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LONG-TIME DECAY FOR THE LINEARIZED SYSTEM OF A CHARGED PARTICLE IN THE KLEIN-GORDON FIELD

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Abstract

We establish a long time decay for linearized dynamics of nonlinear Hamilton system which describes interaction of a charged particle with the Klein-Gordon field. The main contribution is deriving the decay for the frozen linearized system for initial data which are symplectic orthogonal to the root space of the linear differential operator involved.

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1 Introduction. Charged Particle in Klein-Gordon Field

Method of symplectic projection appears to be fruitful in establishing soliton-type asymptotics for a variety of Hamilton systems, [1]–[10]. In the present paper we consider the system of a charged particle interacting with the Klein-Gordon field which reads [8],

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$$\dot\psi(x,t)=\pi(x,t),\qquad \dot\pi(x,t)=\Delta\psi(x,t)-m^2\psi(x,t)-\rho(x-q(t)),\quad x\in\mathbb{R}^3,$$
$$\dot q(t)=p(t)/\sqrt{1+p^2(t)},\quad \dot p(t)=\int\psi(x,t)\nabla\rho(x-q(t))dx, \tag{1.1}$$

where $m>0$. This is a Hamilton system with the Hamilton functional

$$\mathcal{H}(\psi,\pi,q,p)=\frac{1}{2}\int\Big(|\pi(x)|^2+|\nabla\psi(x)|^2+m^2|\psi(x)|^2\Big)dx+\int\psi(x)\rho(x-q)dx+\sqrt{1+p^2}. \tag{1.2}$$

The system (1.1) is translation-invariant and admits soliton solutions

$$Y_{a,v}(t)=(\psi_v(x-vt-a),\pi_v(x-vt-a),vt+a,p_v),\qquad p_v=v/\sqrt{1-v^2} \tag{1.3}$$

for all $a,v\in\mathbb{R}^3$ with $|v|<1$. The states $S_{a,v}:=Y_{a,v}(0)$ form the solitary manifold

$$\mathcal{S}:=\{S_{a,v}:a,v\in\mathbb{R}^3,|v|<1\}. \tag{1.4}$$

Let ρ be a real valued function of the Sobolev class $H^2(\mathbb{R}^3)$, compactly supported, and spherically symmetric, i.e.

$$\rho,\nabla\rho,\nabla\nabla\rho\in L^2(\mathbb{R}^3),\qquad \rho(x)=0\ \text{for}\ |x|\ge R_\rho,\qquad \rho(x)=\rho_1(|x|). \tag{1.5}$$

We require that all “modes” of the wave field are coupled to the particle, this is formalized by the Wiener condition

$$\hat\rho(k)=(2\pi)^{-3/2}\int e^{ikx}\rho(x)dx\neq 0\quad\text{for all}\quad k\in\mathbb{R}^3. \tag{1.6}$$

2 Symplectic Projection, Linearization, and Decay for the Linearized Dynamics

We sketch the derivation of the linearized dynamics of the system (1.1) and make a statement on its time decay. For details see [8].

2.1 Hamilton form and symplectic structure

The system (1.1) reads as the Hamilton system

$$\dot Y=J\mathcal{D}\mathcal{H}(Y),\ J:=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&1\\0&0&-1&0\end{pmatrix},\ Y=(\psi,\pi,q,p)\in\mathcal{E}, \tag{2.1}$$

where $\mathcal{D}\mathcal{H}$ is the Fréchet derivative of the Hamilton functional (1.2), and $\mathcal{E}$ is the phase space of the system consisting of finite energy states $Y=(\psi,\pi,q,p)$, see [8, Definition 2.1].

Let us identify the tangent space to $\mathcal{E}$, at every point, with $\mathcal{E}$. Consider the symplectic form Ω defined on $\mathcal{E}$ by $\Omega = \int d\psi(x)\wedge d\pi(x)\,dx + dq\wedge dp$, i.e.

$$\Omega(Y_1, Y_2) = \langle Y_1, JY_2\rangle, \;\; Y_1, Y_2 \in \mathcal{E}, \tag{2.2}$$

where

$$\langle Y_1, Y_2\rangle := \langle \psi_1, \psi_2\rangle + \langle \pi_1, \pi_2\rangle + q_1 q_2 + p_1 p_2$$

and $\langle \psi_1, \psi_2\rangle = \int \psi_1(x)\psi_2(x)dx$, etc. It is clear that the form Ω is non-degenerate, i.e.

$$\Omega(Y_1, Y_2) = 0 \;\text{ for every }\; Y_2 \in \mathcal{E} \implies Y_1 = 0.$$

Definition 2.1. i) $Y_1 \nmid Y_2$ means that $Y_1 \in \mathcal{E}$, $Y_2 \in \mathcal{E}$, and Y_1 is symplectic orthogonal to Y_2, i.e. $\Omega(Y_1, Y_2) = 0$.

ii) A projection operator $P : \mathcal{E} \to \mathcal{E}$ is called symplectic orthogonal if $Y_1 \nmid Y_2$ for $Y_1 \in \operatorname{Ker} P$ and $Y_2 \in \Im P$.

2.2 Linearization on the solitary manifold

Let us consider a solution to the system (1.1), and split it as the sum

$$Y(t) = S(\sigma(t)) + Z(t), \tag{2.3}$$

where $\sigma(t) = (b(t), v(t))$ is an arbitrary smooth function of $t \in \mathbb{R}$. In detail, denote $Y = (\psi, \pi, q, p)$ and $Z = (\Psi, \Pi, Q, P)$. Then (2.3) means that

$$\left.\begin{array}{rclrcl} \psi(x,t) &=& \psi_{v(t)}(x - b(t)) + \Psi(x - b(t), t), & q(t) &=& b(t) + Q(t) \\ \pi(x,t) &=& \pi_{v(t)}(x - b(t)) + \Pi(x - b(t), t), & p(t) &=& p_{v(t)} + P(t) \end{array}\right| \tag{2.4}$$

Let us substitute (2.4) to (1.1), and linearize the equations in Z. We obtain

$$\dot{Z}(t) = A(t)Z(t) + T(t) + N(t), \;\; t \in \mathbb{R}. \tag{2.5}$$

Here the operator $A(t) = A_{v,w}$ depends on two parameters, $v = v(t)$, and $w = \dot{b}(t)$ and can be written in the form

$$A_{v,w}\begin{pmatrix} \Psi \\ \Pi \\ Q \\ P \end{pmatrix} := \begin{pmatrix} w\cdot\nabla & 1 & 0 & 0 \\ \Delta - m^2 & w\cdot\nabla & \nabla\rho\cdot & 0 \\ 0 & 0 & 0 & B_v \\ \langle\cdot, \nabla\rho\rangle & 0 & \langle\nabla\psi_v, \cdot\nabla\rho\rangle & 0 \end{pmatrix}\begin{pmatrix} \Psi \\ \Pi \\ Q \\ P \end{pmatrix}, \tag{2.6}$$

where $B_v = v(E - v\otimes v)$. Furthermore, $T(t) = T_{v,w}$ is given by

$$T_{v,w} = \begin{pmatrix} (w - v)\cdot\nabla\psi_v - \dot{v}\cdot\nabla_v\psi_v \\ (w - v)\cdot\nabla\pi_v - \dot{v}\cdot\nabla_v\pi_v \\ v - w \\ -\dot{v}\cdot\nabla_v p_v \end{pmatrix}, \tag{2.7}$$

where $v = v(t)$, $w = w(t)$, $\sigma = \sigma(t) = (b(t), v(t))$, $Z = Z(t)$, and $N(t)$ is a second order term with respect to Z.

Note that the formulas (2.7), (3.21) imply that $T(t) \in \mathcal{T}_{S(\sigma(t))}\mathcal{S}$, the tangent space to the manifold $\mathcal{S}$ at the point $\sigma(t)$, $t \in \mathbb{R}$. This fact suggests the unstable character of the nonlinear dynamics along the solitary manifold.

2.3 Symplectic decomposition of dynamics and decay for transversal component

Let us split the dynamics in two components: along the manifold $\mathcal{S}$ and in transversal directions. The equation (2.5) is obtained without any assumption on $\sigma(t)$ in (2.3). We choose

$$S(\sigma(t)) := \Pi Y(t). \tag{2.8}$$

Let us fix some $\beta > 3/2$. $\mathcal{E}_\beta$ it the weighted version of the phase space introduced in [8, Definition 2.1], $\|\cdot\|_\beta$ is the norm in $\mathcal{E}_\beta$.

Proposition 2.2. [8, Proposition 6.4]. *Let (2.8) hold and let initial data be sufficiently close to $\mathcal{S}$ in the norm $\|\cdot\|_\beta$. Then*

$$\|Z(t)\|_{-\beta} \le \frac{C}{(1+|t|)^{3/2}}, \quad t \ge 0. \tag{2.9}$$

Let us comment on two main difficulties in proving (2.9). First, the linear part of the equation (2.5) is non-autonomous, hence we cannot apply directly known methods of scattering theory. So we reduce the problem to the analysis of the *frozen* linear equation,

$$\dot{X}(t) = A_1 X(t), \ \ t \in \mathbb{R}, \tag{2.10}$$

where A_1 is the operator A_{v_1,v_1} defined by (2.6) with $v_1 = v(t_1)$ for a fixed t_1. Then we estimate the error by the method of majorants.

Second, even for the frozen equation (2.10), the decay of type (2.9) for all solutions does not hold without the orthogonality condition of type (2.8). Namely, the equation (2.10) admits the *secular solutions* which arise by differentiation of the soliton (1.3) in the parameters a and v_1 in the moving coordinate $y = x - v_1 t$. Hence, we have to take into account the orthogonality condition (2.8) in order to avoid the secular solutions. For this purpose we will apply this symplectic orthogonal projection which kills the "runaway solutions".

Definition 2.3. i) For $v \in \mathbb{R}^3$ with $|v| < 1$ denote by Π_v the symplectic orthogonal projection of $\mathcal{E}$ onto the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$, and $P_v = \mathbf{I} - \Pi_v$.
ii) Denote by $\mathcal{Z}_v = P_v\mathcal{E}$ the space symplectic orthogonal to $\mathcal{T}_{S(\sigma)}\mathcal{S}$ with $\sigma = (b, v)$ (for an arbitrary $b \in \mathbb{R}$).

Now we have the symplectic orthogonal decomposition

$$\mathcal{E} = \mathcal{T}_{S(\sigma)}\mathcal{S} + \mathcal{Z}_v, \qquad \sigma = (b, v), \tag{2.11}$$

and the symplectic orthogonality (2.8) can be written in the following equivalent forms,

$$\Pi_{v(t)} Z(t) = 0, \quad P_{v(t)} Z(t) = Z(t). \tag{2.12}$$

Note that the tangent space $\mathcal{T}_{S(\sigma)}\mathcal{S}$ is invariant under the operator $A_{v,v}$, hence the space $\mathcal{Z}_v$ is also invariant: $A_{v,v} Z \in \mathcal{Z}_v$ on a dense domain of $Z \in \mathcal{Z}_v$.

Our main result is the following proposition which will be one of the main ingredients for proving (2.9). Let us consider the Cauchy problem for the equation (2.10) with $A = A_{v,v}$ for a fixed $v \in V$ and fix a $\beta > 3/2$.

Proposition 2.4. *Let (1.5) and (1.6) hold,* $|v| \le \tilde{v} < 1$, *and* $X_0 \in \mathcal{E}$. *Then*
i) The equation (2.10), with $A_1 = A = A_{v,v}$, *admits the unique solution* $e^{At}X_0 := X(t) \in C(\mathbb{R}, \mathcal{E})$ *with the initial condition* $X(0) = X_0$.
ii) For $X_0 \in \mathcal{Z}_v \cap \mathcal{E}_\beta$, *the solution* $X(t)$ *has the following decay,*

$$||e^{At}X_0||_{-\beta} \le \frac{C(\tilde{v})}{(1+|t|)^{3/2}} ||X_0||_\beta, \qquad t \in \mathbb{R}. \tag{2.13}$$

3 Proof of Proposition 2.4

Let us discuss our strategy of the proof. We apply the Fourier-Laplace transform

$$\tilde{X}(\lambda) = \int_0^\infty e^{-\lambda t} X(t)dt, \qquad \mathrm{Re}\,\lambda > 0 \tag{3.1}$$

to (2.10) and obtain

$$\lambda \tilde{X}(\lambda) = A\tilde{X}(\lambda) + X_0, \qquad \mathrm{Re}\,\lambda > 0. \tag{3.2}$$

Then the solution $X(t)$ is given by

$$\tilde{X}(\lambda) = -(A-\lambda)^{-1}X_0, \qquad \mathrm{Re}\,\lambda > 0 \tag{3.3}$$

if the resolvent $R(\lambda) = (A-\lambda)^{-1}$ exists for $\mathrm{Re}\,\lambda > 0$.

The analyticity of $\tilde{X}(\lambda)$ and Paley-Wiener arguments (see [11]) should provide the existence of a $\mathcal{E}_{-\beta}$ - valued distribution $X(t)$, $t \in \mathbb{R}$, with a support in $[0, \infty)$. Formally,

$$X(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\omega t} \tilde{X}(i\omega + 0)d\omega, \qquad t \in \mathbb{R}. \tag{3.4}$$

However, to check the continuity of $X(t)$ for $t \ge 0$, we need additionally a bound for $\tilde{X}(i\omega+0)$ at large $|\omega|$. Finally, for the time decay of $X(t)$, we need an additional information on the smoothness and decay of $\tilde{X}(i\omega + 0)$. More precisely, we should prove that the function $\tilde{X}(i\omega+0)$

i) is smooth outside $\omega = 0$ and $\omega = \pm\mu$, where $\mu = \mu(v) > 0$,

ii) decays in a certain sense as $|\omega| \to \infty$.

iii) admits the Puiseux expansion at $\omega = \pm\mu$.

iv) is analytic at $\omega = 0$ if $X_0 \in \mathcal{Z}_v = P_v\mathcal{E}$ and $X_0 \in \mathcal{E}_\beta$.

Then the decay (2.9) would follow from the Fourier-Laplace representation (3.4).

We will check with detail the properties of type i)-iv) only for the last two components $\tilde{Q}(\lambda)$ and $\tilde{P}(\lambda)$ of the vector $\tilde{X}(\lambda) = (\tilde{\Psi}(\lambda), \tilde{\Pi}(\lambda), \tilde{Q}(\lambda), \tilde{P}(\lambda))$. The properties provide the decay (2.9) for the vector components $Q(t)$ and $P(t)$ of the solution $X(t)$.

However, we will not prove the properties of type i)-iv) for the field components $\Psi(x, \lambda)$ and $\Pi(x, \lambda)$. We prove the decay (2.9) for the field components directly from the time-dependent field equations of the system (2.10), using the decay of the component $Q(t)$ and a version of strong Huygens principle for the Klein-Gordon equation, [8].

3.1 Constructing the resolvent

To justify the representation (3.3), we construct the resolvent as a bounded operator in $\mathcal{E}$ for $\mathrm{Re}\,\lambda > 0$. We will write $(\Psi(y), \Pi(y), Q, P)$ instead of $(\tilde\Psi(y,\lambda), \tilde\Pi(y,\lambda), \tilde Q(\lambda), \tilde P(\lambda))$ to simplify the notations. Then (3.2) reads

$$(A-\lambda)\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} = -\begin{pmatrix}\Psi_0\\ \Pi_0\\ Q_0\\ P_0\end{pmatrix}, \text{ where } A\begin{pmatrix}\Psi\\ \Pi\\ Q\\ P\end{pmatrix} = \begin{pmatrix}\Pi + v\cdot\nabla\Psi\\ \Delta\Psi - m^2\Psi + v\cdot\nabla\Pi + Q\cdot\nabla\rho\\ B_vP\\ -\langle\nabla\Psi,\rho\rangle + \langle\nabla\psi_v, Q\cdot\nabla\rho\rangle\end{pmatrix}.$$

It is the system of equations

$$\left.\begin{aligned} \Pi(y) + v\cdot\nabla\Psi(y) - \lambda\Psi(y) &= -\Psi_0(y)\\ \Delta\Psi(y) - m^2\Psi(y) + v\cdot\nabla\Pi(y) + Q\cdot\nabla\rho(y) - \lambda\Pi(y) &= -\Pi_0(y)\\ B_vP - \lambda Q &= -Q_0\\ -\langle\nabla\Psi(y),\rho(y)\rangle + \langle\nabla\psi_v(y), Q\cdot\nabla\rho(y)\rangle - \lambda P &= -P_0 \end{aligned}\right| \quad y\in\mathbb{R}^3. \tag{3.5}$$

Step i) Let us consider the first two equations. In Fourier space they become

$$\left.\begin{aligned} \hat\Pi(k) - ivk\hat\Psi(k) - \lambda\hat\Psi(k) &= -\hat\Psi_0(k)\\ (-k^2 - m^2)\hat\Psi(k) - (ivk+\lambda)\hat\Pi(k) &= -\hat\Pi_0(k) + iQk\hat\rho(k) \end{aligned}\right| \quad k\in\mathbb{R}^3. \tag{3.6}$$

This implies

$$\hat\Psi = \frac{1}{\hat D}((ikv+\lambda)\hat\Psi_0 + \hat\Pi_0 - ikQ\hat\rho), \tag{3.7}$$

$$\hat\Pi = \frac{1}{\hat D}(-(k^2+m^2)\hat\Psi_0 + (ikv+\lambda)\hat\Pi_0 - i(ikv+\lambda)kQ\hat\rho), \tag{3.8}$$

where

$$\hat D = \hat D(\lambda) = k^2 + m^2 + (ikv+\lambda)^2. \tag{3.9}$$

From now on we use the system of coordinates in x-space in which $v = (|v|, 0, 0)$, hence $vk = |v|k_1$. Substitute (3.7) to the 4-th equation of (3.5) and obtain

$$\int \frac{ik}{\hat D}((ikv+\lambda)\hat\Psi_0 + \Pi_0 - ikQ\hat\rho)\overline{\hat\rho}dk + \int k\hat\psi_v kQ\overline{\hat\rho}dk - \lambda P = -P_0.$$

Since

$$\hat\psi_v(k) = -\frac{\hat\rho(k)}{k^2+m^2-(kv)^2},\ \hat\pi_v(k) = i(kv)\hat\psi_v(k), \tag{3.10}$$

see [8], we come to

$$(K - H(\lambda))Q + \lambda P = P_0 + \Phi(\lambda),$$

where

$$\Phi(\lambda) = \Phi(\Psi_0,\Pi_0)(\lambda) := i\int \frac{k}{\hat D}((ikv+\lambda)\hat\Psi_0 + \hat\Pi_0)\overline{\hat\rho}dk. \tag{3.11}$$

Here K and $H(\lambda)$ are 3×3-matrices with the matrix elements

$$K_{ij}=\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+m^2-(|v|k_1)^2},\ \ H_{ij}(\lambda)=\int\frac{k_ik_j|\hat\rho(k)|^2dk}{k^2+m^2+(i|v|k_1+\lambda)^2}. \tag{3.12}$$

The matrix K is diagonal and positive definite since $\hat\rho(k)$ is spherically symmetric and not identically zero by (1.6). The matrix H is well defined for $\mathrm{Re}\,\lambda>0$ since the denominator does not vanish. The matrix H is diagonal similarly to K. Indeed, if $i\neq j$, then at least one of these indexes is not equal to one, and the integrand in (3.12) is odd with respect to the corresponding variable. Finally the 3-rd and the 4-th equations of (3.5) become

$$M(\lambda)\begin{pmatrix} Q\\ P\end{pmatrix}=\begin{pmatrix} Q_0\\ P_0+\Phi\end{pmatrix},\ \text{where}\ M(\lambda)=\begin{pmatrix}\lambda E & -B_v\\ K-H(\lambda) & \lambda E\end{pmatrix}. \tag{3.13}$$

Lemma 3.1. [8] *The matrix $M(\lambda)$ is invertible for* $\mathrm{Re}\,\lambda>0$.

Then we obtain

$$\begin{pmatrix} Q\\ P\end{pmatrix}=M^{-1}(\lambda)\begin{pmatrix} Q_0\\ P_0+\Phi\end{pmatrix},\qquad \mathrm{Re}\,\lambda>0. \tag{3.14}$$

Finally, formula (3.14) and formulas (3.7), (3.8) give the expression of the resolvent $R(\lambda)=(A-\lambda)^{-1}$, $\mathrm{Re}\,\lambda>0$, in Fourier space.

3.2 Time decay of the vector components

Let us prove the decay (2.9) for the components $Q(t)$ and $P(t)$.

Lemma 3.2. *Let $X_0\in\mathcal{Z}_v\cap\mathcal{E}_\beta$. Then $Q(t)$, $P(t)$ are continuous and*

$$|Q(t)|+|P(t)|\le\frac{C(\tilde v)}{(1+|t|)^{3/2}},\qquad t\ge 0. \tag{3.15}$$

Proof. We split the Fourier integral (3.4) for the vector components into three terms using the partition of unity $\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega)=1$, $\omega\in\mathbb{R}$. By (3.14) we obtain:

$$\begin{aligned}\begin{pmatrix} Q(t)\\ P(t)\end{pmatrix} &= \frac{1}{2\pi}\int e^{i\omega t}(\zeta_1(\omega)+\zeta_2(\omega)+\zeta_3(\omega))M^{-1}(i\omega)\begin{pmatrix} Q_0\\ P_0+\Phi(i\omega)\end{pmatrix}d\omega\\ &= I_1(t)+I_2(t)+I_3(t),\end{aligned} \tag{3.16}$$

where the functions $\zeta_k(\omega)\in C^\infty(\mathbb{R})$ are supported by

$$\left.\begin{array}{lll}\mathrm{supp}\,\zeta_1 &\subset& \{\omega\in\mathbb{R}:\varepsilon_0/2<|\omega|<\mu+2\}\\ \mathrm{supp}\,\zeta_2 &\subset& \{\omega\in\mathbb{R}:|\omega|>\mu+1\}\\ \mathrm{supp}\,\zeta_3 &\subset& \{\omega\in\mathbb{R}:|\omega|<\varepsilon_0\}\end{array}\right| \tag{3.17}$$

Then i) The functions $I_1(t)$ and $I_2(t)$ decay at least like $(1+|t|)^{-3/2}$, see [8].

ii) The main contribution of the present paper is providing the direct proof that the function $I_3(t)$ decays like $t^{-\infty}$ if $(\Psi_0,\Pi_0,Q_0,P_0)\in\mathcal{Z}_v$. The proof can be developed to the cases of the scalar wave field and of the Maxwell field, [9, 10, 7]. The result follows from the statement below:

Proposition 3.3. *The vector components*

$$\begin{pmatrix} Q(i\omega) \\ P(i\omega) \end{pmatrix} = M^{-1}(i\omega) \begin{pmatrix} Q_0 \\ P_0 + \Phi(i\omega) \end{pmatrix}$$

as functions of ω *belong to* $C^\infty(-\varepsilon_0;\varepsilon_0)$ *for sufficiently small* ε_0 *if* $(\Psi_0, \Pi_0, Q_0, P_0) \in \mathcal{Z}_v$.

Proof. i) We compute directly the matrix $\mathcal{M} := M^{-1}(i\omega)$ and obtain for $\omega \neq 0$:

$$\mathcal{M} := \begin{pmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{pmatrix},$$

where

$$\mathcal{M}_{11} = -\begin{pmatrix} \frac{i\omega}{\omega^2 + v^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{i\omega}{\omega^2 + v f(\omega)} & 0 \\ 0 & 0 & \frac{i\omega}{\omega^2 + v f(\omega)} \end{pmatrix},$$

$$\mathcal{M}_{12} = -\begin{pmatrix} \frac{v^3}{\omega^2 + v^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{v}{\omega^2 + v f(\omega)} & 0 \\ 0 & 0 & \frac{v}{\omega^2 + v f(\omega)} \end{pmatrix},$$

$$\mathcal{M}_{21} = -\begin{pmatrix} \frac{f_1(\omega)}{\omega^2 + v^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{f(\omega)}{\omega^2 + v f(\omega)} & 0 \\ 0 & 0 & \frac{f(\omega)}{\omega^2 + v f(\omega)} \end{pmatrix},$$

$$\mathcal{M}_{22} = -i\begin{pmatrix} \frac{\omega}{\omega^2 + v^3 f_1(\omega)} & 0 & 0 \\ 0 & \frac{\omega}{\omega^2 + v f(\omega)} & 0 \\ 0 & 0 & \frac{\omega}{\omega^2 + v f(\omega)} \end{pmatrix}.$$

Recall that $f_1(\omega) = F_{11}(\omega)$, $f(\omega) = F_{22}(\omega) = F_{33}(\omega)$, where

$$F_{jj}(\omega) = \int dk |\hat\rho|^2 k_j^2 \left(\frac{1}{m^2 + k^2 - (vk_1 + \omega)^2} - \frac{1}{m^2 + k^2 - (vk_1)^2} \right) = \omega^2 I_j(\omega)$$

with $I_j(\omega) \in C^\infty(-\varepsilon_0;\varepsilon_0)$, since

$$I_j(\omega) = \int dk \frac{|\hat\rho|^2 k_j^2 (2\omega v k_1 + n^2 + 4(vk_1)^2)}{n^4(n^2 - \delta)}, \ n^2 := m^2 + k^2 - (vk_1)^2, \ \delta := \omega(\omega + 2vk_1). \quad (3.18)$$

Further,

$$\Phi(i\omega) = i \int dk \frac{i(k_1 v + \omega)\hat\Psi_0(k) + \hat\Pi_0(k)}{m^2 + k^2 - (vk_1 + \omega)^2} k \overline{\hat\rho}.$$

Lemma 3.4. *The symplectic orthogonality conditions (2.8) (or (2.12)) for the initial data read*

$$P_0 + \Phi(0) = 0, \tag{3.19}$$

$$\zeta(0) + B_v^{-1} Q_0 = 0, \tag{3.20}$$

where ζ is defined below by (3.22).

Proof. The orthogonality conditions (2.8) (or (2.12)) for the initial data X_0 read $\Omega(X_0, \tau_j) = \Omega(X_0, \tau_{j+3}) = 0$, $j = 1,2,3$, where

$$\tau_j = (-\partial_j \psi_v(y), -\partial_j \pi_v(y), e_j, 0), \ \ \tau_{j+3} = (\partial_{v_j}\psi_v(y), \partial_{v_j}\pi_v(y), 0, \partial_{v_j} p_v), \tag{3.21}$$

ψ_v and π_v are given by (3.10), for details see [8].

First let us check that the conditions $\Omega(X_0, \tau_j) = 0$ read $P_0 + \Phi(0) = 0$. One has

$$\Phi(0) = i \int dk \frac{ikv\hat{\Psi}_0 + \Pi_0}{\hat{D}} k\overline{\hat{\rho}}, \ \ \hat{D} = k^2 + m^2 - (kv)^2.$$

For $j = 1,2,3$ we have

$$0 = \Omega(X_0, \tau_j) = -\langle \Psi_0, \partial_j \pi_v \rangle + \langle \Pi_0, \partial_j \psi_v \rangle - P_0 \cdot e_j = -\int dk \hat{\Psi}_0 \overline{-ik_j i(kv)\frac{-\hat{\rho}}{\hat{D}}}$$

$$+ \int dk \hat{\Pi}_0 \overline{-ik_j \frac{-\hat{\rho}}{\hat{D}}} - P_0 \cdot e_j = \int dk \frac{\hat{\Psi}_0 (kv)\overline{\hat{\rho}} k_j}{\hat{D}} - i \int dk \frac{\hat{\Pi}_0 \overline{\hat{\rho}} k_j}{\hat{D}} - (P_0)_j = -(\Phi(0) + P_0)_j$$

and the statement is checked. Further,

$$0 = \Omega(X_0, \tau_{j+3}) = \int dk \hat{\Psi}_0 \cdot \overline{ik_j \frac{k^2 + m^2 + (kv)^2}{\hat{D}} \frac{-\hat{\rho}}{\hat{D}}} - \int dk \hat{\Pi}_0 \cdot \overline{\frac{2(kv)k_j}{\hat{D}} \frac{-\hat{\rho}}{\hat{D}}} + Q_0 \cdot \partial_{v_j} p_v =$$

$$i \int dk \frac{\hat{\Psi}_0 k_j (k^2 + m^2 + (kv)^2)\overline{\hat{\rho}}}{\hat{D}^2} + \int dk \frac{\hat{\Pi}_0 2(kv) k_j \overline{\hat{\rho}}}{\hat{D}^2} + Q_0 \cdot \partial_{v_j} p_v.$$

Note that $Q_0 \cdot \partial_{v_j} p_v = Q_0 \cdot B_v^{-1} e_j = B_v^{-1} Q_0 \cdot e_j$. Then these symplectic orthogonality conditions $\Omega(X_0, \tau_{j+3}) = 0$, $j = 1,2,3$ coincide with

$$\zeta(0) + B_v^{-1} Q_0 = 0,$$

where $\zeta(\omega)$ is defined by (3.22).

□

By (3.19) $P_0 + \Phi(i\omega) = \Phi(i\omega) - \Phi(0) = i\omega\zeta(\omega)$, where $\zeta(\omega) \in C^\infty(-\varepsilon_0; \varepsilon_0)$ because

$$\zeta(\omega) = \int dk \frac{\omega(ik_1 v\hat{\Psi}_0(k) + \hat{\Pi}_0(k)) + i(m^2 + k^2 + (k_1 v)^2)\hat{\Psi}_0(k) + 2vk_1\hat{\Pi}_0(k)}{(m^2 + k^2 - (vk_1 + \omega)^2)(m^2 + k^2 - (vk_1)^2)} k\overline{\hat{\rho}}. \tag{3.22}$$

ii) Let us start with the component $P(i\omega)$.

$$P(i\omega) = \mathcal{M}_{21} Q_0 + \mathcal{M}_{22}(P_0 + \Phi(i\omega)) =$$

$$-\begin{pmatrix} \frac{I_1(\omega)}{1+\nu^3 I_1(\omega)} & 0 & 0 \\ 0 & \frac{I(\omega)}{1+\nu I(\omega)} & 0 \\ 0 & 0 & \frac{I(\omega)}{1+\nu I(\omega)} \end{pmatrix} Q_0 + \begin{pmatrix} \frac{1}{1+\nu^3 I_1(\omega)} & 0 & 0 \\ 0 & \frac{1}{1+\nu I(\omega)} & 0 \\ 0 & 0 & \frac{1}{1+\nu I(\omega)} \end{pmatrix} \zeta(\omega),$$

where $I(\omega) := I_2(\omega) = I_3(\omega)$. Since $I_1(0) > 0$ and $I(0) > 0$ by (3.18), one can observe that $P(i\omega) \in C^\infty(-\varepsilon_0; \varepsilon_0)$.

iii) Now let us proceed to $Q(i\omega)$:

$$Q(i\omega) = \mathcal{M}_{11} Q_0 + \mathcal{M}_{12}(P_0 + \Phi(i\omega)).$$

By (3.20) one has

$$P_0 + \Phi(i\omega) = i\omega\zeta(\omega) = i\omega(\zeta(\omega) - \zeta(0) - B_\nu^{-1} Q_0) = i\omega(\omega J(\omega) - B_\nu^{-1} Q_0),$$

where

$$J(\omega) = \int dk \frac{in^2(\omega + 3\nu k_1)\hat{\Psi}_0(k) + (n^2 + 2\nu k_1(\omega + 2\nu k_1))\hat{\Pi}_0(k)}{n^4(n^2 - \delta)} k\hat{\rho} \in C^\infty(-\varepsilon_0; \varepsilon_0).$$

Finally, since $\mathcal{M}_{11} = i\omega \mathcal{M}_{12} B_\nu^{-1}$, one has

$$Q(i\omega) = \mathcal{M}_{11} Q_0 - i\omega \mathcal{M}_{12} B_\nu^{-1} Q_0 + i\omega \mathcal{M}_{12} \omega J(\omega) =$$

$$i\omega^2 \mathcal{M}_{12} J(\omega) = \begin{pmatrix} \frac{\nu^3}{1+\nu^3 I_1(\omega)} & 0 & 0 \\ 0 & \frac{\nu}{1+\nu I(\omega)} & 0 \\ 0 & 0 & \frac{\nu}{1+\nu I(\omega)} \end{pmatrix} J(\omega)$$

and, similarly, $Q(i\omega) \in C^\infty(-\varepsilon_0; \varepsilon_0)$. The proposition is proved. □

This completes the proof of Lemma 3.2 and Proposition 2.4. □

Remarks 3.5. i) Note that

$$\Phi(\Psi_0, \Pi_0)(\lambda) = \tilde{F}_{t\to\lambda} \langle W^1(t)(\Psi_0, \Pi_0), \nabla\rho \rangle,$$

where $\tilde{F}_{t\to\lambda}$ is Laplace transform in t, $W^1(t)$ is the first component of the dynamical group $W(t)$ of the free Klein-Gordon equation.

ii) Let $f(t) = \langle W^1(t)(\Psi_0, \Pi_0), \nabla\rho \rangle$. Then

$$\Phi(i\omega) = \tilde{F}_{t\to\omega} f(t) = \int_0^\infty e^{-i\omega t} f(t)dt, \quad \Phi(0) = \int_0^\infty f(t)dt = -P_0$$

by (3.19). Set

$$g(t) = -\int_t^\infty f(s)ds, \text{ then } g'(t) = f(t), \; g(0) = -\int_0^\infty f(s)ds = P_0.$$

We have

$$P_0 + \Phi(i\omega) = P_0 + \tilde{f}(\omega) = P_0 + \omega\tilde{g}(\omega) - g(0) = \omega\tilde{g}(\omega),$$

thus, $\tilde{g}(\omega) = i\xi(\omega)$. Further, $P(i\omega) = \mathcal{M}_{21}Q_0 + \mathcal{M}_{22}(P_0 + \Phi(i\omega)) = \mathcal{M}_{21}Q_0 + \mathcal{M}_{22}\omega\tilde{g}(\omega)$, the last combination being smooth in ω.

iii) For $\tilde{g}(\omega)$, where $g(t)$ is introduced in the previous remark, we have

$$\tilde{g}(0) = \int_0^\infty g(t)dt = -iB_v^{-1}Q_0$$

by (3.20). Set

$$h(t) = -\int_t^\infty g(s)ds, \text{ then } h'(t) = g(t), \ h(0) = -\int_0^\infty g(t)dt = iB_v^{-1}Q_0.$$

Note that $\tilde{h}(\omega) = iJ(\omega) \in C^\infty(-\varepsilon_0, \varepsilon_0)$. Further,

$$Q(i\omega) = \mathcal{M}_{11}Q_0 + \omega\mathcal{M}_{12}\tilde{g}(\omega) = \mathcal{M}_{11}Q_0 + \omega\mathcal{M}_{12}(\omega\tilde{h}(\omega) - h(0)) = \omega^2\mathcal{M}_{12}\tilde{h}(\omega)$$

and the last term is again smooth in ω.

These observations are optional in our case of Klein-Gordon equation, where $Q(i\omega)$ and $P(i\omega)$ happen to be infinitely smooth, but this approach becomes crucial in the cases of the wave and Maxwell equations, where the smoothness is of a finite order, [10, 7]. It is worth to note that similar technique can be applied also for the four-wave solitons case (see e.g. [12]-[14])

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