist a finite set $\{\mathbf{x}_1, ..., \mathbf{x}_N\}$ of non-degenerate stationary points of the phase S. Then*

$$\mathbf{I}_\lambda = \sum_{j=1}^N \mathbf{F}_j(\lambda), \tag{2.9}$$

where

$$\mathbf{F}_j(\lambda) = \left(\frac{2\pi}{\lambda}\right)^{\frac{n}{2}} \frac{\exp(i\lambda S(\mathbf{x}_j) + \frac{i\pi}{4} sgn S''(\mathbf{x}_j))}{\left|\det S''(\mathbf{x}_j)\right|^{1/2}} \mathbf{f}(\mathbf{x}_j)(1 + O(\frac{1}{\lambda})) \tag{2.10}$$

and $sgn S''(\mathbf{x}_j)$ is the difference between the number of positive and negative eigenvalue of the matrix $S''(\mathbf{x}_j)$.

3 Acoustic Field of Moving Sources in Dispersive Acoustic Fluids

3.1 Acoustic equation in a dispersive media

Let $\mathbf{x} = (x_1, x_2, x_3)$ be the spatial coordinates, t is the time coordinate, $c = c(\mathbf{x}), \mathbf{x} \in \mathbb{R}^3$ be the sound speed in a fluid, $\rho(\mathbf{x})$ is the density of the fluid, $u(\mathbf{x},t)$ is the acoustic pressure. The pressure u satisfies the acoustic equation

$$\frac{1}{c^2(\mathbf{x})}\frac{\partial^2 u(t,\mathbf{x})}{\partial t^2} - \rho(\mathbf{x})\nabla\cdot\rho^{-1}(\mathbf{x})\nabla u(t,\mathbf{x}) = f(t,\mathbf{x}), \qquad (t,\mathbf{x}) \in \mathbb{R}^4 \tag{3.1}$$

where f is a source of the acoustic vibrations.

In the case of the dispersive fluid $c = c(\omega, \mathbf{x})$ we have to change the equation (3.1) by the pseudodifferential equation

$$c^{-2}(D_t, \mathbf{x})\frac{\partial^2 u(t,\mathbf{x})}{\partial t^2} - \rho(\mathbf{x})\nabla\cdot\rho^{-1}(\mathbf{x})\nabla u(t,\mathbf{x}) = f(t,\mathbf{x}),(t,\mathbf{x}) \in \mathbb{R}^4, \tag{3.2}$$

where

$$c^{-2}(D_t, \mathbf{x})\varphi(t,\mathbf{x}) = \frac{1}{2\pi}\int_{\mathbb{R}} c^{-2}(\omega,\mathbf{x})\hat{\varphi}(\omega,\mathbf{x})e^{-i\omega t}d\omega,$$

and

$$\hat{\varphi}(\omega,\mathbf{x}) = \int_{\mathbb{R}} \varphi(t,\mathbf{x})e^{i\omega t}dt$$

is the Fourier transform understood in the sense of distributions. The principle of causality demands that the function $c^{-2}(\omega, \mathbf{x})$ is a boundary value with respect to the ω of an analytic bounded in the upper complex half-plane function (see for instance [29], [31]).

3.2 Representation of fields of moving sources of the form the time-frequency integrals

We consider now the case the homogeneous dispersive media, that is the sound velocity is $c(\omega)$ and the density $\rho > 0$ are independent of $\mathbf{x}$. Then equation (3.2) after Fourier transform with respect to the time t accepts the form of the Helmholtz equation

$$\left(\Delta + k^2(\omega)\right)\hat{u}(\omega,\mathbf{x}) = -\hat{f}(\omega,\mathbf{x}), \qquad k(\omega) = \frac{\omega}{c(\omega)}, \omega \in \mathbb{R}, \mathbf{x} \in \mathbb{R}^3, \tag{3.3}$$

where $\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ is the Laplace operator.

The typical example of the dispersive fluid is the bubbly water (see for instance [25], Chapter 8) for which

$$k(\omega) = \frac{\sqrt{\omega^2 + \omega_b^2}}{c_0}$$

where c_0 is a constant phase speed in nonperturbed water, ω_b is a bubble frequency, depending on the concentrations of the bubbles in the unite of the volume, radius of the bubbles, etc. (see [25], p. 317-320). We also note the papers [29], [31] containing numerous examples of dispersive acoustic media.

In what follows we consider the lossless fluid, that is we suppose that $k(\omega) > 0$ for all $\omega \in \mathbb{R}$. To find the unique solution of the equation (3.3) we apply the limiting absorption principle. Let

$$g(\omega, \mathbf{x}) = -\frac{e^{ik(\omega)|\mathbf{x}|}}{4\pi |\mathbf{x}|}$$

be the Green function of the Helmholtz operator (3.3) satisfying the limiting absorption principle. Then the solution of the equation (3.2) for $c(\omega, \mathbf{x}) = c(\omega), \rho(\mathbf{x}) = \rho$ is given as

$$u(t, \mathbf{x}) = \frac{1}{8\pi^2} \int_{\mathbb{R}^2} e^{-i\omega(t-\tau)} d\omega d\tau \int_{\mathbb{R}^3} \frac{e^{ik(\omega)|\mathbf{x}-\mathbf{y}|}}{|\mathbf{x}-\mathbf{y}|} f(\tau, \mathbf{y}) d\mathbf{y}. \tag{3.4}$$

For a moving source given as

$$f(t, \mathbf{x}) = A(t)\delta(\mathbf{x} - \mathbf{x}_0(t)), \tag{3.5}$$

formula (3.4) accepts the form

$$u(t, \mathbf{x}) = \frac{1}{8\pi^2} \int_{\mathbb{R}\times\mathbb{R}} \frac{A(\tau) e^{iS(t,\mathbf{x},\omega,\tau)}}{|\mathbf{x} - \mathbf{x}_0(\tau)|} d\omega d\tau, \tag{3.6}$$

where

$$S(t, \mathbf{x}, \omega, \tau) = k(\omega)|\mathbf{x} - \mathbf{x}_0(\tau)| - \omega(t - \tau). \tag{3.7}$$

We denote by

$$v_g(\omega) = \frac{1}{k'(\omega)}$$

the group velocity in the dispersive fluid. In what follows we suppose that $v_g(\omega) > 0$ for all $\omega \in \mathbb{R}$.

Let there exist $R > 0$ large enough such that

$$\inf_{|\tau|+|\omega|>R} \left| \frac{|\mathbf{v}_0(\tau)|}{v_g(\omega)} - 1 \right| > 0,$$
$$\sup_{|\tau|+|\omega|>R} \left| \frac{|\mathbf{v}_0(\tau)|}{c(\omega)} - 1 \right| > 0.$$

where $\mathbf{v}_0(\tau) = \dot{\mathbf{x}}_0(\tau)$ is the velocity of the source. Then there exists $C > 0$ such that

$$\left| \nabla_{(\omega,\tau)} S(t, \mathbf{x}, \omega, \tau) \right| \geq C(|\omega| + |\tau|) \tag{3.8}$$

for $|\tau| + |\omega| > R$. Condition (3.8) provides the existence of the double integral in (3.6) as oscillatory.

In what follows we suppose the moving source is modulated, that is

$$A(t) = a(t)e^{-i\omega_0 t}, a(t) = \tilde{a}(t/\lambda) \tag{3.9}$$

with the eigenfrequency (the carrier frequency) $\omega_0 > 0$ and slowly varying amplitude $a(t)$ where $\tilde{a}(t)$ is a smooth function bounded with all derivatives, $\lambda > 0$ is a large dimensionless parameter characterizing the slowness of the variations of the amplitude a. The law of the motion of the source is

$$\mathbf{x}_0(t) = \lambda \mathbf{X}_0(t/\lambda), \tag{3.10}$$

where $\lambda > 0$ is the same large parameter, $\mathbf{X}_0$ is a smooth vector-function with all bounded derivatives. Formula (3.10) implies that the velocity of the source is

$$\mathbf{v}_0(t) = \dot{\mathbf{X}}_0(t/\lambda), \tag{3.11}$$

and the acceleration is

$$\mathbf{a}_0(t) = \frac{1}{\lambda}\ddot{\mathbf{X}}_0(t/\lambda). \tag{3.12}$$

That is the source moves with an *arbitrary* bounded velocity but with a *small acceleration.*

We make in the integral (3.6) the scale change of the variables:

$$\mathbf{x}{=}\lambda\mathbf{X}, t = \lambda T, \tau = \lambda\iota,$$
$$\mathbf{X}{\in}\mathbb{R}^3, T, \iota \in \mathbb{R}, \lambda > 0.$$

Then we obtain

$$\tilde{u}_\lambda(T,\mathbf{X}) = u(\lambda T, \lambda\mathbf{X}) = \frac{1}{8\pi^2}\int_{\mathbb{R}\times\mathbb{R}} \frac{\tilde{a}(\iota)e^{i\lambda\tilde{S}(T,\mathbf{X},\omega,\iota)}}{|\mathbf{X}-\mathbf{X}_0(\tau)|} d\omega d\iota \tag{3.13}$$

where

$$\tilde{S}(T,\mathbf{X},\omega,\iota) = k(\omega)\,|\mathbf{X}-\mathbf{X}_0(\tau)| - \omega(T-\iota) - \omega_0\iota. \tag{3.14}$$

3.3 Asymptotic analysis of the acoustic field

We apply the stationary phase method for the asymptotic analysis of $\tilde{u}_\lambda(T,\mathbf{X})$ for $\lambda \to +\infty$ and fixed $(T,\mathbf{X}), T > 0$. The stationary points of the phase (3.14) are solutions of the system of the equations with respect to (ω,ι)

$$\tilde{S}'_\omega(T,\mathbf{X},\omega,\iota) = \frac{|\mathbf{X}-\mathbf{X}_0(\iota)|}{v_g(\omega)} - (T-\iota) = 0, \tag{3.15}$$
$$\tilde{S}'_\iota(T,\mathbf{X},\omega,\iota) = -k(\omega)V_0(\mathbf{X},\iota) + (\omega-\omega_0) = 0,$$

where

$$V_0(\mathbf{X},\iota) = \frac{\mathbf{X}-\mathbf{X}_0(\iota)}{|\mathbf{X}-\mathbf{X}_0(\iota)|}\cdot\mathbf{V}_0(\iota),$$

is the value of the projection of $\mathbf{V}_0(\iota) = \dot{\mathbf{X}}_0(\iota)$ on the vector $\mathbf{X}-\mathbf{X}_0(\iota)$.

The Hess matrix of the phase $\tilde{S}$ is defined as

$$\tilde{S}''(T,\mathbf{X},\omega,\iota) = \begin{pmatrix} k''(\omega)\,|\mathbf{X}-\mathbf{X}_0(\iota)| & 1-\frac{V_0(\mathbf{X},\iota)}{v_g(\omega)} \\ 1-\frac{V_0(\mathbf{X},\iota)}{v_g(\omega)} & -k(\omega)\frac{\partial V_0(\mathbf{X},\tau)}{\partial\tau} \end{pmatrix}.$$

Let $\omega_s = \omega_s(T,\mathbf{X}), \iota_s = \iota_s(T,\mathbf{X})$ be a nondegenerate stationary point of the phase $\tilde{S}$. Then the contribution of (ω_s,ι_s) in the asymptotics of $\tilde{u}_\lambda(T,\mathbf{X})$ for fix $(T,\mathbf{X})$ is given by the formula

$$\begin{aligned}&\tilde{u}_\lambda(T,\mathbf{X}) \\ &= \frac{1}{4\pi\lambda}\frac{\exp i\left[\tilde{S}(T,\mathbf{X},\omega_s,\iota_s)+\frac{\pi}{4}sgn\tilde{S}''(T,\mathbf{X},\omega_s,\iota_s)\right]}{\left|\det\tilde{S}''(T,\mathbf{X},\omega_s,\iota_s)\right|^{1/2}|\mathbf{X}-\mathbf{X}_0(\iota_s)|}(1+O(\frac{1}{\lambda})).\end{aligned} \tag{3.16}$$

Coming back to the "old" variables $t,\mathbf{x}$, and τ we obtain

$$u(t,\mathbf{x}) \sim \frac{1}{4\pi}\frac{\exp i\left[S(t,\mathbf{x},\omega_s,\tau_s)+\frac{\pi}{4}sgnS''(t,\mathbf{x},\omega_s,\tau_s)\right]}{|\det S''(t,\mathbf{x},\omega_s,\tau_s)|^{1/2}|\mathbf{x}-\mathbf{x}_0(\tau_s)|}, \tag{3.17}$$

where

$$S(t,\mathbf{x},\omega,\tau) = k(\omega)|\mathbf{x}-\mathbf{x}_0(\tau)| - \omega(t-\tau) - \omega_0\tau,$$

and $(\omega_s,\tau_s) = (\omega_s(t,\mathbf{x}),\tau_s(t,\mathbf{x}))$ are a solution of the system

$$\begin{aligned}\frac{|\mathbf{x}-\mathbf{x}_0(\tau)|}{v_g(\omega)} - (t-\tau) &= 0, \\ -k(\omega)v_0(\tau,\mathbf{x}) + (\omega-\omega_0) &= 0,\end{aligned} \tag{3.18}$$

where

$$v_0(\tau,\mathbf{x}) = \mathbf{v}_0(\tau)\cdot\frac{\mathbf{x}-\mathbf{x}_0(\tau)}{|\mathbf{x}-\mathbf{x}_0(\tau)|}$$

is the value of the projection of the velocity vector $\mathbf{v}_0(\tau)$ on the vector $\mathbf{x}-\mathbf{x}_0(\tau)$,

$$S''(t,\mathbf{x},\omega,\tau) = \begin{pmatrix} k''(\omega)|\mathbf{x}-\mathbf{x}_0(\tau)| & 1-\frac{v_0(t,\mathbf{x})}{v_g(\omega)} \\ 1-\frac{v_0(\tau,\mathbf{x})}{v_g(\omega)} & -k(\omega)\frac{\partial v_0(\tau,\mathbf{x})}{\partial\tau}\end{pmatrix}. \tag{3.19}$$

The equivalence in formula (3.17) means that the right-hand side in (3.17) is the main term of the asymptotics of the acoustic field for

$$\inf_\tau |\mathbf{x}-\mathbf{x}_0(\tau)|\frac{\tilde{\omega}}{c_0} >> 1, \frac{t}{\tilde{t}} >> 1,$$

where $\tilde{t} > 0$ be a time scale, $\tilde{\omega} > 0$ be a frequency scale.

Note that if the following conditions

$$\begin{aligned}\sup_{(\omega,\tau)}\left(\frac{\tilde{\omega}}{\tilde{t}}\left|k''(\omega)\right||\mathbf{x}-\mathbf{x}_0(\tau)| + \left|\frac{v_0(\tau,\mathbf{x})}{v_g(\omega)}\right|\right) &< 1, \\ \sup_{(\omega,\tau)}\left(\frac{\tilde{t}}{\tilde{\omega}}k(\omega)\left|\frac{\partial v_0(\tau,\mathbf{x})}{\partial\tau}\right| + \left|\frac{v_0(\tau,\mathbf{x})}{v_g(\omega)}\right|\right) &< 1\end{aligned} \tag{3.20}$$

hold, then system (3.18) has the unique solution (ω_s,τ_s) which can be find by the method of successive approximations

$$\begin{aligned}&(\omega_s,\tau_s) = \lim_{n\to\infty}(\omega^n,\tau^n), \\ &(\omega^0,\tau^0) = (\omega_0,t), \\ &\left\{\begin{aligned}\tau^{n+1} &= t - \frac{|\mathbf{x}-\mathbf{x}_0(\tau^n)|}{v_g(\omega^n)}, \\ \omega^{n+1} &= \omega_0 - k(\omega^n)v_0(\tau^n,\mathbf{x}).\end{aligned}\right.\end{aligned}$$

Conditions (3.20) provide that the stationary point (ω_s, τ_s) is unique, non degenerate, and

$$sgn S''(t, \mathbf{x}, \omega_s, \tau_s) = 0.$$

Example 3.1. We apply our approach to the case of non dispersive fluids, that is $c(\omega) = c > 0$. Then the first equation in (3.18) independent of ω, and under the subsonic motion

$$\sup_t \frac{|\mathbf{v}(t)|}{c} < 1$$

(3.18) has the unique solution τ_s. We obtain from the second equation in (3.18)

$$\omega_s = \frac{\omega_0}{1 - \frac{v_0(\tau_s, \mathbf{x})}{c}}.$$

It is easy to check that

$$\det S''(t, \mathbf{x}, \omega_s, \tau_s) = \left(1 - \frac{v_0(\tau_s, \mathbf{x})}{c}\right)^2$$

and $sgn S''(t, \mathbf{x}, \omega_s, \tau_s) = 0$. Hence formula (3.17) implies that

$$u(t, \mathbf{x}) \sim \frac{1}{4\pi} \frac{\exp iS\,(t, \mathbf{x}, \omega_s, \tau_s)}{\left(1 - \frac{v_0(\tau_s, \mathbf{x})}{c}\right) |\mathbf{x} - \mathbf{x}_0(\tau_s)|}. \tag{3.21}$$

Note that the right-hand side in (3.21) coincides with the acoustic *Liènard-Wiechert* potential (see for instance [1]). Hence in the case of a homogeneous no-dispersive fluid asymptotic formula (3.21) is exact.

3.4 Doppler effects

Note that for fix point $\mathbf{x}$ formulas (3.17) can be written of the form

$$u(t, \mathbf{x}) \sim \Phi(t) e^{iF(t)} \tag{3.22}$$

where Φ is a bounded function, F is a real-valued function such that $\lim_{t\to\infty} F(t) = \infty$. Hence according to the signal processing theory (see for instance [9]) $F(t)$ is a phase of the wave process $\Phi(t)e^{iF(t)}$, and the instantaneous frequency $\omega_{in}(t)$ of the wave process $\Phi(t)e^{iF(t)}$ is defined as

$$\omega_{in}(t) = -F_j'(t).$$

In our case

$$\begin{aligned} F(t) &= S\,(t, \mathbf{x}, \omega_s(t, \mathbf{x}), \tau_s(t, \mathbf{x})) \\ &= k(\omega_s(t, \mathbf{x}))\,|\mathbf{x} - \mathbf{x}_0(\tau_s(t, \mathbf{x}))| - \omega_s(t, \mathbf{x})(t - \tau_s(t, \mathbf{x})) - \omega_0 \tau_s(t, \mathbf{x}), \end{aligned} \tag{3.23}$$

where $(\omega_s(t, \mathbf{x}), \tau_s(t, \mathbf{x}))$ is the stationary point of the phase S.

By the differentiation of F as a composed function we obtain

$$\begin{aligned} -F'(t) = &-\frac{\partial S\,(t, \mathbf{x}, \omega_s(t, \mathbf{x}), \tau_s(t, \mathbf{x}))}{\partial t} - \frac{\partial S\,(t, \mathbf{x}, \omega_s(t, \mathbf{x}), \tau_s(t, \mathbf{x}))}{\partial \omega} \frac{\partial \omega_s(t, \mathbf{x})}{\partial t} \\ &- \frac{\partial S\,(t, \mathbf{x}, \omega_s(t, \mathbf{x}), \tau_s(t, \mathbf{x}))}{\partial \tau} \frac{\partial \tau_s(t, \mathbf{x})}{\partial t}. \end{aligned}$$

Taking into account that $(\omega_s(t,\mathbf{x}),\tau_s(t,\mathbf{x}))$ is a stationary point of S, we obtain that

$$\omega_{in}(t) = \omega_s(t,\mathbf{x}).$$

It implies that the instantaneous frequency $\omega_{in}(t)$ of the wave processes $u(t,\mathbf{x})$ for fix $\mathbf{x}$ coincides with $\omega_s(t,\mathbf{x})$. Hence the instantaneous Doppler effect is

$$\Delta_{in}(t,\mathbf{x}) = \omega_s(t,\mathbf{x}) - \omega_0 = k(\omega_s(t,\mathbf{x}))v(\mathbf{x},\tau_s(t,\mathbf{x})). \tag{3.24}$$

Formula (3.24) implies if $v(\mathbf{x},\tau_s(t,\mathbf{x})) > 0$ (the source moving to the receiver) $\omega_s(t) > \omega_0$, and if $v(\mathbf{x},\tau_s(t,\mathbf{x})) < 0$ (the source moving from the receiver) $\omega_s(t,\mathbf{x}) < \omega_0$.

We note also that $\tau_s(t,\mathbf{x})$ is the time of the radiation of the signal arrived to the receiver $\mathbf{x}$ at the moment t. Since the group velocity $v_g(\omega) > 0$ we obtain that $\tau_s(t,\mathbf{x}) < t$, that is the causality principle is fulfilled.

The Doppler effect for the time (the retarded time) is defined as

$$\tilde{\Delta}_{in}(t,\mathbf{x}) = t - \tau_s(t,\mathbf{x}) = \frac{|\mathbf{x} - \mathbf{x}_0(\tau_s(t,\mathbf{x}))|}{v_g(\omega_s(t,\mathbf{x}))}. \tag{3.25}$$

3.5 Acoustic wave propagation from the moving sources in stratified dispersive waveguides

We consider the wave propagation from moving sources in the dispersive acoustic waveguides simulating the wave propagation in the ocean. Let

$$\mathbf{x} = (\mathbf{x}',z) \in \mathbb{R}^3, \mathbf{x}' = (x_1,x_2) \in \mathbb{R}^2$$

where $\mathbf{x}'$ is the vector of the horizontal coordinate, z is the vertical coordinate.

We suppose that the sound speed $c = c(z,\omega)$ depends on the depth and the frequency ω

$$c(z,\omega) = \begin{cases} c_0(z,\omega), 0 < z < H \\ c_1(\omega), z \geq H, \end{cases},$$

and the density of the fluid depends of z

$$\rho(z) = \begin{cases} \rho_0(z), 0 < z < H, \\ \rho_1, z \geq H. \end{cases}.$$

We consider the modified acoustic equation for dispersive media in the half-space $\mathbb{R}^3_+ = \left\{\mathbf{x} \in \mathbb{R}^3 : z > 0\right\}$

$$\left(c^{-2}(z,D_t)\frac{\partial^2}{\partial t^2} - \Delta_{\mathbf{x}'} - \rho(z)\frac{\partial}{\partial z}\rho^{-1}(z)\frac{\partial}{\partial z}\right)u(\mathbf{x},t) = f(\mathbf{x},t), \tag{3.26}$$

$$\mathbf{x}=(\mathbf{x}',z) \in \mathbb{R}^3, t \in \mathbb{R}$$

under conditions

$$u(\mathbf{x}',0,t) = 0, \tag{3.27}$$

$$[u(\mathbf{x}',z,t)]_{z=H} = 0, \left[\frac{1}{\rho(z)}\frac{\partial u(\mathbf{x}',z,t)}{\partial z}\right]_{z=H} = 0, \tag{3.28}$$

where

$$[\psi(\mathbf{x}',z,t)]_{z=H} = \psi(\mathbf{x}',H+0,t) - \psi(\mathbf{x}',H-0,t)$$

is the jump of $\psi(\mathbf{x}',z,t)$ on the surface $z = H$. Condition (3.27) means that the acoustic pressure equals zero on the surface $z = 0$ and the acoustic pressure and the normal component of the velocity of particles of the fluid are continuous on the interface $z = H$. In what follows we suppose that

$$c(z,\omega) < c_1(\omega) \tag{3.29}$$

for all (z,ω).

Let

$$f(\mathbf{x},t) = A(t)\delta(\mathbf{x} - \mathbf{x}_0(t)),$$

where $A(t) = a(t)e^{-i\omega_0 t}, a(t) = \tilde{a}(t/\lambda)$ as above,

$$\mathbf{x}_0(t) = (\mathbf{y}_0(t), z_0(t)),$$

The law of the motion of the source is of the form

$$\mathbf{y}_0(t) = \lambda\mathbf{Y}_0(t/\lambda), z_0(t) = Z_0(t/\lambda) \tag{3.30}$$

where $\lambda > 0$ is the same large parameter, $\mathbf{Y}_0, Z_0$ are smooth functions with bounded derivatives. Formulas (3.30) mean that the source moves with an arbitrary horizontal velocity and a small vertical velocity.

Let g_ω be the Green function satisfying the limiting absorption principle of the Helmholtz equation corresponding (3.26)

$$\left(\Delta_y + \rho(z)\frac{\partial}{\partial z}\rho^{-1}(z)\frac{\partial}{\partial z} + k^2(z,\omega)\right)g_\omega(\mathbf{y},z,z_0) = -\delta(\mathbf{y},z-z_0), \tag{3.31}$$

$$\mathbf{y} \in \mathbb{R}^2, z > 0, z_0 \in (0,H)$$

with $k(z,\omega) = \frac{\omega}{c(z,\omega)}$, where

$$g_\omega(\mathbf{y},0,z_0) = 0, \mathbf{y} \in \mathbb{R}^2, \tag{3.32}$$

$$[g_\omega(\mathbf{y},z,z_0)]_{z=H} = 0, \left[\frac{1}{\rho(z)}\frac{\partial g_\omega(\mathbf{y},z,z_0)}{\partial z}\right]_{z=H} = 0, \mathbf{y} \in \mathbb{R}^2. \tag{3.33}$$

We correspond to (3.31)-(3.33) the self-adjoint Sturm–Liouville spectral problem

$$\begin{gathered}-\rho(z)\frac{\partial}{\partial z}\rho^{-1}(z)\frac{\partial\psi_\omega(z)}{\partial z} - (k^2(z,\omega) - k_1^2(\omega))\psi_\omega(z) \\ = \mu^2(\omega)\psi_\omega(z), z \in (0,+\infty), \\ \psi_\omega(0) = 0, [\psi_\omega(z)]_{z=H} = 0, \left[\frac{1}{\rho(z)}\frac{\partial\psi_\omega(z)}{\partial z}\right]_{z=H} = 0,\end{gathered} \tag{3.34}$$

where $k_1(\omega) = \frac{\omega}{c_1}$, in the Hilbert space $L^2_{\rho^{-1}}(\mathbb{R}_+)$ with the norm

$$||u||_{L^2_{\rho^{-1}}(\mathbb{R}_+)} = \left(\int_0^\infty \rho^{-1}(z)\,|u(z)|^2\,dz\right)^{1/2}.$$

A finite discrete spectrum of the problem (3.34) is located on the interval $(-m(\omega),0)$ where

$$m(\omega)=\sup_{z\in(0,H)}(k^2(z,\omega)-k_1^2(\omega))$$

and the continuous spectrum is $[0,+\infty)$.

Let $\left\{\varphi_j(z,\omega)\right\}_{j=1}^{N(\omega)}$ be the orthonormal in $L^2_{\rho^{-1}}(\mathbb{R}_+)$ system of the eigenfunctions of the problem (3.34) corresponding to the eigenvalues $\left\{\beta_j^2(\omega)\right\}_{j=1}^{\infty}$, and $\psi(z,\alpha,\omega)$ be the orthonormal system of generalized eigenfunctions of the problem (3.34), depending on the parameter $\alpha\in(0,+\infty)$.

Applying the spectral decomposition of $\delta(z-z_0)$ on eigenfunction and generalized eigenfunctions of the problem (3.34) (see for instance [2]) we obtain

$$\begin{aligned}g_\omega(\mathbf{y},z,z_0)&=\frac{i}{4\rho(z_0)}\sum_{j=1}^{N(\omega)}H_0^{(1)}(\gamma_j(\omega)\,|\mathbf{y}|)\varphi_j(z,\omega)\varphi_j(z_0,\omega)\\&+\frac{i}{4\rho(z_0)}\int_0^\infty H_0^{(1)}(\sqrt{k_1^2(\omega)-\alpha^2}\,|\mathbf{y}|)\psi(z,\alpha,\omega)\psi(z_0,\alpha,\omega)d\alpha,\end{aligned}$$

where $H_0^{(1)}(w)$ is Hankel function of the first kind and zero order,

$$\gamma_j(\omega)=\sqrt{k_1^2(\omega)+\beta_j^2(\omega)}$$

It is well-known (see for instance [5], [28]) that the part of the Green function g_ω corresponding to the continuous spectrum of the Sturm-Liouville spectral problem (3.34) does not contribute in the main term of the asymptotics of g_ω. Changing the Hankel function $H_0^{(1)}(r)$ by its main term of asymptotics for $r\to+\infty$ we obtain

$$g_\omega(\mathbf{y},z,z_0)\sim\sum_{j=1}^{N(\omega)}\frac{e^{i\pi/4}\varphi_j(\omega,z)\varphi_j(\omega,z_0)}{\rho(z_0)(8\pi\gamma_j(\omega)|\mathbf{y}|)^{1/2}}\exp\left(i\gamma_j(\omega)\,|\mathbf{y}|\right)\tag{3.35}$$
$$k_1(\omega)\,|\mathbf{y}|\to\infty.$$

Applying the formula (3.35), the representation of the acoustic pressure as a double oscillatory integral, and stationary phase method we obtain the following formula (see for instance [28])

$$u(\mathbf{y},z,t)=\sum_{j=1}^{N(\omega_0)}u_j(\mathbf{y},z,t),$$

where

$$\begin{aligned}&u_j(\mathbf{y},z,t)\\&\sim\frac{\sqrt{\frac{\pi}{2}}e^{i\pi/4}}{\rho(z_0)}\frac{a(\tau_j)\varphi_j(\omega_j,z)\varphi_j(\omega_j,z_0(\tau_j))e^{i\lambda S_j(\mathbf{y},t,\omega_j,\tau_j)+i\frac{\pi}{4}sgn\tilde{S}_j''(\mathbf{y},t,\omega_j,\tau_j)}}{(\gamma_j\left(\omega_j\right)|\mathbf{y}-\mathbf{y}_0(\tau_j)|\left|\det S_j''(\mathbf{y},t,\omega_j,\tau_j)\right|)^{1/2}},\end{aligned}\tag{3.36}$$

$$S_j(\mathbf{y},t,\omega,\tau) = \gamma_j(\omega)\,|\mathbf{y}-\mathbf{y}_0(\tau)| - \omega(t-\tau) - \omega_0\tau$$

and $\omega_j = \omega_j(\mathbf{y},t), \tau_j = \tau_j(\mathbf{y},t)$ are the stationary points of the phase, that is the solutions of the system

$$\begin{aligned} &\frac{|\mathbf{y}-\mathbf{y}_0(\tau)|}{v_{jg}(\omega)} - (t-\tau) = 0, \\ &-\gamma_j(\omega)v(\mathbf{y},\tau) + (\omega-\omega_0) = 0, \end{aligned} \tag{3.37}$$

$v(\mathbf{y},\tau)$ is the value of the projection of the vector $\dot{\mathbf{y}}(\tau)$ on the vector $\mathbf{y}-\mathbf{y}_0(\tau)$,

$$v_{jg}(\omega) = \frac{1}{\gamma_j'(\omega)}$$

is the group velocity of the *mode* with number j. Note that the $\omega_j(\mathbf{y},t)$ is the instantaneous frequency of j–mode and $\tau_j(\mathbf{y},t)$ is the time of the excitation of the j–mode arriving at the point $\mathbf{x} = (\mathbf{y},\mathbf{z})$ at the moment $t > 0$. We suppose that $v_{jg}(\omega) > 0$ for all ω. This condition provides the fulfillment of the causality principle that is $\tau_j(\mathbf{y},t) < t$.

Note that the Doppler shift $\Delta\omega_{in}^j(\mathbf{y},t)$ and the retarded time $\Delta t_{in}^j(\mathbf{y},t)$ of the mode with number j are given by the formulas

$$\begin{aligned} \Delta\omega_{in}^j &= \omega_j(\mathbf{y},t) - \omega_0 = \gamma_j(\omega_j(\mathbf{y},t))v(\mathbf{y},\tau_j(\mathbf{y},t)), \\ \Delta t_{in}^j(\mathbf{y},t) &= t - \tau_j(\mathbf{y},t) = \frac{|\mathbf{y}-\mathbf{y}_0(\tau_j(\mathbf{y},t))|}{v_{jg}(\omega_j(\mathbf{y},t))}. \end{aligned}$$

4 Conclusion

The paper is developed the time-frequency integrals and the two-dimensional stationary phase method for the problems of waves propagation from moving sources in acoustic dispersive media. Applying the stationary phase method we obtained the effective formula for the acoustic waves in the dispersive acoustic waveguides simulated the ocean generated by non-uniformly moving modulated source. The explicit formulas for the Doppler effect and the retarded time also was discussed.

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