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MEROMORPHIC FUNCTIONS COMPATIBLE WITH HOMOMORPHISMS OF ACTIONS ON C

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Abstract

We consider homomorphisms $H : G_1 \longrightarrow G_2$ of holomorphic (group or pseudo-group) actions G_1 and G_2 on domains Ω_1 and Ω_2 respectively in **C**, together with meromorphic functions f that are compatible with these homomorphisms in the sense that

$$f(g(z)) = H(g)(f(z))$$

for every $g \in G_1$ and $z \in \Omega_1$. Such situations are rooted in the cases of elliptic and modular functions, modular and automorphic forms, etc... We investigate various aspects of such cases, such as constructions and correspondences between families of functions compatible with different homomorphisms, that transform one family of functions compatible with one homomorphism to another one compatible with a different homomorphism.

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1 Introduction

Consider two holomorphic group actions, G_1 on a domain Ω_1 and G_2 on a domain Ω_2 in **C**, i.e. where G_1 is a group with $g : \Omega_1 \longrightarrow \Omega_1$ a holomorphic function on Ω for every $g \in G_1$, and similarly for G_2 on Ω_2. The question of having meromorphic functions defined in Ω_1 with values in Ω_2 that are compatible with some homomorphism $H : G_1 \longrightarrow G_2$, in the sense of establishing commutative diagrams of the form

$$\begin{tikzcd} \Omega_1 \arrow[r, "g"] \arrow[d, "f"'] & \Omega_1 \arrow[d, "f"] \\ \Omega_2 \arrow[r, "H(g)"'] & \Omega_2 \end{tikzcd} \qquad (A)$$

$\forall g \in G_1$, and thus having

$$f(g(\tau)) = H(g)(f(\tau)) \qquad (1)$$

for every $g \in G_1$ and every $\tau \in \Omega_1$, can be an immensely fruitful question in many situations of specific homomorphisms $H : G_1 \longrightarrow G_2$, where G_1 and G_2 act holomorphically on domains Ω_1 and Ω_2 respectively.

The case of functions f (such as elliptic functions) *invariant* under a group action G_1 acting holomorphically on a domain $\Omega_1 \subset$ **C**, i.e. where $f(g(\tau)) = f(\tau), \forall \tau \in \Omega_1$, present important cases where G_2 (on Ω_2) is the trivial group with only the identity element, and the homomorphism $H : G_1 \longrightarrow G_2$ being the trivial homomorphism.

The elliptic functions on **C** are those meromorphic functions compatible with trivial group homomorphisms on the group actions on **C** offered by lattices $L = n_1 l_1 + n_2 l_2$, where $n_1, n_2 \in$ **Z** and l_1 and l_2 are two complex numbers with l_1/l_2 not real. The corresponding (commutative) group action G on **C** of a lattice L is by $g_l(\tau) = \tau + l$ for every $\tau \in$ **C** and every $l \in L$.

The modular functions on the upper half-plane H of **C**, are those functions that are compatible with trivial group homomorphisms on the group action offered by the group M of all 2×2 matrices $m = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with integer entries and $\det(m) = 1$. The corresponding group action G on H is given by $g_m(\tau) = \frac{a\tau+b}{c\tau+d}$ for every $m \in M$ and $\tau \in H$.

Examples of meromorphic functions in **C** compatible with non-trivial group homomorphisms are offered by functions that commute with the elements of the same group action G_1 (see [5]), i.e. where $f(V(\tau)) = V(f(\tau))$ for every $V \in G_1$, in which case H is the identity homomorphism $I : G_1 \longrightarrow G_1$. In [5], and starting from an automorphic form f of weight r for a function group $\hat{\Gamma} = \left\{ \frac{a\tau+b}{c\tau+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \right\}$ associated with an infinite group Γ of complex 2×2 matrices, a function F commuting with all elements of the function group $\hat{\Gamma}$ is constructed via

$$F(\tau) = r\frac{f(\tau)}{f'(\tau)} + \tau. \qquad (2)$$

Cases of compatibility of a meromorphic function $f : \Omega_1 \longrightarrow \Omega_2$ with a homeomorphism $H : G_1 \longrightarrow G_2$ of two monoids G_1 and G_2 of holomorphic functions, where the elements of G_1 are self-maps of a domain Ω_1 and those of G_2 are self-maps of another domain Ω_2, and which don't necessarily define proper monoid-actions on either of Ω_1 or Ω_2 (with regard to the compatibility of the binary operation in G_1 or G_2 with the "action" on the given domains) but only "act" on them by some operation, this compatibility can take a meaning different from that represented by the commutative diagram of *compositions* in (A) as follows. If one lets $*$ denote the binary operations both in G_1 and G_2, and one denotes by $[\cdot]$ the "actions" of G_1 and G_2 on Ω_1 and Ω_2 respectively (e.g. $g[z] = g(z) + z$ or $g[z] = g(z) \cdot z$ or $g[z] = g(z)$, etc..., where $(g_2 * g_1)[z]$ may not necessarily be the same as $g_2[g_1[z]]$, and this is where the action is only a *pseudo-monoidal* action), then one can have the compatibility of a meromorphic function $f : \Omega_1 \longrightarrow \Omega_2$ with $H : G_1 \longrightarrow G_2$ given in the form

$$f(g[z]) = H(g)[f(z)], \tag{3}$$

leading to

$$f((g_1 * g_2)[z]) = H(g_1 * g_2)[f(z)] = (H(g_1) * H(g_2))[f(z)], \tag{4}$$

for every $g_1, g_2 \in G_1$, which is not necessarily the same as $H(g_1)[H(g_2))[f(z)]]$. In such general cases it is obvious that we get proper group actions as special cases on Ω_1 and Ω_2 if all operations considered are composition operations.

Modular and automorphic forms ([2],[3],[4]) are indeed compatible with homomorphisms of group actions as described above. For these cases one considers a group Γ of 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of determinant 1, with composition as binary operation (and forming possibly a function group) and with corresponding group action G_1 on $\mathbf{C}$ given by $V(\tau) = \frac{a\tau+b}{c\tau+d}$, where $V \in G_1$ corresponds to $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$. While G_2 is a mutiplicative group of functions with multiplication as binary operation, and acting multiplicativly on the points in $\mathbf{C}$, with the (pseudo) homomorphism $H : G_1 \longrightarrow G_2$ on $\mathbf{C}$ given by a power of the (first) derivative operator multiplied by a group homomorphism (the *multiplier system*) denoted by v as below. This is given by:

$$f(V(\tau)) = v(V)(c\tau + d)^r f(\tau), \tag{5}$$

where $v : \Gamma \longrightarrow C(0,1)$, with $C(0,1) = e^{i\theta} : 0 \le \theta < 2\pi$, is a group homomorphism called the multiplier system for Γ. For these cases one has that $f((V_1 * V_2)[\tau]) = f((V_1 \circ V_2)(\tau))$, while

$$\begin{aligned} H(V_1 * V_2)[f(\tau)] &= v(V_1 \circ V_2)((V_1 \circ V_2)'(\tau))^{-r/2} f(\tau) \\ &= v(V_1)v(V_2)(V_1'(V_2(\tau)))^{-r/2}(V_2'(\tau))^{-r/2} f(\tau), \end{aligned} \tag{6}$$

where $'$ denotes derivative with respect to τ, and keeping in mind that $V'(\tau) = (c\tau + d)^{-2}$.

It has to be mentioned, in connection with the previous example and with eq. (5), that (as will be derived in sec. 2) a homomorphism $H : G_1 \longrightarrow G_2$ of group actions on $\mathbf{C}$ that satisfies eq. (3) for functions f, where the binary operation in G_1 is composition of functions, and where the action on $\mathbf{C}$ is by composition on z, and where the action of the elements of G_2 on $\mathbf{C}$ is by multiplication, must be such that it is a derivative operator on the elements of G_1 followed by some group homomorphism of G_1 into $\mathbf{C}$ (with values dependent only on the elements in G_1 and not on $\tau \in \mathbf{C}$).

In this paper we discuss cases of meromorphic functions (together with some of their properties) that are compatible as above with specific homomorphisms of holomorphic group actions on $\mathbf{C}$. In section 2 we consider correspondences between collections of functions where each collection consists of functions compatible with a different homomorphism of group actions as above. Thus we consider the establishment of functions compatible with one homomorphism from other functions that are compatible with different homomorphisms. While in section 3 we consider further constructions related to subgroups Γ of finite index of the inhomogeneous modular group with the corresponding action on $\mathbf{C}$ by linear fractional transformations.

2 Correspondences between Families of Functions Compatible with Different Actions

In this section, correspondences between sets of functions compatible with different homomorphisms of group actions on $\mathbf{C}$ will be given. We shall be interested with some specific constructions of certain functions associated with these actions, and with some general considerations associated with the compatibility question between meromorphic functions and group actions as mentioned above.

We first start with the following. In [5], Theorem 1, an interesting mapping ξ from one set $\mathcal{F}$ of meromorphic functions compatible with one homomorphism $H : G_1 \to G_2$ into another set $\tilde{\mathcal{F}}$ of meromorphic functions compatible with another homomorphism $\tilde{H}$: $G_3 \to G_4$ was introduced. For that case G_1 was any function group (i.e. a group of linear fractional transformations V with an invariant domain whose boundary consists of limit points of the action of this group on $\mathbf{C}$) with $H(V) = v(V)(dV/dz)^{-r/2}$, where $v : G_1 \to e^{i\theta}$ is a group homomorphism into the multiplicative group $e^{i\theta}$ where $0 \leq \theta < 2\pi$, called a multiplier system, i.e. where $f(V(\tau)) = v(V)(c\tau + d)^r f(\tau)$. The other homomorphism $\tilde{H}$ is the identity morphism from G_1 to G_1. This mapping

$$\xi : \mathcal{F} \longrightarrow \tilde{\mathcal{F}}, \tag{7}$$

thus estalishes a correspondence between a set $\mathcal{F}$ of automorphic forms and the set $\tilde{\mathcal{F}}$ of functions that commute with all the elements of the function group G_1. For $f \in \mathcal{F}$, $\xi(f)$ is given by $F(\tau)$ as in (2) above.

In the next theorem, we establish a partial converse to Theorem 1 in [5] in the sense that given a meromorphic function F that commutes with all elements in a function group G_1, then one can construct (from F) a meromorphic function f that satisfies $f(V(\tau)) = v(V)(c\tau + d)^r f(\tau)$ only for a subgroup $\tilde{G}_1$ of G_1, and for a specific multiplier system (i.e. a homomorphism) $v : \tilde{G}_1 \to \mathbf{C}$ (where $\mathbf{C}$ is considered as a multiplicative monoid). This establishes a correspondence between the set $\mathcal{F}$ of all functions compatible with the identity morphism on G_1 and the set $\tilde{\mathcal{F}}$ of functions that are compatible with the "pseudo" homomorphism $H : \tilde{G}_1 \to G_3$ given by a power of the first derivative multiplied by a multiplier system.

We shall need the following proposition, with regard to a certain group of 2×2 matrices.

Proposition 1.

1. *The set Σ of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero determinant such that every element (in Σ) satisfies $a + c = b + d$ (or every element satisfies $a + b = c + d$, or $a - c = b - d$, or $a - b = c - d$) forms a group under the operations of matrix multiplication.*

2. *If for every $V \in \Sigma$ one defines $v(V)$ to be*
$$v(V) = a + c, \tag{8}$$
(or, respectively as above, $v(V) = a + b$, or $v(V) = a - c$, or $v(V) = a - b$) then $v : \Sigma \to \mathbf{C}$ defines a multiplier system for Σ (i.e. a group homomorphism where $v(V \circ V') = v(V)v(V')$).

3. *The matrices in $SL(2, \mathbf{Z})$ that satisfy $a + c = b + d$ are precisely those that satisfy $a + c = 1$ with $b = a - 1$ and $d = c + 1$, or satisfy $a + c = -1$ with $b = a + 1$ and $d = c - 1$. Thus for any such matrix V in $SL(2, \mathbf{Z})$, one has that $v(V) = \pm 1$.*

Proof. If $V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $V' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ with $a + c = b + d$ and $a' + c' = b' + d'$ then

$$V \circ V' = \begin{pmatrix} aa' + bc' & ab' + bd' \\ ca' + dc' & cb' + dd' \end{pmatrix} \tag{9}$$

giving, on the one hand, that

$$(aa' + bc') + (ca' + dc') = a'(a + c) + c'(b + d) = (a + c)(a' + c') \tag{10}$$

(using that $a + c = b + d$), and on the other that

$$(ab' + bd') + (cb' + dd') = b'(a + c) + d'(b + d) = (a + c)(a' + c') \tag{11}$$

(using the above equalities). Thus multiplication is a binary operation on Σ. The identity matrix belongs to Σ, and multiplicative inverses $\frac{l}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ satisfy the same condition and hence also belong to Σ. This proves part 1, while part 2 follows immmediately from the fact that $v(V \circ V') = (aa' + bc') + (ca' + dc') = (a + c)(a' + c') = v(V)v(V')$.

For part 3, with $d = a + c - b$ and $ad - bc = 1$, one has that

$$ad - bc = a(a + c - b) - bc = (a - b)(a + c) = 1,$$

giving that (with all entries integers) either $a + c = 1$ and $a - b = 1$, and consequently that $d = c + 1$, or $a + c = -1$ and $a - b = -1$, and consequently that $d = c - 1$. $\square$

The Theorem is now as follows.

Theorem 1. *Let $\mathcal{F}$ be the family of meromorphic functions F that commute with all the elements of a function group G_1. Then there exists a correspondence between the set $\mathcal{F}$ and the set $\tilde{\mathcal{F}}$ of meromorphic functions (that can be called "pseudo-automorphic forms") compatible with $H(V) = v(V)(cz+d)^k$ where V belongs to the subgroup of G_1 in Σ (Σ as in the proposition above), i.e. $V \in G_1 \cap \Sigma$, and where $v(V) = \frac{1}{(a+c)^k}$. The mapping $\xi : \mathcal{F} \to \tilde{\mathcal{F}}$ is given by*

$$\xi(F)(z) = \left(\frac{F'(z)}{(F(z) \mp 1)^2}\right)^{k/2}. \tag{12}$$

Proof. We only need to prove the result for the case where $k = 2$, and only when the denominator is $(F(z)+1)^2$ as the other case with $(F(z)-1)^2$ is exactly similar. For $V(z) = \frac{az+b}{cz+d}$, and knowing that $F(V(z)) = V(F(z))$, one first has that

$$F(V(z)) = \frac{aF(z)+b}{cF(z)+d}, \tag{13}$$

and second that

$$(F(V(z)))' = F'(V(z))V'(z) = V'(F(z))F'(z), \tag{14}$$

giving that (with $V'(z) = (cz+d)^{-2}$)

$$F'(V(z)) = \frac{(cz+d)^2 F'(z)}{(cF(z)+d)^2}. \tag{15}$$

Thus $\xi(F)(V(z))$ is now given by

$$\begin{aligned} \xi(F)(V(z)) &= \frac{F'(V(z))}{(F(V(z))+1)^2} \\ &= \frac{(cz+d)^2 F'(z)/(cF(z)+d)^2}{[(aF(z)+b)/(cF(z)+d)+1]^2} \\ &= \frac{(cz+d)^2 F'(z)}{(aF(z)+b)^2 + 2(aF(z)+b)(cF(z)+d) + (cF(z)+d)^2} \\ &= \frac{(cz+d)^2 F'(z)}{(a^2+2ac+c^2)F^2(z) + 2(ab+ad+cb+cd)F(z) + (b^2+2bd+d^2)}. \end{aligned} \tag{16}$$

Thus we have

$$\begin{aligned} \xi(F)(V(z)) &= \frac{(cz+d)^2 F'(z)}{(a+b)^2 F^2(z) + 2(a+c)(b+d)F(z) + (b+d)^2} \\ &= \frac{(cz+d)^2 F'(z)}{[(a+c)F(z)+(b+d)]^2}. \end{aligned} \tag{17}$$

Now for the case where V satisfies $a+c = b+d$, one finally obtains that

$$
\begin{aligned}
\xi(F)(V(z)) &= \frac{(cz+d)^2}{(a+c)^2}\frac{F'(z)}{(F(z)+1)^2} = \frac{(cz+d)^2}{(a+c)^2}\xi(F) \\
&= v(V)(cz+d)^2\xi(F), \qquad (18)
\end{aligned}
$$

and the result follows. □

Next, is about Lemma 1 below, and we start by considering the equation

$$f(g[z]) = H(g)[f(z)] \qquad (19)$$

for some cases of groups of "actions" G_1 and G_2 on $\mathbf{C}$, and then identifying the corresponding homomorphisms H that have to be satisfied in such cases so that this equation is satisfied. The first case is when the group G_1 of functions acts in the obvious way of composition, i.e. by $g[z] = g(z)$ for every $g \in G_1$, in which case G_1 offers a (proper) group action on $\mathbf{C}$. While we take the action of G_2 on $\mathbf{C}$ to be defined by multiplication, i.e. by $h[z] = h(z)z$, which offers a pseudo-group action on $\mathbf{C}$. Thus we would need $H : G_1 \to G_2$ to satisfy an equation of the form

$$f(g(z)) = H(g(z))f(z), \qquad (20)$$

with

$$H(g_2(g_1(z))) = H(g_2(z))H(g_1(z)). \qquad (21)$$

For this case, H would have to satisfy the consistency relation arising from the following: On the one hand one has that

$$f(g_2(g_1(z))) = H(g_2(g_1(z)))f(z) = H(g_2(z))H(g_1(z))f(z), \qquad (22)$$

and on the other one has that

$$f(g_2(g_1(z))) = H(g_2)(g_1(z))H(g_1(z))f(z). \qquad (23)$$

Thus H would have to satisfy the consistency relation

$$H(g_2(g_1))(z) = H(g_2)(g_1(z))H(g_1(z)), \qquad (24)$$

and this implies that H has got to be a derivative operator, e.g. $H(g) = dg/dz$, leading to

$$f(g(z)) = g'(z)f(z). \qquad (25)$$

Other possible candidates for H are all related to the derivative operator, such as $H(g) = (dg/dz)^k$, where $k \in \mathbf{Z}$ is any integer.

For other cases of group actions of interest, one finds the following. If G_1 acts by composition on $\mathbf{C}$ as above, and G_2 is an additive group of functions acting on $\mathbf{C}$ by addition, i.e. by $h[z] = h(z) + z$, then it turns out (as was done in the previous case) that $H(g(z)) = \ln(g'(z))$, giving that

$$f(g(z)) = \ln(g'(z)) + f(z). \tag{26}$$

While if G_1 is an additive group of functions that acts additively (i.e. by $g[z] = g(z)+z$) on $\mathbf{C}$, with G_2 a multiplicative group of functions acting multiplicatively on $\mathbf{C}$, then H would have to satisfy $H(g(z)) = e^{g(z)}$, giving that

$$f(g(z)) = e^{g(z)} f(z). \tag{27}$$

Similarly if both groups G_1 and G_2 are additive groups of functions acting additively on $\mathbf{C}$, then $H : G_1 \to G_2$ can be any linear operator, for instance H can be of the form $H(g(z)) = kg(z)$ (where k is any constant), or $H(g(z)) = d^n g(z)/dz^n$, or $H(g(z))$ can be given by the antiderivative of g, etc... leading to situations where

$$f(g(z) + z) = kg(z) + f(z), \quad \text{or} \quad f(g(z) + z) = \frac{d^n g(z)}{dz^n} + f(z), \quad etc... \tag{28}$$

Lemma 1. a) *Let $G_1 = \{az + b\}$ be an additive group of complex linear polynomials that act additively on $\mathbf{C}$ (i.e. if $p \in G_1$, then $p[z] = p(z) + z = (a + 1)z + b$). Let $\mathcal{F}$ be the set of all meromorphic functions f compatible with the group morphism H given by $H(p(z)) = p'(z)$ and acting additively on $\mathbf{C}$. Then there exists a correspondence ξ between the set $\mathcal{F}$ and the set $\tilde{\mathcal{F}}$ of all meromorphic functions that commute with all the elements in $G_3 = \{(a + 1)z + b\}$ (i.e. that are compatible with the identity group morphism on G_1), given by*

$$\xi(f)(z) = \frac{f'(z)}{f''(z)} + z. \tag{29}$$

b) *Let $G_1 = \{az^2\}$ be an additive group of quadratic polynomials that act additively on $\mathbf{C}$. Let $\mathcal{F}$ be the set of all meromorphic functions f compatible with the group morphism H given by $H(p(z)) = e^{p(z)}$ and acting multiplicatively on $\mathbf{C}$. Then there exists a correspondence ξ between the set $\mathcal{F}$ and the set $\tilde{\mathcal{F}}$ of all meromorphic functions F that satisfy $F(g(z)) = g'(z)F(z)$, where $g(z) = p(z) + z = az^2 + z$ ($p \in G_1$). This mapping is given by*

$$\xi(f)(z) = \frac{e^z}{f'(z) - f(z)}. \tag{30}$$

Proof. a) $f(p[z]) = H(p(z))[f(z)]$ gives that

$$f(p(z) + z) = p'(z) + f(z) = a + f(z), \tag{31}$$

i.e. that $f((a+1)z+b) = a + f(z)$. Let

$$g(z) = p(z) + z = (a+1)z + b, \tag{32}$$

then

$$(f(g(z)))' = (a + f(z))' = f'(z). \tag{33}$$

But

$$(f(g(z)))' = f'(g(z))g'(z) = f'(g(z))(a+1). \tag{34}$$

Hence

$$f'(g(z)) = f'(z)/(a+1). \tag{35}$$

We also have that

$$(f(g(z)))'' = (a + f(z))'' = f''(z), \tag{36}$$

which is also equal to

$$(f'(g(z)).g'(z))' = f''(g(z))'g'^2(z) + f'(g(z))g''(z). \tag{37}$$

And since $g''(z) = 0$, this gives that

$$f''(z) = f''(g(z))(a+1)^2, \tag{38}$$

and hence that

$$f''(g(z)) = f''(z)/(a+1)^2. \tag{39}$$

Thus

$$\begin{aligned}
\xi(f)(g(z)) &= \frac{f'(g(z))}{f''(g(z))} + g(z) \\
&= \frac{f'(z)/(a+1)}{f''(z)/(a+1)^2} + (a+1)z + b \\
&= (a+1)\left(\frac{f'(z)}{f''(z)} + z\right) + b \\
&= (a+1)\xi(f)(z) + b \\
&= g(\xi(f(z))),
\end{aligned} \tag{40}$$

which establishes part a.

b) We have

$$f(p(z) + z) = e^{p(z)} f(z) \quad (\text{where } \ p(z) = az^2) \tag{41}$$

which gives that

$$(f(p(z)+z))' = 2aze^{az^2} f(z) + e^{az^2} f'(z) = (2az+1)f'(p(z)+z), \tag{42}$$

i.e. that

$$f'(p(z)+z) = [2aze^{az^2} f(z) + e^{az^2} f'(z)]/(2az+1). \tag{43}$$

Thus, for $g(z) = p(z) + z = az^2 + z$

$$\begin{aligned} \xi(f)(g(z)) &= \frac{e^{az^2+z}}{[2aze^{az^2}f(z) + e^{az^2}f'(z)]/(2az+1) - e^{az^2}f(z)} \\ &= (2az+1)\frac{e^z}{f'(z) - f(z)} \qquad (44) \\ &= g'(z)\xi(f(z)), \qquad (45) \end{aligned}$$

which proves part b. □

3 Constructions Associated with Subgroups of the Inhomogeneous Modular Group

In this section we consider, by straightforward analysis and discussions, constructions related to subgroups of finite index of the inhomogeneous modular group. The analysis could have been done by considering more general and powerful techniques, but we restrict ourselves to more elementary discussions.

Let Γ be such a subgroup, and let $\tilde{\Gamma}$ be the corresponding group of linear fractional transformations. Let $H : \tilde{\Gamma} \longrightarrow \tilde{M}$ be a group homomorphism where $\tilde{M}$ is a group of linear fractional transformations associated with a group M of 2×2 complex matrices. We will assume that $\ker(H)$ is a finite index subgroup of $\tilde{\Gamma}$ (and thus is also of finite index in the inhomogenous modular group), and that $M_1, M_2, \cdots, M_n$ are the images in $\tilde{M}$ under H of the cosets $[\ker(H)]_i$, $i = 1, \cdots, n$, of $\ker(H)$ (where $[\ker(H)]_1 = \ker(H)$, and M_1 =Identity).

In this section, and starting from modular forms g of weight r for $\ker(H)$, we seek functions that behave like

$$f(V(z)) = H(V)(f(z)), \quad \forall V \in \tilde{\Gamma}, \qquad (46)$$

or as close as possible to this equation, e.g. up to multiplicative factors (dependent only on V) of $H(V)(f(z))$. In particular these functions f will be modular functions for $\ker(H)$, i.e. $f(V(z)) = f(z)$ for every $V \in \ker(H) \subset \tilde{\Gamma}$, and behave (i.e. transform) similarly under the elements in $\tilde{\Gamma}$ up to membership in the same cosets of $\ker(H)$.

The extreme cases for this problem are already established: If $H : \tilde{\Gamma} \longrightarrow \tilde{M}$ is such that $\ker(H) = \tilde{\Gamma}$, and f is a modular form of weight 0 for $\ker(H) = \tilde{\Gamma}$, then this gives the case where $f(V(z)) = f(z)$ leading to modular functions f. And the case where $H : \tilde{\Gamma} \rightarrow \tilde{\Gamma}$ is an isomorphism (even though $\ker(H)$ may not be of finite index here) gives functions f satisfying $f(V(z)) = V(f(z))$ and thus commuting with all elements in $\tilde{\Gamma}$, as constructed in [5].

We consider other (in-between) cases. We start with the following.

Lemma 2. *Let* $H : \tilde{\Gamma} \longrightarrow \tilde{M}$ *(be an epimorphism) where* $M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \right\}$ *modulo* $\{I, -I\}$ *(and* $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$*). If*

$$g(z) = \sum_{T \in \ker(H)} \frac{h(T(z))}{\mu_T(z)}, \tag{47}$$

is a modular form of weight r *for* $\ker(H)$*, where* $\mu_T(z) = (cz+d)^{-2r}$ *for* $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *and satisfies* $\mu_{TS}(z) = \mu_T(S(z))\mu_S(z)$*, and* h *is a holomorphic function, then*

$$f(z) = i \frac{\sum_{T \in [\ker(H)]_1} (h(T(z))/\mu_T(z))}{\sum_{S \in [\ker(H)]_2} (h(S(z))/\mu_S(z))} = i \frac{g(z)}{g(L(z))/\mu_L(z)}, \tag{48}$$

(*where* $L \in \Gamma$ *is any element in* $[\ker(H)]_2$) *is a meromorphic function that satisfies* $f(V(z)) = H(V)(f(z))$ *for every* $V \in \tilde{\Gamma}$*, i.e. satisfies*

$$f(V(z)) = f(z) \quad \forall V \in [\ker(H)]_1, \quad \text{and} \quad f(V(z)) = -\frac{1}{f(z)} \quad \forall V \in [\ker(H)]_2 \tag{49}$$

(*where* $-1/f(z) = M_2(f(z))$ *with* $M_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$).

Proof. For $V \in \ker(H)$,

$$\begin{aligned} f(V(z)) &= i \frac{g(V(z))}{g(L(V(z)))/\mu_L(V(z))} \\ &= i \frac{\mu_V(z) g(z)}{g(L(V(z)))/(\mu_{LV}(z)/\mu_V(z))} \\ &= i \frac{g(z)}{g(L'(z))/\mu_{L'}(z)}, \end{aligned} \tag{50}$$

where $L' = LV \in [\ker(H)]_2$ also satisfies $[\ker(H)]_1 L' = [\ker(H)]_2$, and thus $f(V(z)) = f(z)$. While for $V \in [\ker(H)]_2$,

$$\begin{aligned} f(V(z)) &= i \frac{g(V(z))}{g(L(V(z)))/\mu_L(V(z))} \\ &= i \frac{g(V(z))}{g(L'(z))/(\mu_{L'}(z)/\mu_V(z))} \\ &= i \frac{g(V(z))/\mu_V(z)}{g(L'(z))/\mu_{L'}(z)}, \end{aligned} \tag{51}$$

where now $L' = LV \in \ker(H)$, and thus $g(L'(z)) = \mu_{L'}(z) g(z)$. Hence

$$f(V(z)) = i \frac{g(V(z))/\mu_V(z)}{g(z)} = -\frac{1}{ig(z)/(g(V(z))/\mu_V(z))} = -\frac{1}{f(z)}. \tag{52}$$

The result follows. ∎

The next lemma illustrates another aspect of the problem. Let $H : \tilde{\Gamma} \to \tilde{M}$ be a group morphism and assume that $g(z) = \sum_{T \in \ker(H)} h(T(z))/\mu_T(z)$ is a modular form of weight r for $\ker(H)$. We assume that $\ker(H)$ is of finite index in Γ and that $\{L_i\}$, $i = 1, \cdots, n$, form a set of coset representatives for $\ker(H)$ in Γ. For this case we will denote by $\{i\}_h$, or simply by $\{i\}$ whenever h is known, the sum whose value at a point z is given by

$$\{i\}_h(z) = \sum_{S \in [\ker(H)]_i} \frac{h(S(z))}{\mu_S(z)} = \frac{g(L_i(z))}{\mu_{L_i}(z)} \quad \text{for } \ i = 1, 2, 3, 4. \tag{53}$$

Note that it does not matter which coset representatives L_i we have chosen, and that $\{1\}_h(z) = g(z)$ since for L_1 we have that $g(L_1(z)) = \mu_{L_1}(z) g(z)$.

Lemma 3. *Let* $H : \tilde{\Gamma} \longrightarrow \tilde{M}$ (*be an epimorphism*) *where*
$M = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \right\}$ *modulo* $\{I, -I\}$ (*and* I *is the identity*)*. If*

$$g(z) = \sum_{T \in \ker(H)} \frac{h(T(z))}{\mu_T(z)} \tag{54}$$

($\mu_T(z) = (cz + d)^{-2r}$) *is a modular form of weight* r *for* $\ker(H)$*, then*

1. *There does not exist any general linear fractional form*

$$f(z) = \frac{a\{1\} + b\{2\} + c\{3\} + d\{4\}}{e\{1\} + f\{2\} + g\{3\} + h\{4\}} \tag{55}$$

   (*where* $a, b, \cdots h \in \mathbf{C}$) *that satisfies* $f(V(z)) = H(V)(f(z))$ *for every* $V \in \Gamma$. (*In fact we would conjecture that there does not exist any meromorphic function* f *such that* $f(V(z)) = H(V)(f(z))$.)

2. *There exists a linear fractional form*

$$f(z) = i \frac{\{1\} + \{4\}}{\{2\} + \{3\}}, \tag{56}$$

   and a multiplier system given by

$$v(V) = \det(H(V)), \tag{57}$$

   such that

$$f(V(z)) = v(V) H(V)(f(z)) = \det(H(V)) H(V)(f(z)) \quad \forall V \in \Gamma. \tag{58}$$

Proof. 1) We do this part by straightforward elementary analysis although it can be done by other techniques. We first start by considering $\{i\}(V(z))$ for all $i = 1, 2, 3, 4$ and for all 4 cosets $[\ker(H)]_i$ where V can exist. For $V \in [\ker(H)]_1$, we have that

$$\{i\}(V(z)) = \mu_V(z)\{i\} \quad \forall i = 1, 2, 3, 4. \tag{59}$$

While for $V \in [\ker(H)]_2$ we have that

$$\{1\}(V(z)) = \mu_V(z)\{2\}, \quad \{2\}(V(z)) = \mu_V(z)\{1\}, \tag{60}$$

and

$$\{3\}(V(z)) = \mu_V(z)\{4\} \quad \{4\}(V(z)) = \mu_V(z)\{3\}. \tag{61}$$

For $V \in [\ker(H)]_3$ we have that

$$\{1\}(V(z)) = \mu_V(z)\{3\}, \quad \{2\}(V(z)) = \mu_V(z)\{4\}, \tag{62}$$

and

$$\{3\}(V(z)) = \mu_V(z)\{1\}, \quad \{4\}(V(z)) = \mu_V(z)\{2\}. \tag{63}$$

While for $V \in [\ker(H)]_4$ we have that

$$\{1\}(V(z)) = \mu_V(z)\{4\}, \quad \{2\}(V(z)) = \mu_V(z)\{3\}, \tag{64}$$

and

$$\{3\}(V(z)) = \mu_V(z)\{2\}, \quad \{4\}(V(z)) = \mu_V(z)\{1\}. \tag{65}$$

Now assume that (indeed) $f(V(z)) = H(V)(f(z))$ for every $V \in \Gamma$. Then For $V \in [\ker(H)]_1$ we (indeed) have that

$$f(V(z)) = \frac{a\{1\} + b\{2\} + c\{3\} + d\{4\}}{e\{1\} + f\{2\} + g\{3\} + h\{4\}} = f(z) = H(V)(f(z)). \tag{66}$$

While for $V \in [\ker(H)]_2$, and requiring that $f(V(z)) = H(V)(f(z)) = \frac{1}{f(z)}$ we find that

$$f(V(z)) = \frac{a\{2\} + b\{1\} + c\{4\} + d\{3\}}{e\{2\} + f\{1\} + g\{4\} + h\{3\}} = \frac{e\{1\} + f\{2\} + g\{3\} + h\{4\}}{a\{1\} + b\{2\} + c\{3\} + d\{4\}}. \tag{67}$$

And for $V \in [\ker(H)]_3$, and requiring that $f(V(z)) = H(V)(f(z)) = -\frac{1}{f(z)}$ we get that

$$f(V(z)) = \frac{a\{3\} + b\{4\} + c\{1\} + d\{2\}}{e\{3\} + f\{4\} + g\{1\} + h\{2\}} = -\frac{e\{1\} + f\{2\} + g\{3\} + h\{4\}}{a\{1\} + b\{2\} + c\{3\} + d\{4\}}. \tag{68}$$

And finally for $V \in [\ker(H)]_4$, and requiring that $f(V(z)) = H(V)(f(z)) = -f(z)$ we find that

$$f(V(z)) = \frac{a\{4\} + b\{3\} + c\{2\} + d\{1\}}{e\{4\} + f\{3\} + g\{2\} + h\{1\}} = -\frac{a\{1\} + b\{2\} + c\{3\} + d\{4\}}{e\{1\} + f\{2\} + g\{3\} + h\{4\}}. \tag{69}$$

Since at least one of a, b, c, d is not zero, we will assume that $a \neq 0$ and (by dividing $a, \cdots, h$ by a if necessary to normalize these coefficients) we will assume that $a = 1$. Now in the set of new coefficients $1, b, \cdots, h$, and from (67), at least one of e, f, g, h is not zero, and we can assume that (for example) $f \neq 0$. Thus we find (from 67, and keeping in mind that this equation must be satisfied for all z) that $d = g/f$, and that $f = 1/f$ giving that $f^2 = 1$, i.e. $f = \pm 1$. We also conclude that d and g are either both zero or both not zero.

Assume that d and g are both not zero. Then from (68), (keeping $a = 1$ and dividing by g on the right hand side) we find (among other things) that $g = -1/g$ giving $g^2 = -1$, i.e. $g = \pm i$. Now from (69), and after dividing the right hand side by d we find that $d = 1/d$ giving $d^2 = 1$, i.e. $d = \pm 1$. But if $d = g/f$ and $f = \pm 1$, then it cannot be that $d = \pm 1$ and $g = \pm i$. Thus d and g must both be zero.

For this case where $d = g = 0$, one finds (e.g. from ( )) that f must be zero contradicting that $f = \pm 1$. Thus there does not exist a linear fractional form (as in (56)) to satisfy $f(V(z)) = H(V)(f(z))$ for every $V \in \Gamma$. This proves part 1.

2) For $V \in [\ker(H)]_1$ with $\det(H(V)) = 1$,

$$F(V(z)) = i\frac{\{1\} + \{4\}}{\{2\} + \{3\}} = f(z) = \det(H(V))H(V)(f(z)). \tag{70}$$

For $V \in [\ker(H)]_2$ with $\det(H(V)) = -1$,

$$F(V(z)) = i\frac{\{2\} + \{3\}}{\{1\} + \{4\}} = -\frac{1}{f(z)} = \det(H(V))H(V)(f(z)). \tag{71}$$

For $V \in [\ker(H)]_3$ with $\det(H(V)) = 1$,

$$F(V(z)) = i\frac{\{3\} + \{2\}}{\{4\} + \{1\}} = -\frac{1}{f(z)} = \det(H(V))H(V)(f(z)). \tag{72}$$

Finally for $V \in [\ker(H)]_4$ with $\det(H(V)) = -1$,

$$F(V(z)) = i\frac{\{4\} + \{1\}}{\{3\} + \{2\}} = f(z) = \det(H(V))H(V)(f(z)). \tag{73}$$

This proves part 2. $\square$

As was done in the first part of the previous Lemma, one can similarly show that there does not exist a quadratic fractional form

$$f(z) = \frac{\sum_{i,j=1}^{4} a_{ij}\{i\}\{j\}}{\sum_{i,j=1}^{4} b_{ij}\{i\}\{j\}}, \tag{74}$$

that satisfies $f(V(z)) = H(V)(f(z))$ for every $V \in \Gamma$. We thus conjecture that there does not exist any meromorphic fucntion f that satisfies this requirement for this particular case.

Given the above discussions, we can pause the following possibility. Let Γ be a subgroup of finite index of the inhomogeneous modular group, with $\tilde{\Gamma}$ the corresponding group of linear fractional transformations, and let M be the group of 2×2 complex matrices having determinant of modulus 1, with $\tilde{M}$ the corresponding group of linear fractional transformation. Then for every group morphism $H : \tilde{\Gamma} \longrightarrow \tilde{M}$, with $Ker(H)$ of finite index in $\tilde{\Gamma}$, there exists a meromorphic function f in **C** compatible with the product of H and an *appropriate multiplier system* $v : \Gamma \longrightarrow \mathbf{C}$, to give

$$f(V(z)) = v(V)H(V)(f(z)), \quad \forall V \in \tilde{\Gamma}. \tag{75}$$

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