Communications in **M**athematical **A**nalysis
Volume 14, Number 2, pp. 1–12 (2013) www.math-res-pub.org/cma
ISSN 1938-9787

ON DISCRETE q-EXTENSIONS OF CHEBYSHEV POLYNOMIALS

MESUMA ATAKISHIYEVA *
Facultad de Ciencias, Universidad Autónoma del Estado de Morelos,
C.P. 62250 Cuernavaca, Morelos, México

NATIG ATAKISHIYEV †
Instituto de Matemáticas, Unidad Cuernavaca,
Universidad Nacional Autónoma de México,
C.P. 62251 Cuernavaca, Morelos, México

(Communicated by Vladimir Rabinovich)

Abstract

We study in detail main properties of two families of the basic hypergeometric ${}_2\phi_1$-polynomials, which are natural q-extensions of the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$. In particular, we show that they are expressible as special cases of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with some chosen parameters a, b and c. We derive quadratic transformations that relate these polynomials to the little q-Jacobi polynomials $p_n(x;a,b\,|q)$. Explicit forms of discrete orthogonality relations on a finite interval, q-difference equations and Rodrigues-type difference formulas for these q-Chebyshev polynomials are also given.

AMS Subject Classification: 33D45, 39A70, 47B39.

Keywords: Chebyshev polynomials; q-extension; q-Jacobi polynomials.

1 Introduction

The Chebyshev polynomials find frequent and profound applications in many areas of mathematical analysis such as approximation, series expansions, interpolation, quadrature and integral equations [1, 2]. Hence it is of considerable interest to inquire into the defining of explicit q-extensions of the Chebyshev polynomials, which may be similarly useful in analysis of q-special functions. The interest in this study is motivated by the following circumstance. It is well known that the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ may be regarded as special cases of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ with parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$, respectively. Therefore it appears at first that the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|\,q)$ (which evidently represent q-extensions of the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$) with the particular values of the parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$ would be natural q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$. Under closer examination however, it turns out that the continuous q-Jacobi polynomials $P_n^{(-1/2,-1/2)}(x|\,q)$ and $P_n^{(1/2,1/2)}(x|\,q)$ *are only constant (but q-dependent) multiples* of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$. In other words, the continuous q-Jacobi polynomials $P_n^{(-1/2,-1/2)}(x|\,q)$ and $P_n^{(1/2,1/2)}(x|\,q)$ are, in fact, rescalings of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$; therefore the former two polynomial families are just trivial

*E-mail address: mesuma@servm.fc.uaem.mx
†E-mail address: natig@matcuer.unam.mx

q-extensions of the latter ones. This curious "q-degeneracy" of the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|\,q)$ for the values of the parameters $\alpha=\beta=-1/2$ and $\alpha=\beta=1/2$ had been already noticed by R.Askey and J.A.Wilson in their seminal work [3]. Observe also that nothing essentially changes when one tries to use the connection with the *monic form*[1] of the continuous Rogers q-ultraspherical polynomials $C_n^{(M)}(x;q^{\lambda}|\,q)$, rather than with the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|\,q)$. The q-polynomials $C_n^{(M)}(x;1|\,q)$ are known to provide a q-extension of the Chebyshev polynomials $T_n(x)$, whereas the $C_n^{(M)}(x;q|\,q)$ represent a q-extension of the Chebyshev polynomials $U_n(x)$. But both of these q-extensions are trivial in the above-mentioned sense.

This work is an attempt to explore properties of q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ in terms of the basic hypergeometric ${}_2\phi_1$-polynomials, which were introduced in a recent paper [4] devoted to the study of Fourier integral transforms for the q-Fibonacci and q-Lucas polynomials. We prove that these two q-Chebyshev families are expressible as special cases of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with particularly chosen parameters a, b and c. Thus it becomes apparent that the required q-Chebyshev polynomials have been "in hiding" within the Askey q-scheme at one level higher than the continuous q-Jacobi polynomials $P_n^{(\alpha,\beta)}(x|\,q)$. We use this connection with the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ in order to establish an explicit form of the discrete orthogonality relation for these q-Chebyshev polynomials.

The paper is organized as follows. In section 2 we determine three-term recurrence relations for the q-Chebyshev polynomials under study in order to clarify their connections with the big q-Jacobi polynomials. Quadratic transformations, relating them with the little q-Jacobi polynomials are derived in section 3. In section 4 we present explicit forms of discrete orthogonality relations on a finite interval, q-difference equations and Rodrigues-type difference formulas for these q-Chebyshev polynomials. Some conclusions are offered in section 5. The Appendix contains the derivation of two transformation formulas between basic hypergeometric ${}_2\phi_1$ and ${}_3\phi_2$ polynomials, associated with q-extensions of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$.

Throughout this exposition we employ standard notation of the theory of special functions (see, for example, [5]–[7]).

2 Connections with Big q-Jacobi Polynomials

Recall that the Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$ are explicitly given in terms of the hypergeometric ${}_2F_1$-polynomials as

$$T_0(z)=1\,,\qquad T_n(z)={}_2F_1\Big(-n,n;\,1/2\,\Big|\,\frac{1-z}{2}\Big)=2^{n-1}z^n\,{}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};1-n\,\Big|\,1/z^2\right),\qquad n\geq 1, \tag{2.1}$$

and

$$U_n(z)=(n+1)\,{}_2F_1\Big(-n,n+2;\,3/2\,\Big|\,\frac{1-z}{2}\Big)=(2z)^n\,{}_2F_1\left(-\frac{n}{2},\frac{1-n}{2};-n\,\Big|\,1/z^2\right),\qquad n\geq 0\,, \tag{2.2}$$

respectively. The Chebyshev polynomials $T_n(x)$ are generated by the three-term recurrence relation

$$2z\,T_n(z)\,=\,T_{n+1}(z)+T_{n-1}(z)\,,\qquad n\geq 1\,, \tag{2.3}$$

with the initial conditions $T_0(z)=1$ and $T_1(z)=z$; whereas the Chebyshev polynomials $U_n(x)$ are governed by the same recurrence (2.3) but for $n\geq 0$ and initial assignment $U_{-1}(z)=0$ and $U_0(z)=1$.

As was noticed in [4], two q-polynomial families of degree n in the variable x, defined by

$$p_n^{(T)}(x\,|\,q)=2^{n-1}x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};\,q^{2(1-n)}\,\Big|\,q^2;\,q^2x^{-2}\Big),\qquad n\geq 1,\qquad p_0^{(T)}(x\,|\,q)=1\,, \tag{2.4}$$

[1]We recall that an arbitrary polynomial $p_n(x)=\sum_{k=0}^{n}c_{n,k}\,x^k$ of degree n in the variable x can be written in the *monic form* $p_n^{(M)}(x)=c_{n,n}^{-1}\,p_n(x)=x^n+c_{n,n}^{-1}\sum_{k=0}^{n-1}c_{n,k}\,x^k$ just by changing its normalization.

$$p_n^{(U)}(x\,|\,q) = (2x)^n\,{}_2\phi_1\Big(q^{-n}, q^{1-n};\, q^{-2n}\,\Big|\,q^2\,;\, q^2x^{-2}\Big), \qquad n \geq 0\,, \qquad 0 < q < 1\,, \tag{2.5}$$

represent very natural q-extensions of the Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $U_n(x)$, respectively. For checking this statement one just has to bear in mind the well-known limit property

$$\lim_{q\to 1}\,{}_2\phi_1\Big(q^{-n}, q^a\,;\, q^b\,\Big|\,q; z\Big) = {}_2F_1(-n, a\,;\, b\,|z) \tag{2.6}$$

of the q-hypergeometric ${}_2\phi_1$-polynomials (see, for example, section 1.10, p. 15 in [7]). Then from (2.6) it follows at once that the polynomials $p_n^{(T)}(x|\,q)$ and $p_n^{(U)}(x|\,q)$ coincide in the limit as $q \to 1$ with the $T_n(x)$ and $U_n(x)$, given by the second lines in (2.1) and (2.2), respectively.

Note that from (2.4) and (2.5) it is evident that both of these q-polynomials are either reflection symmetric (when degree n is even) or antisymmetric (when degree n is odd), that is,

$$p_n^{(T)}(-x\,|\,q) \,=\, (-1)^n\, p_n^{(T)}(x\,|\,q)\,, \qquad p_n^{(U)}(-x\,|\,q) \,=\, (-1)^n\, p_n^{(U)}(x\,|\,q)\,. \tag{2.7}$$

The best route to determine whether these q-polynomials (2.4) and (2.5) are related to some "named" families of basic hypergeometric orthogonal polynomials from the Askey q-scheme [7], is first to find three-term recurrence relations, associated with them.

Let us start with (2.4) and slightly simplify its explicit form,

$$\begin{aligned} p_n^{(T)}(x\,|\,q) &= 2^{(n-1)}x^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(q^{-n}, q^{1-n}; q^2)_k}{(q^{2(1-n)}, q^2; q^2)_k}\, q^{2k}x^{-2k} = 2^{(n-1)}x^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{(q^{-n}; q)_{2k}\, q^{2k}}{(q^{2(1-n)}, q^2; q^2)_k}\, x^{-2k} \\ &= (q;q)_n\, 2^{(n-1)}x^n \sum_{k=0}^{\lfloor n/2\rfloor} \frac{q^{k(2k-2n+1)}\, x^{-2k}}{(q;q)_{n-2k}\,(q^{2(1-n)}, q^2; q^2)_k}\,, \end{aligned} \tag{2.8}$$

by using the relation $(z, qz; q^2) = (z;q)_{2n}$ at the first step and the identity

$$(q^{-n};q)_{2k} = \frac{(q;q)_n}{(q;q)_{n-2k}}\, q^{k(2k-2n-1)}\,, \qquad 0 \leq k \leq \lfloor n/2 \rfloor\,,$$

at the second one. Observe that the symbol $\lfloor x \rfloor$ in (2.8) denotes the greatest integer in x and we have employed the conventional notation $(z_1, z_2, ..., z_k\,; q)_n := \prod_{j=1}^k (z_j\,; q)_n$ for products of q-shifted factorials $(z_j\,; q)_n$, $j = 1, 2, ..., k$.

Let us assume now that n is odd, $n = 2m+1$. Then from (2.8) one obtains that

$$\begin{aligned} p_{2m+1}^{(T)}(x\,|\,q) \,&=\, (q;q)_{2m+1}\, x(2x)^{2m} \sum_{k=0}^{m} \frac{q^{k(2k-4m-1)}\, x^{-2k}}{(q;q)_{2m+1-2k}\,(q^{-4m}, q^2; q^2)_k} \\ &= (q;q)_{2m}\, x(2x)^{2m} \sum_{k=0}^{m} \frac{(1-q^{2m+1})(1-q^{2k-4m})}{(1-q^{-4m})(1-q^{2m-2k+1})}\, \frac{q^{k(2k-4m-1)}\, x^{-2k}}{(q;q)_{2(m-k)}\,(q^{2(1-2m)}, q^2; q^2)_k}\,, \end{aligned} \tag{2.9}$$

upon employing the relations

$$(1-z)(zq;q)_k \,=\, (z;q)_{k+1} \,=\, (1-zq^k)(z;q)_k\,. \tag{2.10}$$

Finally, use a readily verified identity

$$\frac{(1-q^{2m+1})(1-q^{2k-4m})}{(1-q^{-4m})(1-q^{2m-2k+1})} \,=\, q^{2k} \,+\, \frac{(1-q^{1-2m})(1-q^{2k})}{(1-q^{-4m})(1-q^{2m-2k+1})}\,, \qquad 0 \leq k \leq m\,,$$

to represent (2.9) as

$$p_{2m+1}^{(T)}(x\,|\,q) = (q;q)_{2m}\,x(2x)^{2m}\sum_{k=0}^{m}\frac{q^{k(2k-4m+1)}\,x^{-2k}}{(q;q)_{2(m-k)}\,(q^{2(1-2m)},q^2;q^2)_k}$$

$$-\frac{q^{6m-1}\,(q;q)_{2m-1}}{(1+q^{2m})(1+q^{2m-1})}\,x(2x)^{2m}\sum_{k=1}^{m}\frac{q^{k(2k-4m-1)}\,x^{-2k}}{(q;q)_{2(m-k)+1}\,(q^{4(1-m)},q^2;q^2)_{k-1}}$$

$$= 2x\,p_{2m}^{(T)}(x\,|\,q) - \frac{2\,q^{2m}\,(q;q)_{2m-1}\,(2x)^{2m-1}}{(1+q^{2m})(1+q^{2m-1})}\sum_{l=0}^{m-1}\frac{q^{l[2l-2(2m-1)+1]}\,x^{-2l}}{(q;q)_{2m-1-2l}\,(q^{2[1-(2m-1)]},q^2;q^2)_l}$$

$$= 2x\,p_{2m}^{(T)}(x\,|\,q) - \frac{4\,q^{2m}}{(1+q^{2m})(1+q^{2m-1})}\,p_{2m-1}^{(T)}(x\,|\,q)\,. \tag{2.11}$$

Similarly, if one assumes that the degree n in (2.8) is even, $n = 2m$, then by the same reasoning one arrives at the three-term recurrence relation between the polynomials $p_{2m}^{(T)}(x|\,q)$, $p_{2m-1}^{(T)}(x|\,q)$ and $p_{2m-2}^{(T)}(x|\,q)$. Thus we conclude that the general (*i.e.*, valid for both even and odd degrees n) recurrence formula for the q-polynomials (2.4) is

$$p_{n+1}^{(T)}(x\,|\,q) = 2x\,p_n^{(T)}(x\,|\,q) - \frac{4\,q^n}{(1+q^n)(1+q^{n-1})}\,p_{n-1}^{(T)}(x\,|\,q), \qquad n\geq 1\,. \tag{2.12}$$

Using the same considerations *mutatis mutandis*, one derives the three-term recurrence relation for the second family of q-polynomials (2.5):

$$p_{n+1}^{(U)}(x\,|\,q) = 2x\,p_n^{(U)}(x\,|\,q) - \frac{4\,q^{n-1}}{(1+q^n)(1+q^{n+1})}\,p_{n-1}^{(U)}(x\,|\,q)\,, \qquad n\geq 0\,, \qquad p_{-1}^{(U)}(x\,|\,q) = 0\,. \tag{2.13}$$

Now we are in a position to establish that the q-extensions (2.4) and (2.5) of the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are in fact connected with the big q-Jacobi polynomials

$$P_n(x;a,b,c;q) := {}_3\phi_2\left(q^{-n},ab\,q^{n+1},x;aq,cq\,\middle|\,q;q\right) \tag{2.14}$$

with some particularly chosen parameters a, b and c. Indeed, recall that the *monic form*

$$P_n^{(M)}(x;a,a,-a;q) = \frac{(a^2q^2;q^2)_n}{(a^2q^{n+1};q)_n}\,P_n(x;a,a,-a;q) \tag{2.15}$$

of the big q-Jacobi polynomials (2.14) with the parameters $a = b = -c$ satisfies the three-term recurrence relation

$$P_{n+1}^{(M)}(x;a,a,-a;q) = x\,P_n^{(M)}(x;a,a,-a;q) - \gamma_n(a;q)\,P_{n-1}^{(M)}(x;a,a,-a;q) \tag{2.16}$$

with the coefficients (see (14.5.4), p. 439 in [7])

$$\gamma_n(a;q) = \frac{a^2\,q^{n+1}(1-q^n)(1-a^2q^n)}{(1-a^2q^{2n-1})(1-a^2q^{2n+1})}\,.$$

For $a = q^{-1/2}$ the recurrence (2.16) clearly reduces to

$$P_{n+1}^{(M)}(x;q^{-1/2},q^{-1/2},-q^{-1/2};q) = x\,P_n^{(M)}(x;q^{-1/2},q^{-1/2},-q^{-1/2};q)$$

$$-\frac{q^n}{(1+q^n)(1+q^{n-1})}\,P_{n-1}^{(M)}(x;q^{-1/2},q^{-1/2},-q^{-1/2};q)\,, \tag{2.17}$$

whereas the choice of $a = q^{1/2}$ in (2.16) leads to

$$P_{n+1}^{(M)}(x;q^{1/2},q^{1/2},-q^{1/2};q) = x\,P_n^{(M)}(x;q^{1/2},q^{1/2},-q^{1/2};q) - \frac{q^{n-1}}{(1+q^n)(1+q^{n+1})}\,P_{n-1}^{(M)}(x;q^{1/2},q^{1/2},-q^{1/2};q)\,. \tag{2.18}$$

On comparing (2.17) and (2.18) with (2.12) and (2.13), respectively, one thus concludes that

$$p_0^{(T)}(x\,|\,q) = 1, \qquad p_n^{(T)}(x\,|\,q) = 2^{n-1}\,P_n^{(M)}(x;q^{-1/2},q^{-1/2},-q^{-1/2};q) = 2^{n-1}\,\frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\Big(q^{-n},\,q^n,x;q^{1/2},-q^{1/2}\,\Big|\,q;q\Big)\,, \qquad n \geq 1\,, \tag{2.19}$$

and

$$p_n^{(U)}(x\,|\,q) = 2^n\,P_n^{(M)}(x;q^{1/2},q^{1/2},-q^{1/2};q) = 2^n\,\frac{(q^3;q^2)_n}{(q^{n+2};q)_n}\,{}_3\phi_2\Big(q^{-n},\,q^{n+2},x;q^{3/2},-q^{3/2}\,\Big|\,q;q\Big)\,, \quad n \geq 0\,. \tag{2.20}$$

Evidently, these representations (2.19) and (2.20) in terms of the big q-Jacobi polynomials (2.14) agree with the initial definitions (2.4) and (2.5) of the q-polynomials $p_n^{(T)}(x|\,q)$ and $p_n^{(U)}(x|\,q)$, only if two transformation formulas

$$x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{2(1-n)}\Big|\,q^2;q^2x^{-2}\Big) = \frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\Big|\,q;q\Big), \tag{2.21}$$

$$x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{-2n}\Big|\,q^2;q^2x^{-2}\Big) = \frac{(q^3;q^2)_n}{(q^{n+2};q)_n}\,{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\Big|\,q;q\Big), \tag{2.22}$$

between ${}_2\phi_1$ (with the base q^2) and ${}_3\phi_2$ (with the base q) basic polynomials are valid. Direct proofs of these identities are given in Appendix.

3 Quadratic Transformations

It turns out that, in addition to (2.19) and (2.20), both symmetric or antisymmetric cases of the q-polynomial families (2.4) and (2.5) can be separately expressed in terms of the little q-Jacobi polynomials, defined as (see, for example, (14.12.1), p. 482 in [7])

$$p_n(x;a,b\,|\,q) := {}_2\phi_1(q^{-n},ab\,q^{n+1};aq\,|\,q;qx)\,. \tag{3.1}$$

Indeed, let us apply first the transformation of terminating ${}_2\phi_1$ series (see (1.13.15), p. 20 in [7])

$${}_2\phi_1(q^{-n},a;b|\,q;z) = \frac{(a;q)_n}{(b;q)_n}\,q^{-n(n+1)/2}(-z)^n\,{}_2\phi_1\left(q^{-n},q^{1-n}/b;q^{1-n}/a\,\middle|\,q;\frac{bq^{n+1}}{az}\right) \tag{3.2}$$

to the q-polynomials of even degree $p_{2m}^{(T)}(x|\,q)$, where m is an arbitrary nonnegative integer. This results in the relation

$$\begin{aligned} p_{2m}^{(T)}(x\,|\,q) &= x(2x)^{2m-1}\,{}_2\phi_1\Big(q^{-2m},q^{1-2m};\,q^{2(1-2m)}\,\Big|\,q^2\,;\,q^2\,x^{-2}\Big) \\ &= (-4)^m\,q^{-m(m-1)}\,\frac{(q^{1-2m};q^2)_m}{2(q^{2(1-2m)};q^2)_m}\,{}_2\phi_1\Big(q^{-2m},q^{2m};\,q\,\Big|\,q^2\,;qx^2\Big) \\ &= (-4\,q^m)^m\,\frac{(q;q^2)_m}{2(q^{2m};q^2)_m}\,p_m\Big(q^{-1}x^2;q^{-1},q^{-1}\Big|\,q^2\Big)\,, \qquad m \geq 1\,. \end{aligned} \tag{3.3}$$

Similarly, in the case of the q-polynomials of odd degree $p^{(T)}_{2m+1}(x|q)$ one obtains, by using (3.2), that

$$\begin{aligned} p^{(T)}_{2m+1}(x\,|\,q) &= x(2x)^{2m}\,{}_2\phi_1\Big(q^{-2m-1},q^{-2m};\,q^{-4m}\,\Big|\,q^2;\,q^2x^{-2}\Big) \\ &= (-4)^m\,q^{-m(m-1)}\,\frac{(q^{-1-2m};q^2)_m}{(q^{-4m};q^2)_m}\,x\,{}_2\phi_1\Big(q^{-2m},q^{2m};\,q\,\Big|\,q^2;q^2x^2\Big) \\ &= (-4\,q^m)^m\,\frac{(q^3;q^2)_m}{(q^{2(m+1)};q^2)_m}\,x\,p_m\Big(q^{-1}x^2;q,q^{-1}\Big|\,q^2\Big)\,, \qquad m\geq 0\,. \end{aligned} \tag{3.4}$$

Thus, q-extensions (2.4) of the Chebyshev polynomials $T_n(x)$ can be written it terms of the little q-Jacobi polynomials (3.1) as

$$\begin{aligned} p^{(T)}_{2m}(x\,|\,q) &= (-4\,q^m)^m\,\frac{(q;q^2)_m}{2(q^{2m};q^2)_m}\,p_m\Big(q^{-1}x^2;q^{-1},q^{-1}\Big|\,q^2\Big)\,, \\ p^{(T)}_{2m+1}(x\,|\,q) &= (-4\,q^m)^m\,\frac{(q^3;q^2)_m}{(q^{2(m+1)};q^2)_m}\,x\,p_m\Big(q^{-1}x^2;q,q^{-1}\Big|\,q^2\Big)\,. \end{aligned} \tag{3.5}$$

Exactly in the same manner one obtains that q-extensions (2.5) of the Chebyshev polynomials $U_n(x)$ can be represented as

$$\begin{aligned} p^{(U)}_{2n}(x\,|\,q) &= (-4)^n\,q^{n(n+2)}\,\frac{(q;q^2)_n}{(q^{2(n+1)};q^2)_n}\,p_n\Big(q^{-3}x^2;q^{-1},q\Big|\,q^2\Big)\,, \\ p^{(U)}_{2n+1}(x\,|\,q) &= (-4)^n\,q^{n(n+2)}\,\frac{2(q^3;q^2)_n}{(q^{2(n+2)};q^2)_n}\,x\,p_n\Big(q^{-3}x^2;q,q\Big|\,q^2\Big)\,. \end{aligned} \tag{3.6}$$

Notice that from the well-known limit property (*cf.* (14.12.15) on p. 485 in [7])

$$\lim_{q\to 1}\,p_n\Big(x;\,q^a,q^b\,\Big|\,q\Big) \;=\; \frac{n!}{(\alpha+1)_n}\,P_n^{(\alpha,\beta)}(1-2x) \tag{3.7}$$

of the little q-Jacobi polynomials (3.1), it follows that in the limit as $q\to 1$ the quadratic transformations (3.5) and (3.6) reduce to the relations

$$T_{2m}(x) \;=\; \frac{m!}{(1/2)_m}\,P_m^{(-1/2,-1/2)}(2x^2-1)\,, \qquad T_{2m+1}(x) \;=\; \frac{m!}{(1/2)_m}\,x\,P_m^{(-1/2,1/2)}(2x^2-1)\,, \tag{3.8}$$

and

$$U_{2m}(x) \;=\; \frac{m!}{(1/2)_m}\,P_m^{(1/2,-1/2)}(2x^2-1)\,, \qquad U_{2m+1}(x) \;=\; \frac{2(m+1)!}{(3/2)_m}\,x\,P_m^{(1/2,1/2)}(2x^2-1)\,, \tag{3.9}$$

respectively. It should also be observed that the transformations (3.8) and (3.9) for the Chebyshev polynomials $T_n(x)$ and $U_n(x)$ are special cases of the quadratic transformation (*cf.* Remarks on p. 224 in [7])

$$C^{(\lambda;M)}_{2n}(x) \;=\; \frac{n!}{(\lambda+n)_n}\,P_n^{(\lambda-1/2,-1/2)}(2x^2-1)\,, \qquad C^{(\lambda;M)}_{2n+1}(x) \;=\; \frac{n!}{(\lambda+n+1)_n}\,x\,P_n^{(\lambda-1/2,1/2)}(2x^2-1)\,, \tag{3.10}$$

for the *monic* Gegenbauer (or ultraspherical) polynomials $C_n^{(\lambda;M)}(x)$, defined as (see (9.8.19) and (9.8.22) on p. 222 in [7])

$$C_n^{(\lambda;M)}(x) := \frac{n!}{2^n(\lambda)_n}\,C_n^{(\lambda)}(x) = \frac{(\lambda+n)_\lambda}{2^{2\lambda+n-1}(1/2)_\lambda}\,{}_2F_1\left(-n,n+2\lambda;\lambda+1/2\,\middle|\,\frac{1-x}{2}\right). \tag{3.11}$$

Indeed, taking into account that $C_n^{(0;M)}(x) = 2^{1-n}\,T_n(x)$ and $C_n^{(1;M)}(x) = 2^{-n}\,U_n(x)$ by the defintion (3.11), it is readily checked that (3.8) is a special case of (3.10) with $\lambda = 0$ and (3.9) is a special case of (3.10) with $\lambda = 1$.

It should also be noted that the quadratic transformations (3.5) and (3.6) in terms of the little q-Jacobi polynomials were already mentioned in [4], but without proofs and their limits (3.8) and (3.9) as $q\to 1$; a brief proof of (3.5) and (3.6) is given above for the sake of completeness.

4 Main Characteristics of q-Chebyshev Polynomials

A benefit from establishing the representations (2.19) and (2.20) for the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$ in terms of the big q-Jacobi polynomials (2.14) is that these connections enable one to deduce their main properties from the well-known properties of the latter ones, $P_n(x;a,b,c;q)$. To illustrate this point, we touch on here only three important characteristics of the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$: explicit forms of q-difference equations, discrete orthogonality relations and Rodrigues-type formulas.

It is known that the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with the parameters $a=b=-c$ are solutions of a q-difference equation:

$$\Big[\Big(a^2q^{n+1}+q^{-n}\Big)x^2-a^2q(1+q)\Big]\,p_n(x)=a^2q(x^2-1)\,p_n(qx)+(x^2-a^2q^2)\,p_n(q^{-1}x)\,, \tag{4.1}$$

where $p_n(x)=P_n(x;a,b,c;q)$ (see (14.5.5) on p. 439 in [7]). Hence q-difference equations for the q-Chebyshev polynomials $p_n^{(T)}(x|q)$ and $p_n^{(U)}(x|q)$ are special cases of (4.1) with the parameter $a=q^{-1/2}$ and $a=q^{1/2}$, respectively; that is,

$$\Big[(q^n+q^{-n})\,x^2-(1+q)\Big]\,p_n^{(T)}(x\,|\,q)=(x^2-1)\,p_n^{(T)}(qx\,|\,q)+(x^2-q)\,p_n^{(T)}(q^{-1}x\,|\,q)\,,$$

$$\Big[(q^{n+2}+q^{-n})\,x^2-q^2(1+q)\Big]\,p_n^{(U)}(x\,|\,q)=q^2(x^2-1)\,p_n^{(U)}(qx\,|\,q)+(x^2-q^3)\,p_n^{(U)}(q^{-1}x\,|\,q). \tag{4.2}$$

Recall also that the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ with the parameters $a=b=-c$ satisfy the discrete orthogonality relation

$$\int_{-aq}^{aq}\frac{(x^2/a^2;q^2)_\infty}{(x^2;q^2)_\infty}\,P_m(x;a,a,-a;q)\,P_n(x;a,a,-a;q)\,d_qx$$
$$=2(1-q^2)\,q^{(n+1)(n+2)/2}\,\frac{(q^2;q^2)_\infty}{(a^2q^2;q^2)_\infty^2}\,(a^2q^2,-q^2;q)_\infty\,\frac{a^{2n+1}(1-a^2q)(q;q)_n}{(1-a^2q^{2n+1})(a^2q;q)_n}\,\delta_{mn}\,, \tag{4.3}$$

where the q-integral is defined as (see (14.5.2) and (1.15.7) in [7])

$$\int_{-a}^{a}f(x)\,d_qx:=a(1-q)\sum_{n=0}^{\infty}\Big[f(aq^n)+f(-aq^n)\Big]q^n\,.$$

For $a=q^{-1/2}$ from (4.3) one now gets at once, by employing (2.19) and (2.15), that

$$\int_{-q^{1/2}}^{q^{1/2}}\frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty}\,p_m^{(T)}(x\,|\,q)\,p_n^{(T)}(x\,|\,q)\,d_qx=2q^{1/2}\,\frac{(-q;q)_\infty}{(q^3;q^2)_\infty}\,(q^2;q^2)_\infty^2\,c_n\,\delta_{mn}\,, \tag{4.4}$$

where

$$c_0=1\,,\qquad c_n=4^{n-1}q^{n(n+1)/2}\,\frac{(1-q^n)(q;q^2)_n^2}{(1+q^n)(q^n;q)_n^2}\,,\qquad n\ge 1\,.$$

In a like manner, when $a=q^{1/2}$ one finds from (4.3), by employing (2.20) and (2.15), that

$$\int_{-q^{3/2}}^{q^{3/2}}\frac{(q^{-1}x^2;q^2)_\infty}{(x^2;q^2)_\infty}\,p_m^{(U)}(x\,|\,q)\,p_n^{(U)}(x\,|\,q)\,d_qx=2\,q^{3/2}\,\frac{(-q;q)_\infty}{(q^3;q^2)_\infty}\,(q^2;q^2)_\infty^2\,c_n\,\delta_{mn}\,, \tag{4.5}$$

where

$$c_n=4^nq^{n(n+5)/2}\,\frac{(q;q^2)_{n+1}^2}{(1+q^{n+1})(q^{n+1};q)_{n+1}^2}\,,\qquad n\ge 0\,.$$

Another important property of the big q-Jacobi polynomials $P_n(x;a,b,c;q)$ is described by the Rodrigues-type formula

$$P_n(x;a,b,c;q)\,w(x;a,b,c;q) \,=\, \frac{[ac(1-q)]^n}{(aq,cq;q)_n}\,q^{n(n+1)}\left(\mathcal{D}_q\right)^n w(x;aq^n,bq^n,cq^n;q)\,, \tag{4.6}$$

where $\mathcal{D}_q$ is the q-derivative operator (see (1.15.1) on p. 24 in [7]) and the orthogonality weight function $w(x;a,b,c;q)$ is defined as ((14.5.10), p. 440 in [7])

$$w(x;a,b,c;q) \,:=\, \frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty}\,. \tag{4.7}$$

Hence, from (4.6) and (4.7) it follows, upon using (2.19) and (2.20), that the Rodrigues-type formulas for the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ are

$$p_n^{(T)}(x\,|\,q)\,\frac{(qx^2;q^2)_\infty}{(x^2;q^2)_\infty} = \Big(-2q^n\Big)^n\,\frac{(1-q)^n}{2(q^n;q)_n}\Big(\mathcal{D}_q\Big)^n\,\frac{(q^{1-2n}x^2;q^2)_\infty}{(x^2;q^2)_\infty}\,, \qquad n\geq 1\,,$$

$$p_n^{(U)}(x\,|\,q)\,\frac{(q^{-1}x^2;q^2)_\infty}{(x^2;q^2)_\infty} = \Big(-2q^{n+2}\Big)^n\,\frac{(1-q)^n}{(q^{n+2};q)_n}\Big(\mathcal{D}_q\Big)^n\,\frac{(q^{-1-2n}x^2;q^2)_\infty}{(x^2;q^2)_\infty}, \qquad n\geq 0\,. \tag{4.8}$$

In closing this section, we remark of the following. First, note that it is not difficult to determine also forward and backward shift operators and generating functions for the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ in exactly the same way as above, but this task is left to the reader. Second, since the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$ satisfy the same three-term recurrence relation (2.3) but with different initial assignments, they are known to be interconnected by the relation

$$2\,T_n(x) \,=\, U_n(x) - U_{n-2}(x)\,, \qquad n\geq 1\,, \qquad U_{-1}(x)=0\,. \tag{4.9}$$

Hence one may wonder whether the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ also enjoy the similar property of type (4.9), although they are governed by two distinct three-term recurrence relations (2.12) and (2.13), respectively. A link in question between the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ turns out to be of the form

$$2\,p_n^{(T)}(x\,|\,q) \,=\, p_n^{(U)}(x\,|\,q) - \frac{4q}{(1+q^n)(1+q^{n-1})}\,p_{n-2}^{(U)}(x\,|\,q)\,, \qquad n\geq 1\,, \qquad p_{-1}^{(U)}(x\,|\,q)=0\,. \tag{4.10}$$

This q-extension of the classical relation (4.9) is not difficult to derive by using the explicit forms (2.4) and (2.5) of the q-polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$, and the identities

$$\Big(q^{-n};q\Big)_{2l+2} = \Big(1-q^{-n}\Big)\Big(1-q^{1-n}\Big)\Big(q^{2-n};q\Big)_{2l}\,,$$

$$\Big(q^{-2n};q^2\Big)_{l+2} = \Big(1-q^{-2n}\Big)\Big(1-q^{2(1-n)}\Big)\Big(q^{2(2-n)};q^2\Big)_{l}\,,$$

for the q-shifted factorial $(z;q)_n$.

5 Concluding Remarks

We have studied in detail the main properties of two families of the basic hypergeometric ${}_2\phi_1$-polynomials, defined by (2.4) and (2.5), which represent compact forms of q-extensions of the classical Chebyshev polynomials $T_n(x)$ and $U_n(x)$. They are shown to satisfy the discrete orthogonality relations (4.4) and (4.5) on a finite interval. It should be noted that although these discrete q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ are of clear interest on their own, there is an additional motivation to study them. As we have already remarked, the q-polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ were first arisen in a paper [4], devoted mainly to the evaluation of Fourier integral transforms for q-Fibonacci and q-Lucas polynomials. It is worthwhile to emphasize that the q-Chebyshev polynomials $p_n^{(T)}(x\,|\,q)$ and $p_n^{(U)}(x\,|\,q)$ had emerged in [4] only because they are intimately associated with the very natural extensions of the Fibonacci and Lucas polynomials $p_n^{(F)}(x)$ and $p_n^{(L)}(x)$, defined as

$$p_n^{(F)}(x\,|\,q) = \mathrm{i}^{-n}\, p_n^{(U)}(\mathrm{i}x\,|\,q)\,, \qquad p_n^{(L)}(x\,|\,q) = \mathrm{i}^{-n}\, p_n^{(T)}(\mathrm{i}x\,|\,q)\,, \tag{5.1}$$

respectively. These q-extensions of the Fibonacci and Lucas polynomials are different from and simpler than those q-families, introduced and studied recently by Cigler and Zeng in [8]-[10]. Obviously, the present results also provide us with an insight into corresponding properties of the q-Fibonacci and q-Lucas polynomials $p_n^{(F)}(x\,|\,q)$ and $p_n^{(L)}(x\,|\,q)$, which are direct consequences of the links (5.1).

6 Appendix

I. In order to give a direct proof of a transformation formula

$$x^n\,{}_2\phi_1\left(q^{-n}, q^{1-n}; q^{2(1-n)}\,\middle|\, q^2; q^2x^{-2}\right) = \frac{(q;q^2)_n}{(q^n;q)_n}\,{}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2}\,\middle|\, q;q\right) \tag{6.1}$$

between ${}_2\phi_1$ (with the base q^2) and ${}_3\phi_2$ (with the base q) basic polynomials, which was stated in section 2, we start with the defining relation for the hypergeometric ${}_3\phi_2$-polynomial on the right-hand side of (6.1) and represent it first as

$${}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2}\,\middle|\, q;q\right) := \sum_{k=0}^{n} \frac{(q^{-n}, q^n, x;q)_k}{(q^{1/2}, -q^{1/2}, q;q)_k}\, q^k = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(q^n, x;q)_k}{(q;q^2)_k}\, q^{k(k+1-2n)/2}, \tag{6.2}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}_q$ stands for the q-binomial coefficient,

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}\,, \tag{6.3}$$

and we have employed the identities $(z,-z;q)_n = (z^2;q^2)_n$ and

$$\frac{(q^{-n};q)_k}{(q;q)_k} = (-1)^k \begin{bmatrix} n \\ k \end{bmatrix}_q q^{k(k-1-2n)/2}\,. \tag{6.4}$$

The next step is to use the expansion

$$(x;q)_k = \sum_{l=0}^{k} q^{l(l-1)/2} \begin{bmatrix} k \\ l \end{bmatrix}_q (-x)^l \tag{6.5}$$

on the right-hand side of (6.2) and then to reverse the order of summation in it with respect to the indices k and l. This results in the relation

$$\begin{aligned} {}_3\phi_2\left(q^{-n}, q^n, x; q^{1/2}, -q^{1/2}\,\middle|\, q;q\right) &= (q;q)_n \sum_{k=0}^{n} \frac{(-1)^k\,(q^n;q)_k}{(q;q)_{n-k}\,(q;q^2)_k}\, q^{k(k+1-2n)/2} \sum_{l=0}^{k} \frac{(-x)^l\, q^{l(l-1)/2}}{(q;q)_l\,(q;q)_{k-l}} \\ &= (q;q)_n \sum_{l=0}^{n} \frac{(-x)^l}{(q;q)_l}\, q^{l(l-1)/2} \sum_{k=l}^{n} \frac{(-1)^k\,(q^n;q)_k\, q^{k(k+1-2n)/2}}{(q;q)_{n-k}\,(q;q)_{k-l}\,(q;q^2)_k} \\ &= (q;q)_n \sum_{l=0}^{n} \frac{q^{l(l-n)}}{(q;q)_l}\, x^l \sum_{j=0}^{n-l} \frac{(-1)^j\, q^{j[j+1-2(n-l)]/2}\,(q^n;q)_{l+j}}{(q;q)_j\,(q;q)_{n-l-j}\,(q;q^2)_{l+j}}\,. \end{aligned} \tag{6.6}$$

The last sum over the index j in (6.6) can be simplified by use of the identity (see, for example, (1.8.10) on p. 12 in [7]) $(z;q)_{n+k}=(z;q)_n\,(zq^n;q)_k$ in order to represent factors $(q^n;q)_{l+j}$ and $(q;q^2)_{l+j}$ as

$$(q^n;q)_{l+j} = (q^n;q)_l\,(q^{n+l};q)_j\,, \qquad (q;q^2)_{l+j} = (q;q^2)_l\,(q^{2l+1};q^2)_j\,.$$

Consequently,

$$\begin{aligned}
{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\Big|q;q\Big) &= \sum_{l=0}^{n}\begin{bmatrix} n\\ l\end{bmatrix}_q \frac{(q^n;q)_l}{(q;q^2)_l}\Big(x\,q^{l-n}\Big)^l\sum_{j=0}^{n-l}\frac{(q^{n+l},q^{l-n};q)_j\,q^j}{(q^{l+1/2},-q^{l+1/2},q;q)_j}\\
&= \sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{(q^n;q)_{n-k}}{(q;q^2)_{n-k}}\Big(x\,q^{-k}\Big)^{n-k}\sum_{j=0}^{k}\frac{(q^{2n-k},q^{-k};q)_j\,q^j}{(q^{n-k+1/2},-q^{n-k+1/2},q;q)_j}\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\sum_{k=0}^{n} q^{k(k+1-2n)/2}\begin{bmatrix} n\\ k\end{bmatrix}_q \frac{(q^{1-2n};q^2)_k}{(q^{1-2n};q)_k}\,x^{n-k}\sum_{j=0}^{k}\frac{(q^{2n-k},q^{-k};q)_j\,q^j}{(q^{n-k+1/2},-q^{n-k+1/2},q;q)_j}\,,
\end{aligned} \tag{6.7}$$

where at the last step we have employed the identity

$$(z;q)_{n-k} = (-1)^k\,q^{k(k+1-2n)/2}\,\frac{(z;q)_n\,z^{-k}}{(q^{1-n}/z;q)_k}\,.$$

The sum over the index j in (6.7) can be now evaluated by an Andrew's terminating q-analogue of ${}_3F_2$ sum (see (II.17), p. 355 in [5])

$${}_3\phi_2\Big(q^{-k},\,a^2q^{k+1},\,0;\,a\,q,-a\,q\,\Big|\,q;q\Big) = \begin{cases} \Big(-a^2q^{m+1}\Big)^m\,\dfrac{(q;q^2)_m}{(a^2q^2;q^2)_m}\,, & k=2m\,,\\[2ex] 0\,, & k=2m+1\,,\end{cases} \tag{6.8}$$

with $a=q^{n-k-1/2}$ in the case of (6.7). Thus in the sum over the index k on the right-hand side of (6.7) only terms with the even $k=2m$, $0\le m\le\lfloor n/2\rfloor$, do give nonzero contributions and therefore

$$\begin{aligned}
{}_3\phi_2\Big(q^{-n},q^n,x;q^{1/2},-q^{1/2}\Big|q;q\Big) &= \frac{(q^n;q)_n}{(q;q^2)_n}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\,q^{m(1-m)}\begin{bmatrix} n\\ 2m\end{bmatrix}_q \frac{(q^{1-2n};q^2)_{2m}}{(q^{1-2n};q)_{2m}}\,\frac{(q;q^2)_m\,x^{n-2m}}{(q^{2n-4m+1};q^2)_m}\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\,q^{m(2n-3m)}\,\frac{(q^{2(m-n)+1};q^2)_m}{(q^{2n-4m+1};q^2)_m}\,\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{2(1-n)},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{2(1-n)},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m\\
&= \frac{(q^n;q)_n}{(q;q^2)_n}\,x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{2(1-n)}\Big|q^2;q^2x^{-2}\Big)\,,
\end{aligned} \tag{6.9}$$

where we have repeatedly used the relation $(z;q)_{2m}=(z,qz;q^2)_m$ at the second step and a readily verified identity

$$(-1)^m\,q^{m(2n-3m)}\,(q^{2(m-n)+1};q^2)_m = (q^{2n-4m+1};q^2)_m \tag{6.10}$$

at the third one. This completes the proof of required transformation formula (6.1).

II. In a similar vein, to prove a second transformation formula

$$x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{-2n}\Big|q^2;q^2x^{-2}\Big)=\frac{(q^3;q^2)_n}{(q^{n+2};q)_n}\,{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\Big|q;q\Big), \tag{6.11}$$

we start with the defining relation for the basic hypergeometric polynomial ${}_3\phi_2$ on the right-hand side of (6.11) and evaluate first that

$$\begin{aligned}{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\Big|q;q\Big)&:=\sum_{k=0}^{n}\frac{(q^{-n},q^{n+2},x;q)_k}{(q^{3/2},-q^{3/2},q;q)_k}\,q^k\\&=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(q^{n+2},x;q)_k}{(q^3;q^2)_k}\,q^{k(k+1-2n)/2},\end{aligned} \tag{6.12}$$

by using the relations (6.3) and (6.4). So the next step is to employ the expansion (6.5) on the right-hand side of (6.12) and then to reverse the order of summation in it with respect to the indices k and l. This gives

$$\begin{aligned}{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\Big|q;q\Big)&=(q;q)_n\sum_{k=0}^{n}\frac{(-1)^k\,(q^{n+2};q)_k}{(q;q)_{n-k}\,(q^3;q^2)_k}\,q^{k(k+1-2n)/2}\sum_{l=0}^{k}\frac{(-x)^l\,q^{l(l-1)/2}}{(q;q)_l\,(q;q)_{k-l}}\\&=(q;q)_n\sum_{l=0}^{n}\frac{(-x)^l}{(q;q)_l}\,q^{l(l-1)/2}\sum_{k=l}^{n}\frac{(-1)^k\,(q^{n+2};q)_k\,q^{k(k+1-2n)/2}}{(q;q)_{n-k}\,(q;q)_{k-l}\,(q^3;q^2)_k}\\&=(q;q)_n\sum_{l=0}^{n}\frac{q^{l(l-n)}}{(q;q)_l}\,x^l\sum_{j=0}^{n-l}\frac{(-1)^j\,q^{j[j+1-2(n-l)]/2}\,(q^{n+2};q)_{l+j}}{(q;q)_j\,(q;q)_{n-l-j}\,(q^3;q^2)_{l+j}}\\&=\sum_{l=0}^{n}\begin{bmatrix}n\\l\end{bmatrix}_q\frac{(q^{n+2};q)_l}{(q^3;q^2)_l}\Big(x\,q^{l-n}\Big)^l\sum_{j=0}^{n-l}\frac{(q^{n+l+2},q^{l-n};q)_j\,q^j}{(q^{l+3/2},-q^{l+3/2},q;q)_j}\\&=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(q^{n+2};q)_{n-k}}{(q^3;q^2)_{n-k}}\Big(x\,q^{-k}\Big)^{n-k}\sum_{j=0}^{k}\frac{(q^{2n+2-k},q^{-k};q)_j\,q^j}{(q^{n-k+3/2},-q^{n-k+3/2},q;q)_j}\\&=\frac{(q^{n+2};q)_n}{(q^3;q^2)_n}\sum_{k=0}^{n}q^{k(k+1-2n)/2}\begin{bmatrix}n\\k\end{bmatrix}_q\frac{(q^{-2n-1};q^2)_k}{(q^{-2n-1};q)_k}\,x^{n-k}\sum_{j=0}^{k}\frac{(q^{2n+2-k},q^{-k};q)_j\,q^j}{(q^{n-k+3/2},-q^{n-k+3/2},q;q)_j}\,.\end{aligned} \tag{6.13}$$

The last sum over the index j represents

$${}_3\phi_2\Big(q^{-k},\,q^{2n+2-k},\,0\,;\,q^{n-k+3/2},-q^{n-k+3/2}\,\Big|\,q;q\Big)$$

and can be therefore evaluated by (6.8), but with the parameter $a=q^{n-k+1/2}$. Hence only terms with the even $k=2m$ do contribute into the second sum over the index k in (6.13) and it thus reduces to the expression

$$\begin{aligned}{}_3\phi_2\Big(q^{-n},q^{n+2},x;q^{3/2},-q^{3/2}\Big|q;q\Big)&=\frac{(q^{n+2};q)_n}{(q^3;q^2)_n}\sum_{m=0}^{\lfloor n/2\rfloor}(-1)^m\,q^{m(3-m)}\begin{bmatrix}n\\2m\end{bmatrix}_q\frac{(q^{-2n-1};q^2)_{2m}}{(q^{-2n-1};q)_{2m}}\,\frac{(q;q^2)_m\,x^{n-2m}}{(q^{2n-4m+3};q^2)_m}\\&=\frac{(q^{n+2};q)_n}{(q^3;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}\Big(-q^{2n+2-3m}\Big)^m\frac{(q^{2m-2n-1};q^2)_m}{(q^{2n-4m+3};q^2)_m}\,\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{-2n},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m\\&=\frac{(q^{n+2};q)_n}{(q^3;q^2)_n}\,x^n\sum_{m=0}^{\lfloor n/2\rfloor}\frac{(q^{-n},q^{1-n};q^2)_m}{(q^{-2n},q^2;q^2)_m}\left(\frac{q^2}{x^2}\right)^m=\frac{(q^{n+2};q)_n}{(q^3;q^2)_n}\,x^n\,{}_2\phi_1\Big(q^{-n},q^{1-n};q^{-2n}\Big|q^2;q^2\,x^{-2}\Big)\,,\end{aligned} \tag{6.14}$$

where at the penultimate step we have used the same identity (6.10), but with n replaced by $n+1$. This completes the proof of the transformation formula (6.11).

Acknowledgements

We are grateful to Decio Levi, Orlando Ragnisco and Tom Koornwinder for valuable discussions. The participation of NA in this work has been supported by the DGAPA-UNAM IN101011-3 and SEP-CONACYT 79899 projects " Óptica Matemática"; this work has also been partially supported by the "Proyecto de Redes", PROMEP (México).

References

[1] T. G. Rivlin, *Chebyshev Polynomials : From Approximation Theory to Algebra and Number Theory*, John Wiley, New York 1990.

[2] J. C. Mason and D. C. Handscomb, *Chebyshev Polynomials*, Chapman&Hall/CRC, USA 2003.

[3] R. Askey and J. A. Wilson, Some basic hypergeometric orthogonal polynomials that generalize Jacobi polynomials. *Mem. Am. Math. Soc.*, **54** (1985), pp 1-55.

[4] N. Atakishiyev, P. Franco, D. Levi, and O. Ragnisco, On Fourier integral transforms for q-Fibonacci and q-Lucas polynomials. *J. Phys. A: Math. Theor.* **45** (2012), Art.No.195206, 11pp (for a more detailed version of this work see arXiv:1112.2073v2 [math-ph] 13 March 2012).

[5] G. Gasper and M. Rahman, *Basic Hypergeometric Functions*, Second Edition, Cambridge University Press, Cambridge 2004.

[6] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, Cambridge University Press, Cambridge 1999.

[7] R. Koekoek, P. A. Lesky, and R. F. Swarttouw, *Hypergeometric Orthogonal Polynomials and Their q-Analogues*, Springer Monographs in Mathematics, Springer-Verlag, Berlin Heidelberg 2010.

[8] J. Cigler, q-Fibonacci polynomials. *Fibonacci Quarterly*, **41** (2003), pp 31-40.

[9] J. Cigler, A new class of q-Fibonacci polynomials. *The Electronic Journal of Combinatorics* **10** (2003), pp 1-15.

[10] J. Cigler and J. Zeng, A curious q-analogue of Hermite polynomials. *Journal of Combinatorial Theory A* **118** (2011), pp 9-26.