Communications in Mathematical Analysis
Volume 13, Number 2, pp. 56–81 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

A Class of Marcinkiewicz Type Integral Operators

Ahmad Al-Salman *
Department of Mathematics
Yarmouk University
Irbid, Jordan

(Communicated by Palle Jorgensen)

Abstract

We introduce a class of integral operators related to parametric Marcinkiewicz operators. We present a multiplier formula characterizing the L^2 boundedness of such class of operators. Also, we prove $\mathcal{L}^p_{-\beta}$ (inhomogeneous Sobolev space)$\to L^p$ estimates provided that the kernels are in $L(\log L)(\mathbf{S}^{n-1})$. In fact, we show that the global parts of the introduced operators are bounded on the Lebesgue spaces $L^p(1 < p < \infty)$ while the local parts are bounded on certain Sobolev spaces $\mathcal{L}^p_{-\beta}$.

AMS Subject Classification: Primary 42B20; Secondary 42B15, 42B25.

Keywords: Marcinkiewicz integrals, Rough integral operators, Fourier transform, Sobolev spaces, Triebel-Lizorkin space, Bessel functions.

1 Introduction and Statement of Results

Let $\mathbf{R}^n$, $n \geq 2$ be the n-dimensional Euclidean space and $\mathbf{S}^{n-1}$ be the unit sphere in $\mathbf{R}^n$ equipped with the induced Lebesgue measure $d\sigma$. Let $h : (0,\infty) \to \mathbf{R}$ be a measurable function and $\Omega \in L^1(\mathbf{S}^{n-1})$ be homogeneous of degree zero on $\mathbf{R}^n$ and satisfies

$$\int_{\mathbf{S}^{n-1}} \Omega(y) d\sigma(y) = 0. \tag{1.1}$$

Define the operator $\mu^{(\rho)}_{\Omega,h}$ by

$$\mu^{(\rho)}_{\Omega,h} f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\rho t} \int_{|y|\leq 2^t} f(x-y) \frac{\Omega(y)h(|y|)}{|y|^{n-\rho}} dy \right|^2 dt \right)^{\frac{1}{2}} \tag{1.2}$$

where $\Re(\rho) > 0$. When $h = 1$, the operator $\mu^{(\rho)}_{\Omega} = \mu^{(\rho)}_{\Omega,1}$ is the well known parametric Marcinkiewicz function of higher dimension introduced by Hörmander in 1960 ([14]).

*E-mail address: alsalman@yu.edu.jo

When $h = 1$ and $\rho = 1$, the corresponding operator $\mu_\Omega = \mu^{(1)}_{\Omega,1}$ is the classical Marcinkiewicz integral operator which was introduced by E. M. Stein in ([17]). E. M. Stein proved that if $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \le 1)$, then μ_Ω is bounded on L^p for all $1 < p \le 2$ ([17]). Subsequently, A. Benedek, A. Calderón, and R. Panzone proved the L^p boundedness of μ_Ω for all $1 < p < \infty$ provided that Ω is continuously differentiable on $\mathbf{S}^{n-1}$ ([6]). In 1972, T. Walsh ([19]) showed that μ_Ω is bounded on $L^2(\mathbf{R}^n)$ provided that $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$. In the same paper, Walsh showed the optimality of the condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$. In fact, he showed that the L^2 boundedness of μ_Ω may fail if the condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$ is replaced by $\Omega \in L(\log L)^{\frac{1}{2}-\varepsilon}(\mathbf{S}^{n-1})$ for some $\varepsilon > 0$. Later, Al-Salman, et al. ([4]) showed that the condition $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$ is also sufficient for the L^p boundedness of μ_Ω for all $p \in (1,\infty)$. For further results concerning the operator μ_Ω we cite, among others, the articles ([4], [6], [11], [15]).

On the other hand, the L^p mapping properties of the parametric operator $\mu^{(\rho)}_{\Omega,h}$ have received a considerable amount of attention during the last few years. When $h = 1$ and $\rho > 0$, Hörmander proved that μ^{ρ}_{Ω} is bounded on L^p for all $1 < p < \infty$ provided that $\Omega \in Lip_\alpha(\mathbf{S}^{n-1})$, $(0 < \alpha \le 1)$ [14]. When h satisfies the integrability condition $\sup_{j\in\mathbf{Z}}(\int_{2^{j-1}}^{2^j} |h(r)|^q \frac{dr}{r})^{\frac{1}{q}} < C < \infty$ for some $1 \le q \le \infty$ and $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$, Ding, Lu, and Yabuta ([10]) proved that $\mu^{(\rho)}_{\Omega,h}$ is bounded on L^2. Subsequently, Al-Salman and Al-Qassem [3] showed that $\mu^{(\rho)}_{\Omega,h}$ is bounded on L^p for all $1 < p < \infty$ provided that Ω and h satisfy the same conditions in ([10]). For the latest developments concerning the operator $\mu^{(\rho)}_{\Omega,h}$, we advise readers to consult the very recent papers [1], [2].

The main aim of this paper is trying to understand the L^p mapping properties of the operators $\mu^{(\rho)}_{\Omega,h}$ in (1.2) when the radial functions h satisfy certain point wise size conditions rather than an integrability condition such as that given in ([10]). We are interested in considering operators in the form of (1.2) with functions h with certain growth conditions. We shall show that such class of operators is related to various integral operators such as the operators $\mu^{(\rho)}_{\Omega,h}$ in (1.2) and the fractional Marcinkiewicz integral operators in [16]. To this end, we start by introducing the following class of radial functions h:

Definition 1.1. *A function $h : (0,\infty) \to \mathbf{R}$ is said to be of order $\nu \in \mathbf{R}$ if there exist numbers $0 < \varepsilon_\nu < 1, \beta_\nu > \max\{0,-\nu\}$, $C_{1,\nu} > 0$, $C_{2,\nu} > 0$, and constant A_ν such that the following conditions hold:*

$$|h(t)| \le C_{1,\nu} t^\nu \text{ for } 0 < t < 1; \tag{1.3}$$

$$\left|h(t) - A_\nu t^\nu\right| \le C_{2,\nu} t^{\beta_\nu} \text{ for } 0 < t < 1; \tag{1.4}$$

$$h(t) = O(t^{-\varepsilon_\nu}) \text{ for } t \ge 1. \tag{1.5}$$

For each real ν, we let $\mathcal{B}_\nu$ be the class of all functions of order ν. We let $\mathcal{B}^{(0)}_\nu$ be the class of functions satisfying the conditions (1.3) and (1.5).

It is clear that $\mathcal{B}_\nu \subset \mathcal{B}^{(0)}_\nu$. Moreover, $\mathcal{B}^{(0)}_\nu \subset L^\infty(0,\infty)$ for $\nu \ge 0$ while $\mathcal{B}_\nu$ and $L^\infty(0,\infty)$ are different for $\nu < 0$. Model examples of functions in $\mathcal{B}_\nu$ are $t^\nu (\nu < 0)$, $(1+t)^{-\alpha} (\alpha > 0)$, and $t^\nu h(t)$ where $\nu < 0$ and $h \in L^\infty(0,\infty)$ with $\lim_{t\to 0^+} h(t)$ exists. Another interesting example of a function in the class $\mathcal{B}_\nu$ is the Bessel functions J_ν. This allows one to consider integral operators with kernels involving Bessel functions (See Section 6 for detailed results). Further examples of functions in $\mathcal{B}_\nu$ can be constructed. In particular, for any $\varepsilon > 0, \nu > -1$, $a, b \in \mathbf{R}$ and $k_1(t), k_2(t) \in L^\infty(0,\infty)$, we have

$$h_\nu(t) = (at^{\nu+1}k_1(t) + bt^\nu)\chi_{(0,1)}(t) + t^{-\varepsilon}k_2(t)\chi_{[1,\infty)}(t) \in \mathcal{B}_\nu.$$

Now, we introduce the class of operators related to the class $\mathcal{B}_\nu$. For $\nu, \alpha \in \mathbf{R}$, $h_\nu \in \mathcal{B}_\nu$, and a suitable function $\Gamma : \mathbf{R}^n \to \mathbf{R}$, we let $S_{\Gamma,h_\nu,\alpha}$ be the integral operator defined by

$$S_{\Gamma,h_\nu,\alpha}(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y) \frac{\Gamma(y)}{|y|^{n-1}} h_\nu(2^{\alpha t}|y|) dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}. \tag{1.6}$$

It is clear that if $\alpha = 0$ and $\Gamma = \Omega$, then the operator $S_{\Gamma,h_\nu,0}$ is the Marcinkiewicz integral operator $\mu^{(1)}_{\Omega,h_\nu}$. Moreover, if $\alpha = -1$, $h_\nu(r) = I(r) = \chi_{[0,1)}(r)$, and $\Gamma = \Omega$, then $S_{\Gamma,I,-1}$ is the classical Marcinkiewicz integral operator μ_Ω. Here, $\chi_{[0,1)}$ is the characteristic function of the interval $[0,1)$. In order to state the results of this paper, we cite the following related remarks:

(i) In [9], Chen, Fan, and Ying considered the following fractional Marcinkiewicz integral operator

$$\mu_{\Omega,\alpha} f(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-t(1+\alpha)} \int_{|y|\le 2^t} f(x-y) \frac{\Omega(y)}{|y|^{n-1}} dy \right|^2 dt \right)^{\frac{1}{2}}. \tag{1.7}$$

A particular result in [9] is the following:

Theorem 1.2([9]). *Let* $\Omega \in L^r(\mathbf{S}^{n-1}), r > 1$ *and* $1 < p < \infty$. *If* $|\alpha| < (r-1)/r \max(p, \frac{p}{p-1})$, *then* $\left\| \mu_{\Omega,\alpha} f \right\|_p \le C_p \|f\|_{\mathcal{L}^p_\alpha}$.

Here, $\mathcal{L}^p_\alpha$ is the inhomogeneous Sobolev space (See Section 2 for definition). It should be noticed here that when $\alpha = 0$, the L^p inequality in Theorem 1.2 is simply $\|\mu_\Omega f\|_p \le C_p \|f\|_{L^p}$. This is due to the observation that $\mu_{\Omega,0} = \mu_\Omega$ and $\mathcal{L}^p_0 = L^p$. In light of this result, it is natural to ask whether similar result exists for the parametric Marcinkiewicz integral operator $\mu^{(\rho)}_{\Omega,h}$ in (1.2). An answer of this problem will follow from our discussion of the operators $S_{\Gamma,h_\nu,\alpha}$. In fact, if $-1 < \nu < 0, \beta \in \mathbf{R}$, $\alpha = -1 - \beta/\nu$, $\rho = 1 + \nu$, $h_\nu(t) = t^\nu$, and $\Gamma(y) = \Omega(y)$, then the corresponding operator $S_{\Gamma,h_\nu,\alpha}$ reduces to the following operator

$$\mu^{(\rho)}_{\Omega,\beta}(f)(x) = \left(\int_{-\infty}^{\infty} \left| 2^{-\beta t} \left(\frac{1}{2^{\rho t}} \int_{|y|<2^t} f(x-y) \frac{\Omega(y)}{|y|^{n-\rho}} dy \right) \right|^2 dt \right)^{\frac{1}{2}} \tag{1.8}$$

which is the parametric analogous of the operator (1.7).

(ii) In [16], Si, Wang, and Jiang studied the following fractional Marcinkiewicz integral operator

$$M_{\Omega,\delta}(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y) \frac{\Omega(y)}{|y|^{n-(1+\delta)}} dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}. \tag{1.9}$$

They showed that the fractional operator $M_{\Omega,\delta}$ $(0<\delta<n)$ satisfies certain Herz type Hardy space estimates provided that Ω satisfies a logarithmic type Lipschitz condition. However, when $\delta<0$, the L^p mapping properties of $M_{\Omega,\delta}$ have not discussed in [16]. By specializing the operator $S_{\Gamma,h_\nu,\alpha}$ in (1.6) to the case $\alpha=0$ and $h_\delta(t)=t^\delta$, $\delta<0$, and $\Gamma(y)=\Omega(y)$, the corresponding operator $S_{\Gamma,h_\delta,0}$ reduces to the operator $M_{\Omega,\delta}$. Therefore, the problem of characterizing the L^p mapping properties of the class of operators $M_{\Omega,\delta}$ $(\delta<0)$ is a special case of the corresponding problem concerning the operators $S_{\Gamma,h_\nu,\alpha}$ in (1.6).

(*iii*) By the observation that $\mathcal{B}_\nu^{(0)}\subset L^\infty(0,\infty)$ for $\nu\geq 0$, it can be shown using the same argument in [4] that the operator $S_{\Gamma,h_\nu,\alpha}$ is bounded on L^p for $1<p<\infty$ provided that $\Gamma=\Omega\in L(\log^+L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$ and $\nu\geq 0$. Therefore, it is a natural problem to determine whether a similar result would hold in the case $\nu<0$. Furthermore, in light of the conditions (1.3) and (1.5), it can be claimed that weaker conditions on Ω are needed to guarantee the L^p boundedness of $S_{\Gamma,h_\nu,\alpha}$ for the case $\nu\geq 0$. The later is indeed the content of Theorem 1.3 below.

In light of the aforementioned discussion, we will investigate the L^p mapping properties of the class of operators $S_{\Gamma,h_\nu,\alpha}$. In order to state the results of this paper, we let $S^{(0)}_{\Gamma,h_\nu,\alpha}$ and $S^{(\infty)}_{\Gamma,h_\nu,\alpha}$ be the operators defined by

$$S^{(0)}_{\Gamma,h_\nu,\alpha}f(x)=\left(\int_0^1\left|\int_{|y|\leq u}f(x-y)\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha|y|)dy\right|^2\frac{du}{u^3}\right)^{\frac{1}{2}} \tag{1.10}$$

$$S^{(\infty)}_{\Gamma,h_\nu,\alpha}f(x)=\left(\int_1^\infty\left|\int_{|y|\leq u}f(x-y)\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha|y|)dy\right|^2\frac{du}{u^3}\right)^{\frac{1}{2}}. \tag{1.11}$$

It is clear that

$$\frac{1}{2}\left(S^{(0)}_{\Gamma,h_\nu,\alpha}+S^{(\infty)}_{\Gamma,h_\nu,\alpha}\right)f(x)\leq S_{\Gamma,h_\nu,\alpha}f(x)\leq\left(S^{(0)}_{\Gamma,h_\nu,\alpha}+S^{(\infty)}_{\Gamma,h_\nu,\alpha}\right)f(x). \tag{1.12}$$

For a given $\Gamma:\mathbf{R}^n\to\mathbf{R}$, we let $\Gamma^*(y')=\sup_{r>0}|\Gamma(ry')|$. Under the sole integrability condition of Γ^* on $\mathbf{S}^{n-1}$, we prove the following:

Theorem 1.3. *Suppose that* $\Gamma^*\in L^1(\mathbf{S}^{n-1})$ *and that* $h_\nu\in\mathcal{B}_\nu^{(0)}$. *Let* $\alpha>-1$. *Then*
(a) $\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|_p\leq C\|\Gamma^*\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}\|f\|_p$ *for all* $1<p<\infty$ *whenever* $\nu>-1$.
(b) $\left\|S^{(0)}_{\Gamma,h_\nu,\alpha}(f)\right\|_p\leq C\|\Gamma^*\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}\|f\|_p$ *for all* $1<p<\infty$ *whenever* $\nu>0$.
(c) $\left\|S_{\Gamma,h_\nu,\alpha}(f)\right\|_p\leq C\|\Gamma^*\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}\|f\|_p$ *for all* $1<p<\infty$ *whenever* $\nu>0$.

It should be remarked here that the assumption $\Gamma^*\in L^1(\mathbf{S}^{n-1})$ in Theorem 1.3 allows the function Γ to be very general. In particular, if $b\in L^\infty(0,\infty)$ and $\Omega(y)=\Omega(y')\in L^1(\mathbf{S}^{n-1})$, then $\Gamma^*\in L^1(\mathbf{S}^{n-1})$ where $\Gamma(y)=b(|y|)\Omega(y)$. Some applications of Theorem 1.3 shall be highlighted in Section 6. In particular, we shall present two model results showing the use of Theorem 1.3 to prove L^p estimates of certain operators of Marcinkiewicz type with oscillating kernels.

By Theorem 1.3(c), the problem of investigating the L^p mapping properties of $S_{\Gamma,h_\nu,\alpha}$ reduces to the case $\nu\leq 0$. However, when $\nu=0$ and $\Gamma=\Omega$, it can be shown that the operator

$S_{\Omega,h_0,\alpha}$ has L^p mapping properties similar to those for the operator μ_Ω. In fact, if $h_0 \in \mathcal{B}_0$, then $h_0 \in L^\infty(0,\infty)$ and thus by the same argument in [4] it follows that $S_{\Omega,h_0,\alpha}$is bounded on L^p provided that $\Omega \in L(\log^+ L)^{\frac{1}{2}}(\mathbf{S}^{n-1})$. Hence, it remains to investigate the L^p mapping properties of $S_{\Gamma,h_\nu,\alpha}$ for $-1 < \nu < 0$. We shall start by characterizing the L^2 boundedness. First, we shall establish the following L^2 multiplier theorem:

Theorem 1.4. *Suppose that $-1<\nu<0$, $h_\nu \in \mathcal{B}_\nu$, and that $\Gamma = \Omega \in L^1(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ satisfying (1.1). Let $m_{\Omega,\nu,\alpha}$ be the L^2 multiplier of the operator $S_{\Omega,h_\nu,\alpha}$, i.e.,*

$$\left\|S_{\Omega,h_\nu,\alpha}(f)\right\|_2^2 = \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 m_{\Omega,\nu,\alpha}(\xi)d\xi. \tag{1.13}$$

Let $\mathcal{E}_{\nu,\alpha}(t) = it^{-1}e^{-i\frac{\pi(1-2\nu(1+\alpha))sig(t)}{2}}, t \in \mathbf{R}\backslash\{0\}$. Then there exists a $H_{\Omega,\alpha} \in L^\infty(\mathbf{R}^n)$ such that

$$m_{\Omega,\nu,\alpha}(\xi) = H_{\Omega,\alpha}(\xi) + C_\nu\,|\xi|^{-2\nu(1+\alpha)}\,\tilde{m}_{\Omega,\nu,\alpha}(\xi')$$

where

$$\tilde{m}_{\Omega,\nu,\alpha}(\xi') = \int_{|y|<1}\int_{|z|<1} \frac{|\xi'\cdot(y-z)|^{1-2\nu(1+\alpha)}\,\Omega(y')\overline{\Omega(z')}}{(|y|\,|z|)^{n-\nu(1+\alpha)-1}}\mathcal{E}_{\nu,\alpha}(\xi'\cdot(y-z))dydz$$

for $\nu(1+\alpha) \in (-1,0)\backslash\{-1/2\}$ and

$$\tilde{m}_{\Omega,\frac{-1}{2(1+\alpha)},\alpha}(\xi') = \int_{|y|<1}\int_{|z|<1} \frac{|\xi'\cdot(y-z)|\,\Omega(y')\overline{\Omega(z')}}{(|y|\,|z|)^{n-\frac{1}{2}}}\left\{\frac{-\pi}{2} - i\log^+\frac{1}{|\xi'\cdot(y-z)|}\right\}dydz.$$

where $C_{\nu,\alpha} = -\Gamma(1+2\nu(1+\alpha))/2\nu(1+\alpha)$ for $-1/2 < \nu(1+\alpha) < 0$ and $C_{\nu,\alpha} = \Gamma(2+2\nu(1+\alpha))/2\nu(1+\alpha)(1+2\nu(1+\alpha))$ for $-1 < \nu(1+\alpha) < -1/2$.

By observing that $\sup_{\xi\in\mathbf{R}^n}\left|\tilde{m}_{\Omega,\nu,\alpha}(\xi')\right| \le C_{\nu,\alpha}\,\|\Omega\|_1^2$, we immediately obtain

$$\left\|S_{\Omega,h_\nu,\alpha}(f)\right\|_2^2 \le 2\max(\left\|H_{\Omega,\alpha}\right\|_\infty, C_{\nu,\alpha}\,\|\Omega\|_1^2)\int_{\mathbf{R}^n}\left|\hat{f}(\xi)\right|^2(1+|\xi|^{-2\nu(1+\alpha)})d\xi. \tag{1.14}$$

Hence,

Corollary 1.5. *Suppose that $-1 < \nu < 0, \alpha > -1$, $h_\nu \in \mathcal{B}_\nu$, and that $\Gamma = \Omega \in L^1(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ satisfying (1.1). Then, $S_{\Omega,h_\nu,\alpha}$ is bounded operator from the inhomogeneous Sobolev space $\mathcal{L}^2_{-\nu(1+\alpha)}$ into the Lebesgue space L^2.*

In light of the result of Corollary 1.5, it is natural to conjecture whether the operator $S_{\Omega,h_\nu,\alpha}$ maps $\mathcal{L}^p_{-\nu(1+\alpha)}$ into L^p under the sole condition $\Omega \in L^1(\mathbf{S}^{n-1})$ for $1 < p < \infty$, $\nu > -1$, and $\alpha > -1$. In the following theorem, we give partial answer to this problem:

Theorem 1.6. *Suppose that $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ is a homogeneous function of degree zero on $\mathbf{R}^n$ satisfying (1.1). Suppose that $-1 < \nu < 0$, $0 < 1+\alpha < -1/\nu$, and $h_\nu \in \mathcal{B}_\nu$. Then for $2/(2+\nu(1+\alpha)) < p < -2/\nu(1+\alpha)$, there exists a constant $C_p > 0$ such that*

$$\left\|S_{\Omega,h_\nu,\alpha}(f)\right\|_{L^p} \le C_p\,\|f\|_{\mathcal{L}^p_{-\nu(1+\alpha)}}. \tag{1.15}$$

By Theorem 1.6 and the discussion in Remark (i) above, we immediately obtain the following analogues of Theorem 1.2 for the parametric operator

Corollary 1.7. *Suppose that* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ *is a homogeneous function of degree zero on* $\mathbf{R}^n$ *satisfying (1.1). Suppose that* $0 < \rho < 1$ *and* $0 < \beta < 1$. *Then for* $2/(2-\beta) < p < 2/\beta$, *there exists a constant* $C_p > 0$ *such that*

$$\left\| \mu^{(\rho)}_{\Omega,\beta}(f) \right\|_{L^p} \le C_p \|f\|_{\mathcal{L}^p_\beta}. \tag{1.16}$$

Another consequence of Theorem 1.6 is the following result concerning the fractional Marcinkiewicz integral operator $M_{\Omega,\delta}$ in (1.9):

Corollary 1.8. *Suppose that* $\Omega \in L(\log^+ L)(\mathbf{S}^{n-1})$ *is a homogeneous function of degree zero on* $\mathbf{R}^n$ *satisfying (1.1). Suppose that* $-1 < \delta < 0$. *Then for* $2/(2+\delta) < p < -2/\delta$, *there exists a constant* $C_p > 0$ *such that*

$$\left\| M_{\Omega,\delta}(f) \right\|_{L^p} \le C_p \|f\|_{\mathcal{L}^p_{-\delta}}. \tag{1.17}$$

This paper is organized as follows. In Section 2, we shall recall the definition of Sobolev spaces. In Section 3, we shall prove Theorem 1.3. Section 4 is devoted to the proof of Theorem 1.4. The proof of Theorem 1.6 will be given in Section 5. Finally, Section 6 is devoted for further results.

Throughout this paper the letter C will stand for a constant that may vary at each occurrence, but it is independent of the essential variables. Also, we shall write $d\sigma(y',z')$ to denote $d\sigma(y')d\sigma(z')$. Finally, we write $\Re(a)$ to denote the real part of a.

2 Sobolev Spaces

Sobolev spaces have several equivalent definitions. In the following, we present a definition through Triebel-Lizorkin spaces (a Littlewood-Paly characterization) [13]. Let β be a real number and $1 < p < \infty$. The homogeneous Sobolev space L^p_α is defined to be the space of all tempered distributions modulo polynomials in $\mathcal{S}'(\mathbf{R}^n)/\mathcal{P}$ for which $(|\xi|^\beta \hat{u})\check{} \in L^p(\mathbf{R}^n)$ where $\mathcal{S}(\mathbf{R}^n)$ is the class of Schwartz functions. The norm in L^p_β is defined by

$$\|u\|_{L^p_\beta} = \left\| (|.|^\beta \hat{u})\check{} \right\|_{L^p}.$$

In particular,

$$\|u\|_{L^2_\beta} \approx \left(\int_{\mathbf{R}^n} |\hat{u}(\xi)|^2 |\xi|^{2\beta} d\xi \right)^{\frac{1}{2}}.$$

The following Littlewood-Paley characterization of L^p_β which is contained in Theorem 6.2.7 of [13]) will be useful:

Lemma 2.1 *Let* Ψ *be a radial function in* $\mathcal{S}(\mathbf{R}^n)$ *whose Fourier transform is nonnegative and supported in an annulus. Let* β *be a real number and* $1 < p < \infty$. *Then there exists a constant* C *depends only on* n, β, p, *and* Ψ *such that for all* $f \in L^p_\beta$, *we have*

$$\left\| \left(\sum_{j \in Z} 2^{2\beta k} \left| \Psi_j * f(x) \right|^2 \right)^{\frac{1}{2}} \right\|_p \le C \|f\|_{L^p_\beta}.$$

We remark here that the space L^p_β is also known as the homogenous Triebel-Lizorkin space $\dot{F}^{\beta,2}_p(\mathbf{R}^n)$.

The inhomogeneous Sobolev space $\mathcal{L}^p_\beta$ is defined to be the set of all functions f satisfying

$$\|f\|_{\mathcal{L}^p_\beta} = \|f\|_{L^p_\beta} + \|f\|_{L^p} < \infty.$$

3 The Kernel Sole Integrability Condition

In this section, we shall present a proof of Theorem 1.3. Our argument will be mainly based on Plancherel's theorem and duality argument. The detailed proof is as follows:

Proof of Theorem 1.3. We shall start by the proof of (a). Assume that $\nu > -1$. Then

$$\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|_2^2 = \int_{\mathbf{R}^n} \left|\hat{f}(\xi)\right|^2 m^{(\infty)}_{\Gamma,\nu,\alpha}(\xi)d\xi \tag{3.1}$$

where

$$m^{(\infty)}_{\Gamma,\nu,\alpha}(\xi) = \int_1^\infty \left|\int_{|y|<u} e^{-i\xi\cdot y}\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha |y|)dy\right|^2 \frac{du}{u^3}.$$

Now we can easily write

$$m^{(\infty)}_{\Gamma,\nu,\alpha}(\xi) \le m^{(\infty,1)}_{\Gamma,\nu,\alpha}(\xi) + m^{(\infty,2)}_{\Gamma,\nu,\alpha}(\xi) \tag{3.2}$$

where

$$m^{(\infty,1)}_{\Gamma,\nu,\alpha}(\xi) = \int_1^\infty \left(\int_{\mathbf{S}^{n-1}} \left|\int_0^{u^{-1-\alpha}} e^{-i\xi\cdot y\prime ur}\Gamma(ury')h_\nu(u^{1+\alpha}r)dr\right| d\sigma(y')\right)^2 \frac{du}{u}$$

and

$$m^{(\infty,2)}_{\Gamma,\nu,\alpha}(\xi) = \int_1^\infty \left(\int_{\mathbf{S}^{n-1}} \left|\int_{u^{-1-\alpha}}^1 e^{-i\xi\cdot y\prime ur}\Gamma(ury')h_\nu(u^{1+\alpha}r)dr\right| d\sigma(y')\right)^2 \frac{du}{u}.$$

The aim is to show that the functions $m^{(\infty,1)}_{\Gamma,\nu,\alpha}$ and $m^{(\infty,2)}_{\Gamma,\nu,\alpha}$ are essentially bounded. To see that $m^{(\infty,1)}_{\Gamma,\nu,\alpha}$ is bounded, we use the local behavior of the function h_ν. In fact, by the condition (1.3), we obtain

$$m^{(\infty,1)}_{\Gamma,\nu,\alpha}(\xi) \le \left\|\Gamma^*\right\|^2_{L^1} \int_1^\infty \left(\int_0^{\frac{1}{u^{1+\alpha}}} (u^{1+\alpha}r)^\nu dr\right)^2 \frac{du}{u} = \left(\frac{\|\Gamma^*\|_{L^1}}{\sqrt{2\alpha+2(\nu+1)}}\right)^2. \tag{3.3}$$

On the other hand, to treat the function $m^{(\infty,2)}_{\Gamma,\nu,\alpha}$ we make use of the asymptotic behavior of the function h_ν at ∞. In fact, by making use of the condition (1.5), we obtain

$$m^{(\infty,2)}_{\Gamma,\nu,\alpha}(\xi) \le C\left\|\Gamma^*\right\|^2_{L^1} \int_1^\infty \left(\int_{u^{-1-\alpha}}^1 (u^{1+\alpha}r)^{-\varepsilon_\nu} dr\right)^2 \frac{du}{u} \le \left(\frac{\|\Gamma^*\|_{L^1}}{\sqrt{2\varepsilon_\nu(1+\varepsilon_\nu)(1+\alpha)}}\right)^2. \tag{3.4}$$

By (3.2), (3.3), and (3.4), we get

$$\left\|m^{(\infty)}_{\Gamma,\nu,\alpha}\right\|_{\infty} = \sup_{\xi\in\mathbf{R}^n} m^{(\infty)}_{\Gamma,\nu,\alpha}(\xi) < C_{\nu,\alpha}\left\|\Gamma^*\right\|^2_{L^1} < \infty \tag{3.5}$$

where $C_{\nu,\alpha} = \left(\varepsilon_\nu(1+\varepsilon_\nu)+(\nu+1)^2\right)/\left(2(\alpha+1)\varepsilon_\nu(1+\varepsilon_\nu)(\nu+1)^2\right)$. Thus, by (3.1) and (3.5), we get

$$\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|_2 \le \sqrt{C_{\nu,\alpha}}\left\|\Gamma^*\right\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}\|f\|_2\,. \tag{3.6}$$

Next, for $p>2$, choose a non-negative function $g\in L^{(\frac{p}{2})'}$ with $\|g\|_{(\frac{p}{2})'}=1$ such that

$$\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|^2_p = \int_{\mathbf{R}^n}\int_1^\infty\left|\int_{|y|\le u} f(x-y)\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha|y|)dy\right|^2 g(x)\frac{dudx}{u^3}.$$

Thus,

$$\begin{aligned}
&\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|^2_p \\
= &\int_1^\infty\int_{\mathbf{R}^n}\left|\int_{|y|\le u} f(x-y)\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha|y|)dy\right|^2 g(x)\frac{dxdu}{u^3} \\
\le &\int_1^\infty\left(\int_{|y|\le u}\left(\int_{\mathbf{R}^n}\left|f(x-y)\frac{\Gamma(y)}{|y|^{n-1}}h_\nu(u^\alpha|y|)\right|^2 g(x)dx\right)^{\frac{1}{2}}dy\right)^2\frac{du}{u^3} \\
= &\int_1^\infty\left(\int_{|y|\le u}\frac{|\Gamma(y)|}{|y|^{n-1}}\left|h_\nu(u^\alpha|y|)\right|\left(\int_{\mathbf{R}^n}|f(x-y)|^2 g(x)dx\right)^{\frac{1}{2}}dy\right)^2\frac{du}{u^3} \\
\le &\|g\|_{(\frac{p}{2})'}\|f\|^2_p\int_1^\infty\left(\int_{|y|\le u}|y|^{-n+1}\left|\Gamma(y)h_\nu(u^\alpha|y|)\right|dy\right)^2\frac{du}{u^3} \\
= &\|f\|^2_p\,\tilde{m}^{(\infty)}_{\Gamma,\nu,\alpha}(0) \le C_{\nu,\alpha}\left\|\Gamma^*\right\|^2_{L^1}\|f\|^2_p
\end{aligned} \tag{3.7}$$

where $\tilde{m}^{(\infty)}_{\Gamma,\nu,\alpha}(0)$ has the same definition as $m^{(\infty)}_{\Gamma,\nu,\alpha}(0)$ with Γ and h_ν are replaced by $|\Gamma|$ and $\left|h_\nu\right|$ respectively. Therefore,

$$\left\|S^{(\infty)}_{\Gamma,h_\nu,\alpha}(f)\right\|_p \le \sqrt{C_{\nu,\alpha}}\left\|\Gamma^*\right\|_{\mathbf{L}^1(\mathbf{S}^{n-1})}\|f\|_p\,. \tag{3.8}$$

By (3.6), (3.8), and duality, the proof of (a) is complete.

Now, we prove (b). We shall prove (b) for $p=2$. The proof for other $p's$ follows by similar argument as that used in the proof of (a). However, the case $p=2$ follows by similar argument as in the corresponding case in (a) and noticing that

$$\int_0^1 \left| \int_{|y|<u} e^{-i\xi\cdot y} |y|^{-n+1} \Gamma(y) h_\nu(u^\alpha |y|) dy \right|^2 \frac{du}{u^3}$$

$$\leq \; C \left\|\Gamma^*\right\|^2_{\mathbf{L}^1(\mathbf{S}^{n-1})} \int_0^1 u^{2(1+\alpha)\nu-1} \left(\int_0^1 r^\nu dr \right)^2 du \leq \frac{C \left\|\Gamma^*\right\|^2_{\mathbf{L}^1(\mathbf{S}^{n-1})}}{2(1+\alpha)\nu(\nu+1)^2}.$$

Finally, the L^p inequality in Theorem 1.3(c) follows by Theorem 1.3(a), Theorem 1.3(b), and (1.12). This completes the proof of Theorem 1.3.

4 An $\mathbf{L}^2$ Characterization

In order to establish the multiplier formula in Theorem 1.4, we start by establishing the following decomposition:

Proposition 4.1. *Suppose that $\Omega \in L^1(\mathbf{S}^{n-1})$ that satisfies (1.1) and that $-1 < \nu < 0$* and *$\alpha > -1$. Let $m_{\Omega,\nu,\alpha}$ be the L^2 multiplier of the operator $S_{\Omega,h_\nu,\alpha}$ in (1.13). Then*

$$m_{\Omega,\nu,\alpha}(\xi) = H_{\Omega,\alpha}(\xi) + B_{\Omega,\alpha}(\xi) \tag{4.1}$$

where $H_{\Omega,\alpha}$ is an $L^\infty(\mathbf{R}^n)$ function and $B_{\Omega,\alpha}$ is a function given by

$$B_{\Omega,\alpha}(\xi) = A_\nu^2 \int_0^\infty \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} |y|^\nu dy \right|^2 \frac{du}{u^{1-2\nu(1+\alpha)}}.$$

Proof. We start by writing h_ν as

$$h_\nu^*(t) = h_\nu(t) - A_\nu t^\nu.$$

Then

$$\begin{aligned} m_{\Omega,\nu,\alpha}(\xi) &= \int_0^\infty \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h_\nu(u^{1+\alpha}|y|) dy \right|^2 \frac{du}{u} \\ &= \int_0^\infty \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} \left(A_\nu \left|u^{1+\alpha} y\right|^\nu + h_\nu^*(u^{1+\alpha}|y|) \right) dy \right|^2 \frac{du}{u} \\ &= A_\nu^2 \int_0^\infty \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{|y|^\nu \Omega(y')}{|y|^{n-1}} dy \right|^2 \frac{du}{u^{1-2\nu(1+\alpha)}} + H_{\Omega,\alpha}(\xi) \end{aligned} \tag{4.2}$$

where

$$H_{\Omega,\alpha}(\xi) = H^{(1)}_{\Omega,\alpha}(\xi) + H^{(2)}_{\Omega,\alpha}(\xi),$$

$$H^{(1)}_{\Omega,\alpha}(\xi) = \int_0^{\infty} \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h^*_\nu(u^{1+\alpha}|y|)dy \right|^2 \frac{du}{u}, \tag{4.3}$$

$$H^{(2)}_{\Omega,\alpha}(\xi) = 2A_\nu \Re\Big(\int_0^{\infty} \int_{|y|<1} \int_{|z|<1} \frac{E_{\Omega,\nu,\alpha}(\xi,y,z,u)}{(|y||z|)^{n-1}} \frac{dydzdu}{u}\Big),$$

and

$$E_{\Omega,\nu,\alpha}(\xi,y,z,u) = \left(e^{-iu\xi\cdot y}\Omega(y')\left|u^{1+\alpha}y\right|^{\nu}\right)\left(e^{iu\xi\cdot z}\overline{\Omega(z')}\overline{h^*_\nu(u^{1+\alpha}|z|)}\right).$$

We show that $H^{(1)}_{\Omega,\alpha}, H^{(2)}_{\Omega,\alpha} \in L^\infty(\mathbf{R}^n)$. By (1.5), it follows that

$$\left|h^*_\nu(r)\right| = \left|h_\nu(r) - A_\nu r^\nu\right| \le C(r^{-\varepsilon_\nu} + |A_\nu| r^\nu) \text{ for } r \ge 1. \tag{4.4}$$

Therefore, for $u \ge 1$, we have

$$\left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h^*_\nu(u^{1+\alpha}|y|)dy \right| \le I + II$$

where

$$I = \left| \int_{u^{-1-\alpha}<|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h^*_\nu(u^{1+\alpha}|y|)dy \right|$$

and

$$II = \left| \int_{|y|<u^{-1-\alpha}} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h^*_\nu(u^{1+\alpha}|y|)dy \right|.$$

It is not hard to see that

$$I \le C \int_{u^{-1-\alpha}<|y|<1} \frac{|\Omega(y')|}{|y|^{n-1}} \left((u^{1+\alpha}|y|)^{-\varepsilon_\nu} + |A_\nu|(u^{1+\alpha}|y|)^\nu\right) dy$$

and

$$II \le C \int_{|y|<u^{-1-\alpha}} \frac{|\Omega(y')|}{|y|^{n-1}} (u^{1+\alpha}|y|)^{\beta_\nu} dy.$$

Therefore, we immediately obtain

$$I + II \le C\Big(\frac{1}{u^{(1+\alpha)\varepsilon_\nu}} + u^{\nu(1+\alpha)}\Big)\|\Omega\|_1.$$

Thus,

$$\begin{aligned} &\int_1^{\infty} \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h^*_\nu(u^{1+\alpha}|y|)dy \right|^2 \frac{du}{u} \\ \le\ & C\|\Omega\|_1^2 \int_1^{\infty} \frac{u^{-2(1+\alpha)\varepsilon_\nu} + u^{2\nu(1+\alpha)} + 2u^{(\nu-\varepsilon_\nu)(1+\alpha)}}{u} du \le C\|\Omega\|_1^2. \end{aligned} \tag{4.5}$$

On the other hand,

$$\begin{aligned}
&\int_0^1 \left| \int_{|y|<1} e^{-iu\xi\cdot y} \frac{\Omega(y')}{|y|^{n-1}} h_\nu^*(u^{1+\alpha}|y|)dy \right|^2 \frac{du}{u} \\
\leq\ & C_\nu \int_0^1 \left(\int_{|y|<1} |y|^{-n+\beta_\nu+1} \left|\Omega(y')\right| u^{(1+\alpha)\beta_\nu} dy \right)^2 \frac{du}{u} \\
\leq\ & C_\nu \|\Omega\|_1^2 \left(\int_0^1 u^{2(1+\alpha)\beta_\nu-1} du \right) \left(\int_0^1 r^{\beta_\nu} du \right)^2 \leq C\|\Omega\|_1^2 .
\end{aligned} \tag{4.6}$$

By (4.5) and (4.6), we have

$$\sup_{\xi\in\mathbf{R}^n} H_{\Omega,\alpha}^{(1)}(\xi) \leq C_\nu \|\Omega\|_1^2 . \tag{4.7}$$

Next, by the observation

$$\int_{|y|<1} \frac{|\Omega(y')|\,|y|^\nu}{|y|^{n-1}} dy \leq C\|\Omega\|_1$$

and (4.6), it can be shown that

$$2\Re\Big(\int_1^\infty \int_{|y|<1} \int_{|z|<1} \frac{E_{\Omega,\nu,\alpha}(\xi,y,z,u)}{u(|y|\,|z|)^{n-1}} dydzdu\Big) \leq C\|\Omega\|_1^2 . \tag{4.8}$$

The verification of (4.8) is straightforward. In fact,

$$\begin{aligned}
&2\Re\Big(\int_1^\infty \int_{|y|<1} \int_{|z|<1} \frac{E_{\Omega,\nu,\alpha}(\xi,y,z,u)}{u(|y|\,|z|)^{n-1}} dydzdu\Big) \\
\leq\ & \int_1^\infty \int_{|y|<1} \int_{|z|<1} \frac{|\Omega(y')|\,|\,y|^\nu\,|\Omega(z')|\,(\left|u^{1+\alpha}z\right|^{-\varepsilon_\nu} + |A_\nu|\left|u^{1+\alpha}z\right|^\nu)}{u^{-\nu(1+\alpha)}(\,|y|\,|z|\,)^{n-1}} dydzdu \\
\leq\ & 2C\|\Omega\|_1^2 \int_1^\infty (u^{(\nu-\varepsilon_\nu)(1+\alpha)-1} + u^{2\nu(1+\alpha)-1})du \leq C\|\Omega\|_1^2 .
\end{aligned}$$

Now, we observe that

$$\begin{aligned}
&2\Re\Big(\int_0^1 \int_{|y|<1} \int_{|z|<1} \frac{E_{\Omega,\nu,\alpha}(\xi,y,z,u)}{(|y|\,|z|)^{n-1}} \frac{dydzdu}{u}\Big) \\
\leq\ & 2\|\Omega\|_1^2 \int_0^1 u^{(\nu+\beta_\nu)(1+\alpha)-1} du = \frac{\|\Omega\|_1^2}{(\nu+\beta_\nu)(1+\alpha)}.
\end{aligned} \tag{4.9}$$

By (4.8) and (4.9), we get

$$\sup_{\xi\in\mathbf{R}^n} H^{(2)}_{\Omega,\alpha}(\xi) \le C\,\|\Omega\|_1^2. \tag{4.10}$$

Hence, by (4.2), (4.7), and (4.10), the proof is complete.

Now, in order to give an explicit expression of the function $B_{\Omega,\alpha}$ in terms of the function Ω, we need the following proposition:

Proposition 4.2. *Suppose that* $1<\alpha<3$ *and that* $a\in\mathbf{R}$. *Then*

$$\int_0^\infty \frac{\sin(at)-a\sin t}{t^\alpha}dt = \frac{\left(-a^2+|a|^\alpha\right)C_1}{a(1-\alpha)}, 1<\alpha<2. \tag{4.11}$$

$$\int_0^\infty \frac{\sin(at)-a\sin t}{t^\alpha}dt = \frac{(-a^2+|a|^\alpha)C_2}{a(\alpha-1)(\alpha-2)}, 2<\alpha<3. \tag{4.12}$$

$$\int_0^\infty \frac{\sin(at)-a\sin t}{t^2}dt = a\log\frac{1}{|a|}. \tag{4.13}$$

$$\int_0^\infty \frac{\cos(at)-a\cos t+a-1}{t^\alpha}dt = \frac{\left(a-|a|^{\alpha-1}\right)C_3}{(1-\alpha)}, 1<\alpha<2. \tag{4.14}$$

$$\int_0^\infty \frac{\cos(at)-a\cos t+a-1}{t^\alpha}dt = \frac{\left(a-|a|^{\alpha-1}\right)C_4}{(\alpha-1)(\alpha-2)},\ 2<\alpha<3. \tag{4.15}$$

$$\int_0^\infty \frac{\cos(at)-a\cos t+a-1}{t^2}dt = \frac{\pi}{2}(a-|a|). \tag{4.16}$$

where $C_1=\Gamma(2-\alpha)\cos(\frac{\pi\alpha}{2}), C_2=\Gamma(3-\alpha)\cos(\frac{\pi\alpha}{2}), C_3=\sin(\frac{\pi\alpha}{2})\Gamma(2-\alpha)$, *and* $C_4=\sin(\frac{\pi\alpha}{2})\Gamma(3-\alpha)$.

In order to verify the formulas (4.11)-(4.16) above, one might be able to use certain computer algebra system or consult reference tables of integrals such as [12]. However, for reader's convenience, an elementary proof is carried out below:

Proof . We start by proving (4.13). By integration by parts, we have

$$\begin{aligned}
\int_0^\infty \frac{\sin(at)-a\sin t}{t^2}dt &= \lim_{\epsilon\to0^+,\ N\to\infty}\int_\epsilon^N \frac{\sin(at)-a\sin t}{t^2}dt\\
&= a\lim_{\epsilon\to0^+,\ N\to\infty}\int_\epsilon^N \frac{\cos(at)-\cos t}{t}dt\\
&= a\lim_{\epsilon\to0^+,\ N\to\infty}\left(\int_\epsilon^N \frac{\cos(at)}{t}dt-\int_\epsilon^N\frac{\cos t}{t}dt\right)\\
&= a\lim_{\epsilon\to0^+,\ N\to\infty}\left(\int_{\epsilon|a|}^{N|a|} \frac{\cos t}{t}dt-\int_\epsilon^N\frac{\cos t}{t}dt\right)\\
&= a\lim_{\epsilon\to0^+}\int_{\epsilon|a|}^{\epsilon}\frac{\cos t}{t}dt-\lim_{N\to\infty}\int_N^{N|a|}\frac{\cos t}{t}dt = a\log\frac{1}{|a|}
\end{aligned}$$

where the last limit follows by an application of Lebesgue dominated convergence theorem.

Next, we prove (4.11) and (4.12). Assume that $1 < \alpha < 2$. By integration by parts, we have

$$\int_0^\infty \frac{\sin(at) - a\sin t}{t^\alpha}dt = \frac{a}{\alpha - 1}\int_0^\infty \frac{\cos(at) - \cos t}{t^{\alpha-1}}. \tag{4.17}$$

Since the integrals of $\cos(at)/t^{\alpha-1}$ and $\cos(t)/t^{\alpha-1}$ over $(0,\infty)$ are finite, the integral in (4.17) can be written as

$$\int_0^\infty \frac{\sin(at) - a\sin t}{t^\alpha}dt = \frac{a(|a|^{\alpha-2} - 1)}{\alpha - 1}\int_0^\infty \frac{\cos t}{t^{\alpha-1}}dt. \tag{4.18}$$

By integrating the analytic branch $e^{iz}/z^{\alpha-1}, -\pi/2 < \arg(z) < 3\pi/2$ of the multi-valued function $e^{iz}/z^{\alpha-1}$ over the contour consisting of of the horizontal line segment from r to R, the first quadrant arc of the circle with center at the origin and radius R, the vertical line segment from iR to ir, and the first quadrant arc of the circle with center at the origin and radius r where $0 < r < R < \infty$ ($r \to 0^+, R \to \infty$), we obtain

$$\int_0^\infty \frac{\cos t}{t^{\alpha-1}}dt = -\cos(\frac{\alpha\pi}{2})\Gamma(2-\alpha) \tag{4.19}$$

and

$$\int_0^\infty \frac{\sin t}{t^{\alpha-1}}dt = \sin\frac{\alpha\pi}{2}\Gamma(2-\alpha). \tag{4.20}$$

Thus, by (4.18) and (4.19), we obtain (4.11) for $1 < \alpha < 2$.

Next, assume that $2 < \alpha < 3$. By integration by parts twice, we have

$$\begin{aligned}\int_0^\infty \frac{\sin(at) - a\sin t}{t^\alpha}dt &= \frac{a}{(\alpha-1)(2-\alpha)}\int_0^\infty \frac{a\sin(at) - \sin t}{t^{\alpha-2}} \\ &= \frac{a\left(|a|^{\alpha-2} - 1\right)}{(\alpha-1)(2-\alpha)}\int_0^\infty \frac{\sin t}{t^{\alpha-2}}dt.\end{aligned}$$

Thus by (4.20) with $\alpha - 1$ is replaced by $\alpha - 2$, we obtain (4.12).

Now, we prove (4.14). By integration by parts, we have

$$\begin{aligned}&\int_0^\infty \frac{\cos(at) - a\cos t + a - 1}{t^\alpha}dt \\ &= \frac{-a}{1-\alpha}\int_0^\infty \frac{\sin(at) - \sin t}{t^{\alpha-1}}dt \\ &= \frac{-a}{1-\alpha}(\frac{a}{|a|^{-\alpha+3}} - 1)\int_0^\infty \frac{\sin t}{t^{\alpha-1}}dt\end{aligned}$$

which by (4.20) implies (4.14).

To prove (4.15), we integrate by parts twice and make use of (4.19). Similarly, we can prove (4.16). This completes the proof.

Now, we are ready to prove Theorem 1.4:

Proof of Theorem 1.4. By Plancherel's theorem, Preposition 4.1, and scaling, it follows that

$$m_{\Omega,\nu,\alpha}(\xi) = H_{\Omega,\alpha}(\xi) + C_\nu |\xi|^{-2\nu(1+\alpha)} \tilde{m}_{\Omega,\nu,\alpha}(\xi') \tag{4.21}$$

where

$$\tilde{m}_{\Omega,\nu,\alpha}(\xi') = \int_0^\infty \left| \int_{|y|<1} e^{-iu\xi'\cdot y} \frac{|y|^\nu \Omega(y')}{|y|^{n-1}} dy \right|^2 \frac{du}{u^{1-2\nu(1+\alpha)}}.$$

By (1.1), we must have

$$\tilde{m}_{\Omega,\nu,\alpha}(\xi') = \lim_{\varepsilon\to 0^+, N\to\infty} \int_{|y|<1} \int_{|z|<1} \frac{\Omega(y')\overline{\Omega(z')}}{(|y||z|)^{n-\nu-1}} I_\alpha(\varepsilon, N, \xi', y, z) dydz \tag{4.22}$$

where

$$I_\alpha(\varepsilon, N, \xi', y, z) = \int_\varepsilon^N \frac{e^{-iu\xi'\cdot(y-z)} - \xi'\cdot(y-z)e^{-iu} + \xi'\cdot(y-z) - 1}{u^{1-2\nu(1+\alpha)}} du.$$

Set

$$l_\nu^\alpha(\xi', y, z) = \frac{\sin(\frac{\pi(1-2\nu(1+\alpha))}{2})}{|\xi'\cdot(y-z)|} + i\frac{\cos(\frac{\pi}{2}(1-2\nu(1+\alpha)))}{\xi'\cdot(y-z)}.$$

By an application of Proposition 4.2 we obtain

$$\tilde{m}_{\Omega,\nu,\alpha}(\xi') = C_{\nu,\alpha} \int_{|y|<1} \int_{|z|<1} \frac{|\xi'\cdot(y-z)|^{1-2\nu(1+\alpha)} \Omega(y')\overline{\Omega(z')}}{(|y||z|)^{n-\nu-1}} l_\nu^\alpha(\xi', y, z) dydz$$

for $\nu(1+\alpha) \in (-1,0)\backslash\{-\frac{1}{2}\}$ and

$$\tilde{m}_{\Omega,-\frac{1}{2(1+\alpha)},\alpha}(\xi')$$
$$= \int_{|y|<1} \int_{|z|<1} \frac{|\xi'\cdot(y-z)|\Omega(y')\overline{\Omega(z')}}{(|y||z|)^{n-\frac{1}{2}}} \left\{ \frac{-\pi}{2} - i\log\frac{1}{|\xi'\cdot(y-z)|} \right\} dydz.$$

Here $C_{\nu,\alpha} = -\Gamma(1+2\nu(1+\alpha))/2\nu(1+\alpha)$for $-1/2 < \nu(1+\alpha) < 0$ and $C_{\nu,\alpha} = \Gamma(2+2\nu(1+\alpha))/2\nu(1+\alpha)(1+2\nu(1+\alpha))$ for $-1 < \nu(1+\alpha) < -1/2$. This completes the proof.

5 Sobolev Space Estimates

This section is devoted to the proof of Theorem 1.6. The proof involves very delicate argument which is based on good L^2 estimates and crude L^p estimates. The difficulty that arises here is that the Fourier transform estimates are not good enough to imply sufficient decay of the L^2 norm of the particular operators involved. Therefore, we have to work a bit harder on the corresponding crude L^p estimates (below see (5.23)-(5.30) for $p > 2$ and (5.32)-(5.36) for $p < 2$). The detailed proof is below:

Proof of Theorem 1.6. By Theorem 1.3(a), (1.12), and the definition of $\mathcal{L}^p_{-\nu}$, it suffices to show that (1.15) holds for the operator

$$S^{(0)}_{\Omega,h_\nu,\alpha}f(x) = \left(\int_0^1 \left|\int_{|y|\le t} f(x-y)\frac{\Omega(y)}{|y|^{n-1}}h_\nu(t^\alpha |y|)dy\right|^2 \frac{du}{t^3}\right)^{\frac{1}{2}}. \tag{5.1}$$

We start by decomposing the function Ω. As in [5], it is not hard to show that there exist a subset $\mathbf{A}(\Omega) \subset \mathbf{N}$, a sequence of numbers $\{b_m : m \in \mathbf{A}(\Omega)\}$, and a sequence of functions $\{\Lambda_m : m \in \mathbf{A}(\Omega)\}$ with the following properties

$$\int_{\mathbf{S}^{n-1}} \Lambda_m(y')d\sigma(y') = 0; \tag{5.2}$$

$$\|\Lambda_m\|_1 \le C,\ \|\Lambda_m\|_2 \le C2^{4(m+2)} \tag{5.3}$$

$$\Omega(x') = \sum_{m\in\mathbf{A}(\Omega)} b_m\Lambda_m(x') \tag{5.4}$$

$$\sum_{m\in\mathbf{A}(\Omega)} (m+2)b_m \approx \|\Omega\|_{L\log L}. \tag{5.5}$$

Therefore, by definition of $S^{(0)}_{\Omega,h_\nu,\alpha}$ and (5.4), it follows

$$S^{(0)}_{\Omega,h_\nu,\alpha}(f)(x) \le \sum_{m\in\mathbf{A}(\Omega)} b_m S^{(0)}_{\Lambda_m,h_\nu,\alpha}(f)(x) \tag{5.6}$$

where $S^{(0)}_{\Lambda_m,h_\nu,\alpha}$ has the same definition as $S^{(0)}_{\Omega,h_\nu,\alpha}$ with Ω is replaced by Λ_m. Let $\sigma_{m,\nu,\alpha,t}$ be the measure defined by

$$\int f d\sigma_{m,\nu,\alpha,t} = \frac{1}{t}\int_{|y|\le t} f(x-y)\frac{\Lambda_m(y)}{|y|^{n-1}}h_\nu(t^\alpha |y|)dy. \tag{5.7}$$

Then, we have

$$S^{(0)}_{\Lambda_m,h_\nu,\alpha}(f)(x) = \left(\int_0^1 \left|\sigma_{m,\nu,\alpha,t} * f(x)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}. \tag{5.8}$$

Now, as in [8] we choose a C^∞ function φ on $\mathbf{R}$ which is supported in $[\frac{1}{2},2]$, $0 \le \varphi \le 1$, and $\varphi(x) \ge c > 0$ if $3/5 \le x \le 5/3$. Notice that $\varphi(2^j x)$ is supported in $(2^{-j-1}, 2^{-j+1})$ and that $c \le \sum_{j\in Z}\varphi(2^j x) \le 3$. Set

$$\psi(x) = \frac{\varphi(x)}{\sqrt{\sum_{j\in Z}\varphi(2^j x)}} \quad \text{and } \psi_j(x) = \psi(2^j x).$$

Then, $\{\psi_j\}$ is a smooth partition of unity adapted to the intervals $(2^{-j-1}, 2^{-j+1})$ with

$$\sum_{j\in Z} \psi_j(x) = 1. \tag{5.9}$$

Let Φ_j be defined on $\mathbf{R}^n$ by $\hat{\Phi}_j(\xi) = \psi_j(|\xi|^2)$. Then by (5.9) and the definition of $S^{(0)}_{\Lambda_m,h_\nu,\alpha}$, we get

$$S^{(0)}_{\Lambda_m,h_\nu,\alpha}(f)(x) \le \sum_{j\in Z} S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)(x) \tag{5.10}$$

where

$$S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)(x) = \left(\sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \left|\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x)\right|^2 \frac{dt}{t} \right)^{\frac{1}{2}}. \tag{5.11}$$

Now, we claim the following:

$$\left\|\sigma_{m,\nu,\alpha,t}\right\| \le Ct^{\nu(1+\alpha)}; \tag{5.12}$$

$$\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}\,(\xi)\right|^2 \frac{dt}{t} \le C|\xi|^2 2^{2(\nu(1+\alpha)+1)k}; \tag{5.13}$$

$$\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}\,(\xi)\right|^2 \frac{dt}{t} \le 2^{2\nu(1+\alpha)k} \left|2^k\xi\right|^{-\frac{\delta_\nu}{m+2}} C \tag{5.14}$$

for some $0 < \delta_\nu$ which will be suitably chosen. The verification of (5.12) is straightforward. In fact,

$$\begin{aligned}
&\left\|\sigma_{m,\nu,\alpha,t}\right\| \\
\le\ & \frac{1}{t} \int_{|y|\le t} \frac{|\Lambda_m(y)|}{|y|^{n-1}} \left|h_\nu(t^\alpha |y|)\right| dy \\
=\ & \frac{1}{t} \|\Lambda_m\|_1 \int_0^t \left|h_\nu(t^\alpha r)\right| dr \le Ct^{\nu(1+\alpha)} \|\Lambda_m\|_1 \le Ct^{\nu(1+\alpha)}.
\end{aligned}$$

Next, by (5.2) and (5.3), we have

$$\begin{aligned}
\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}\,(\xi)\right|^2 \frac{dt}{t} &\le C \int_{2^k}^{2^{k+1}} \left(\int_{|y|\le t} \frac{|y|\,|\xi|\,|\Lambda_m(y)|}{|y|^{n-1}} \left|t^\alpha y\right|^\nu dy \right)^2 \frac{dt}{t^3} \\
&\le C|\xi|^2 2^{2(\nu(1+\alpha)+1)k}
\end{aligned}$$

which proves (5.13).

Now, we prove (5.14). Notice that

$$\left(\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}(\xi)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

$$= \left(\int_{2^k}^{2^{k+1}} \left|\int_{|y|\le 1} e^{-i\xi\cdot yt} \frac{\Lambda_m(y)}{|y|^{n-1}} h_\nu(t^{1+\alpha}|y|)dy\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

$$\le \int_0^1 \left(\int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} e^{-i\xi\cdot y' tr} \Lambda_m(y) d\sigma(y')\right|^2 \left|h_\nu(t^{1+\alpha}r)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} dr$$

$$\le 2^{\nu(1+\alpha)k} \int_0^1 \left(\int_{2^k}^{2^{k+1}} \left|\int_{\mathbf{S}^{n-1}} e^{-i\xi\cdot y' tr} \Lambda_m(y) d\sigma(y')\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} r^\nu dr$$

$$\le 2^{\nu(1+\alpha)k} \int_0^1 \left(\int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left|\Lambda_m(y')\Lambda_m(z')\right| I_k(r\xi, y', z') d\sigma d\sigma\right)^{\frac{1}{2}} r^\nu dr \qquad (5.15)$$

where

$$I_k(\eta, y', z', r) = \left|\int_1^2 e^{-i\eta\cdot(y'-z')2^k t} \frac{dt}{t}\right|.$$

By integration by parts and making use of the estimate $I_k(r\xi, y', z') \le 1$, we get

$$I_k(r\xi, y', z') \le \left|\xi\cdot(y'-z')2^k r\right|^{-\delta_{\nu,\alpha}} \qquad (5.16)$$

for some positive $\delta_{\nu,\alpha} < \min\{\frac{1}{8}, 2(1+\nu(1+\alpha))\}$. Thus, by (5.15) and (5.16), we get

$$\left(\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}(\xi)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}$$

$$\le \frac{2^{\nu k(1+\alpha)+1}\left|2^k\xi\right|^{-\frac{1}{2}\delta_{\nu,\alpha}}}{2\nu - \delta_{\nu,\alpha} + 2} \left(\int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left|\Lambda_m(y')\Lambda_m(z')\right| \left|\xi'\cdot(y'-z')\right|^{-\delta_{\nu,\alpha}} d\sigma(y', z')\right)^{\frac{1}{2}}$$

$$\le 2^{\nu(1+\alpha)k}\left|2^k\xi\right|^{-\frac{1}{2}\delta_{\nu,\alpha}} \|\Lambda_m\|_2 \left(\sup_{\xi'} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left|\xi'\cdot(y'-z')\right|^{-2\delta_{\nu,\alpha}} d\sigma d\sigma\right)^{\frac{1}{4}}.$$

By observing that

$$\sup_{\xi'} \int_{\mathbf{S}^{n-1}} \int_{\mathbf{S}^{n-1}} \left|\xi'\cdot(y'-z')\right|^{-2\delta_{\nu,\alpha}} d\sigma(y', z') \le C < \infty,$$

the last inequality implies

$$\left(\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}(\xi)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}} \leq 2^{\nu(1+\alpha)k}\left|2^k\xi\right|^{-\frac{1}{2}\delta_{\nu,\alpha}} \|\Lambda_m\|_2 C_\nu. \tag{5.17}$$

By (5.12) and (5.17), we have

$$\begin{aligned}
&\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}(\xi)\right|^2 \frac{dt}{t} \\
\leq\ & \left(2^{2\nu(1+\alpha)k}\right)^{1-\frac{1}{m+2}} \left(2^{2\nu(1+\alpha)k}\left|2^k\xi\right|^{-\delta_{\nu,\alpha}} \|\Lambda_m\|_2\right)^{\frac{1}{m+2}} C \\
\leq\ & 2^{2\nu(1+\alpha)k}\left|2^k\xi\right|^{-\frac{\delta_{\nu,\alpha}}{m+2}} C.
\end{aligned}$$

Then, by Plancherel's theorem, we have

$$\begin{aligned}
&\left\|S^{(0)}_{\Lambda_m,\nu,\alpha,j}(f)\right\|_2^2 \\
=\ & \sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}} \left|\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x)\right|^2 \frac{dt}{t} dx \\
\leq\ & \sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}(\xi)\right|^2 \frac{dt}{t}\left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi. \qquad (5.18)
\end{aligned}$$

Thus, we consider three cases.

Case 1. $j \leq -2$. By (5.18), (5.14), and making use of the support of the function ψ_{j+k}, we have

$$\begin{aligned}
&\left\|S^{(0)}_{\Lambda_m,\nu,\alpha,j}(f)\right\|_2^2 \\
\leq\ & \sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n} 2^{2\nu(1+\alpha)k}\left|2^k\xi\right|^{-\frac{\delta_{\nu,\alpha}}{m+2}} \left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi \\
\leq\ & C2^{\frac{\delta_{\nu,\alpha}}{m+2}(j+1)} \sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n} 2^{2\nu(1+\alpha)k}\left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi \\
\leq\ & C2^{\frac{\delta_{\nu,\alpha}}{m+2}(j+1)} 2^{-2\nu(1+\alpha)(j+1)} \sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n} |\xi|^{-2\nu(1+\alpha)}\left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi \\
\leq\ & C2^{\frac{\delta_{\nu,\alpha}}{m+2}(j+1)} 2^{-2\nu(1+\alpha)(j+1)} \|f\|^2_{L^2_{-\nu(1+\alpha)}}.
\end{aligned}$$

Thus,

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_2 \leq C2^{\frac{\delta_{\nu,\alpha}}{2(m+2)}(j+1)} 2^{-\nu(1+\alpha)(j+1)} \|f\|_{L^2_{-\nu(1+\alpha)}}. \tag{5.19}$$

Case 2. $j \geq 2$. By (5.18), (5.13), and similar argument as in Case 1, we have

$$\begin{aligned}
&\left\| S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f) \right\|_2^2 \\
\leq\ & \sum_{k=-\infty}^{-1} \int_{\mathbf{R}^n} |\xi|^2 \, 2^{2(\nu(1+\alpha)+1)k} \left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi \\
\leq\ & C2^{-2(1+\nu(1+\alpha))(j-1)} \sum_{k=-\infty}^{-1} \int_{\mathbf{R}^n} |\xi|^{-2\nu(1+\alpha)} \left|\hat{f}(\xi)\right|^2 \left(\psi_{j+k}(|\xi|^2)\right)^2 d\xi \\
\leq\ & C2^{-2(1+\nu(1+\alpha))(j-1)} \|f\|^2_{L^2_{-\nu(1+\alpha)}}.
\end{aligned}$$

Thus,

$$\left\| S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f) \right\|_2 \leq C2^{-(1+\nu(1+\alpha))(j-1)} \|f\|_{L^2_{-\nu(1+\alpha)}}. \tag{5.20}$$

Case 3. $-1 \leq j \leq 1$. Since $0 < -\nu(1+\alpha) < 1$ and $\nu(1+\alpha)k(1+\nu(1+\alpha)) + (1+\nu(1+\alpha))k(-\nu(1+\alpha)) = 0$, the estimates (5.12) and (5.13) imply that

$$\int_{2^k}^{2^{k+1}} \left|(\sigma_{m,\nu,\alpha,t})\hat{}\,(\xi)\right|^2 \frac{dt}{t} \leq |\xi|^{-2\nu(1+\alpha)}. \tag{5.21}$$

Thus, by (5.18) and (5.21), we obtain

$$\left\| S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f) \right\|_2 \leq C_{\nu,\alpha} \|f\|_{L^2_{-\nu(1+\alpha)}}. \tag{5.22}$$

Now, we consider the L^p estimates of $S^{(0)}_{\Lambda_k,h_\nu,\alpha,j}$. For $r > 2$, let $s = (r/2)' > 1$. Choose a nonnegative function $g \in L^s$ with $\|g\|_s = 1$ such that

$$\left\| S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f) \right\|_r^2 = \sum_{k=-\infty}^{-1} \int_{\mathbf{R}^n} \int_{2^k}^{2^{k+1}} \left|\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x)\right|^2 g(x) \frac{dt}{t} dx. \tag{5.23}$$

For every real t and integer l, we let $tI_l = [t2^{-l-1}, t2^{-l}] \subset \mathbf{R}$. Notice that

$$\begin{aligned}
&\int_{2^k}^{2^{k+1}} \left|\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x)\right|^2 \frac{dt}{t} \\
= &\int_{2^k}^{2^{k+1}} \left|\frac{1}{t}\int_{|y|<t} \Phi_{j+k} * f(x-y)\frac{\Lambda_m(y)h_\nu(t^\alpha|y|)}{|y|^{n-1}}dy\right|^2 \frac{dt}{t} \\
\leq &\; C\int_{2^k}^{2^{k+1}} \left(\int_{|y|<t} \left|\Phi_{j+k} * f(x-y)\right| \frac{|\Lambda_m(y)|\left|h_\nu(t^\alpha|y|)\right|}{|y|^{n-1}}dy\right)^2 \frac{dt}{t^3} \\
\leq &\; C\int_{2^k}^{2^{k+1}} \int_{|y|<t} \left|\Phi_{j+k} * f(x-y)\right|^2 \frac{|\Lambda_m(y)|\left|h_\nu(t^\alpha|y|)\right|}{|y|^{n-1}} \frac{dydt}{t^{2-\nu(1+\alpha)}} \\
= &\; C\sum_{l=0}^{\infty}\int_{2^k}^{2^{k+1}} \int_{|y|\in tI_l} \left|\Phi_{j+k} * f(x-y)\right|^2 \frac{|\Lambda_m(y)|\left|h_\nu(t^\alpha|y|)\right|}{|y|^{n-1}} \frac{dydt}{t^{2-\nu(1+\alpha)}} \\
\leq &\; C\sum_{l=0}^{\infty} 2^{-\nu(1+\alpha)(l+1)}\int_{2^k}^{2^{k+1}} \int_{|y|\in tI_l} \left|\Phi_{j+k} * f(x-y)\right|^2 \frac{|\Lambda_m(y)|}{|y|^{n-1}} \frac{dydt}{t^{2-\nu(1+\alpha)}}.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_r^2 \\
\leq &\sum_{l=0}^{\infty}\sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n}\int_{2^k}^{2^{k+1}}\int_{|y|\in tI_l} \left|\Phi_{j+k} * f(x-y)\right|^2 \frac{|\Lambda_m(y)|}{|y|^{n-1}} dy g(x) w_{\nu,\alpha,l}(t)dtdx \\
\leq &\sum_{l=0}^{\infty}\sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n} \left|\Phi_{j+k} * f(x)\right|^2 \int_{2^k}^{2^{k+1}}\left(\frac{1}{t2^{-l}}\int_{|y|\in tI_l} g(x+y)\frac{|\Lambda_m(y)|}{|y|^{n-1}}dy\right) w_{\nu,\alpha,l}(t)dtdx \\
\leq &\sum_{l=0}^{\infty}\sum_{k=-\infty}^{-1}\int_{\mathbf{R}^n} 2^{2\nu(1+\alpha)k}2^{-\nu(1+\alpha)(l+1)-l}\left|\Phi_{j+k} * f(x)\right|^2 R_m^*(g)(x)dx \\
= &\; 2^{-\nu(1+\alpha)}\sum_{l=0}^{\infty} 2^{-l(\nu(1+\alpha)+1)}\int_{\mathbf{R}^n}\sum_{k=-\infty}^{-1} 2^{2\nu(1+\alpha)k}\left|\Phi_{j+k} * f(x)\right|^2 R_m^*(g)(x)dx \\
= &\; C2^{-2\nu(1+\alpha)j}\int_{\mathbf{R}^n}\sum_{k=-\infty}^{-1+j} 2^{2\nu(1+\alpha)k}|\Phi_k * f(x)|^2 R_k^*(g)(x)dx \\
\leq &\; C2^{-2\nu(1+\alpha))j}\left\|\left(\sum_{k\in Z} 2^{2\nu(1+\alpha)k}|\Phi_k * f(x)|^2\right)^{\frac{1}{2}}\right\|_r^2 \left\|R_m^*(g)\right\|_s
\end{aligned}$$

where

$$R_m^*(g)(x) = \sup_{j\in\mathbf{Z}} \int_{2^j<|y|<2^{j+1}} g(x+y)\frac{|\Lambda_m(y)|}{|y|^n}dy \tag{5.24}$$

and

$$w_{\nu,\alpha,l}(t) = \frac{t^{2\nu(1+\alpha)}2^{-\nu(1+\alpha)(l+1)}}{t^2}$$

By Hölder's inequality and the boundedness of Hardy-Littlewood maximal function, we have

$$\left\|R_m^*(g)\right\|_s \le C_s \|g\|_s \le C_s; \tag{5.25}$$

along with Lemma 2.1 imply that

$$\left\|S^{(0)}_{\Lambda_m,\nu,\alpha,j}(f)\right\|_r \le C2^{-\nu(1+\alpha)j}\|f\|_{L^r_{-\nu(1+\alpha)}}. \tag{5.26}$$

Now, for $j \le -2$, by interpolation between (5.19) and (5.26), we get

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_p \le 2^{(\frac{\theta\delta_{\nu,\alpha}}{2(m+2)}-\nu(1+\alpha))j}C\|f\|_{L^p_{-\nu(1+\alpha)}} \tag{5.27}$$

for all $p > 2$.

For $j \ge 2$, we choose $r > p$ and we interpolate between (5.26) and (5.20) to get

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_p \le C_{\nu,\alpha}2^{-(\theta_r+\nu(1+\alpha))j}\|f\|_{L^p_{-\nu(1+\alpha)}} \tag{5.28}$$

where

$$\theta_r = \frac{\frac{1}{p}-\frac{1}{r}}{\frac{1}{2}-\frac{1}{r}}.$$

Clearly, if $r\to\infty$, we have $\theta_r \to \frac{2}{p}$ and thus $-\theta_r-\nu(1+\alpha)\to -\frac{2}{p}-\nu(1+\alpha)$ which is negative provided that $p < -2/\nu(1+\alpha)$. Thus, for $2 < p < -2/\nu(1+\alpha)$, there exists $r(p) > 2$ such that

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_p \le C_{\nu,\alpha}2^{-(\theta_r+\nu(1+\alpha))j}\|f\|_{L^p_{-\nu(1+\alpha)}} \tag{5.29}$$

with $-(\theta_{r(p)}+\nu(1+\alpha)) < 0$.

On the other hand, by interpolation between (5.22) and (5.26), we get

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_p \le C_{\nu,\nu}2^{(1-\theta)\nu(1+\alpha)j}\|f\|_{L^p_{-\nu(1+\alpha)}} \tag{5.30}$$

for $2 < p < \infty$.

By (5.10), and (5.27), (5.29), (5.30), we get

$$\begin{aligned}\left\|S^{(0)}_{\Lambda_k,h_\nu,\alpha}(f)\right\|_p &\le C_{\nu,\alpha}\{S_1+S_2+S_3\}\|f\|_{L^p_{-\nu(1+\alpha)}}\\ &\le (m+2)C_{\nu,\alpha}\|f\|_{L^p_{-\nu(1+\alpha)}}\end{aligned} \tag{5.31}$$

where

$$\begin{aligned} S_1 &= \sum_{j=-\infty}^{-2} 2^{(\frac{\theta\delta_{\nu,\alpha}}{2(m+2)}-\nu(1+\alpha))j} \\ S_2 &= \sum_{j=2}^{\infty} 2^{-(\theta_r+\nu(1+\alpha))j} \\ S_3 &= \sum_{j=-1}^{1} 2^{(1-\theta)\nu(1+\alpha)j}. \end{aligned}$$

Now, we consider the case $(-2/\nu(1+\alpha))' < p < 2$. For $(-2/\nu(1+\alpha))' < r < 2$, we have $r' > 2$. We choose a sequence of functions $g_k(x,t)$ on $\mathbf{R}^n \times \mathbf{R}_+$ with

$$\left\| \left(\sum_{k\in Z} \int_{2^k}^{2^{k+1}} |g_k(x,t)|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{r'} \leq 1 \tag{5.32}$$

such that

$$\left\| S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f) \right\|_r = \int_{\mathbf{R}^n} \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \Big(\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x) \Big) g_k(x,t) \frac{dt}{t} dx.$$

Now,

$$\begin{aligned} &\int_{\mathbf{R}^n} \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \Big(\Phi_{j+k} * \sigma_{m,\nu,\alpha,t} * f(x) \Big) g_k(x,t) \frac{dt}{t} dx \\ \leq\ & \sum_{l=0}^{\infty} \int_{\mathbf{R}^n} \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \left(\frac{1}{t} \int_{|y|\in tI_l} \left| \Phi_{j+k} * f(x-y) \right| \frac{|\Lambda_m(y)| \left| h_\nu(t^\alpha |y|) \right|}{|y|^{n-1}} dy \right) g_k(x,t) \frac{dt}{t} dx \\ \leq\ & \sum_{l=0}^{\infty} B_{l,\nu} \int_{\mathbf{R}^n} \sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} 2^{\nu(1+\alpha)k} \left| \Phi_{j+k} * f \right| * \tilde{\sigma}_{m,l,t}(x) g_k(x,t) \frac{dt}{t} dx \\ \leq\ & \sum_{l=0}^{\infty} B_{l,\nu} \int_{\mathbf{R}^n} \sum_{k=-\infty}^{-1} \Big(2^{\nu(1+\alpha)k} \left| \Phi_{j+k} * f \right| \Big) \left(\int_{2^k}^{2^{k+1}} \left| \tilde{\sigma}_{m,l,t} * g_k(x,t) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \\ \leq\ & \sum_{l=0}^{\infty} B_{l,\nu} \int_{\mathbf{R}^n} \left(\sum_{k=-\infty}^{-1} 2^{2\nu(1+\alpha)k} \left| \Phi_{j+k} * f \right|^2 \right) \left(\sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \left| \tilde{\sigma}_{m,l,t} * g_k(x,t) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} dx \\ \leq\ & \sum_{l=0}^{\infty} B_{l,\nu} \left\| \left(\sum_{k=-\infty}^{-1} 2^{2\nu(1+\alpha)k} \left| \Phi_{j+k} * f \right|^2 \right)^{\frac{1}{2}} \right\|_r \left\| \left(\sum_{k=-\infty}^{-1} \int_{2^k}^{2^{k+1}} \left| \tilde{\sigma}_{m,l,t} * g_k(x,t) \right|^2 \frac{dt}{t} \right)^{\frac{1}{2}} \right\|_{r'} \end{aligned} \tag{5.33}$$

where $B_{l,\nu,\alpha} = 2^{-(\nu+1)l-\nu}$ and

$$\int f d\tilde{\sigma}_{m,l,t} = \frac{1}{t2^{-l}} \int\limits_{|y|\in tI_l} f(y)\frac{|\Lambda_m(y')|}{|y|^{n-1}}dy.$$

By observing that

$$\sup_{t,l}\left|\left|\tilde{\sigma}_{m,l,t}\right| * f(x)\right| = R_m^*(f)$$

where R_m^* is given by (5.24), it follows by (5.25), (5.32), and duality argument that

$$\left\|\left(\sum_{k=-\infty}^{-1}\int\limits_{2^k}^{2^{k+1}}\left|\tilde{\sigma}_{m,l,t} * g_k(x,t)\right|^2 \frac{dt}{t}\right)^{\frac{1}{2}}\right\|_{r'} \leq C_{r'}. \tag{5.34}$$

Therefore, by (5.33), (5.34), and Lemma 2.1, we get

$$\left\|S^{(0)}_{\Lambda_m,h_\nu,\alpha,j}(f)\right\|_r \leq C2^{-\nu(1+\alpha)j}\,\|f\|_{L^r_{-\nu(1+\alpha)}}. \tag{5.35}$$

Now, by repeating the steps (5.27) to (5.31) with (5.26) is replaced by (5.35) (here r is close to 1), we obtain

$$\left\|S^{(0)}_{\Lambda_k,h_\nu,\alpha}(f)\right\|_p \leq (m+2)C_{\nu,\alpha}\,\|f\|_{L^p_{-\nu(1+\alpha)}}. \tag{5.36}$$

the case $(-2/\nu(1+\alpha))' < p < 2$. Hence, the proof is concluded by (5.31), (5.36), and (5.5).

6 Further Results

As pointed out in the introduction section, Theorem 1.3 can be used to consider operators with oscillating kernels. This is due to the observation that the Bessel function J_ν is in the class $\mathcal{B}^{(0)}_\nu$. A particular result in this direction is the following:

Corollary 6.1. *Suppose that* $\Gamma^* \in L^1(\mathbf{S}^{n-1})$. *Let* $\vec{\nu} = (\nu_1,...,\nu_m) \in ((-1,\infty))^m$, $\vec{a} = (a_1,...,a_m) \in \mathbf{R}^m_+$, $\alpha > -1$, *and* $\gamma \in \mathbf{R}$ *be such that*
(i) $\gamma + \sum_{j=1}^m a_j\nu_j > 0$ *and (ii)* $\gamma - \frac{1}{2}\sum_{j=1}^m a_j < 0$. *Then the operator*

$$S_{\Gamma,\vec{\nu},\vec{a},\alpha,\gamma}(f)(x) = \left(\int_{-\infty}^{\infty}\left|\int_{|y|<2^t} f(x-y)\left(\prod_{j=1}^m J_{\nu_j}(2^{\alpha t}|y|^{a_j})\right)\frac{\Gamma(y)}{|y|^{n-\gamma-1}}dy\right|^2\frac{dt}{2^{2t}}\right)^{\frac{1}{2}}$$

satisfies

$$\left\|S_{\Gamma,\vec{\nu},\vec{a},\alpha,\gamma}(f)\right\|_p \leq C\,\|f\|_p \tag{6.1}$$

for all $1 < p < \infty$.

Proof of Corollary 6.1. The main idea of the proof is applying Theorem 1.1 (c) with $\nu = \gamma + \sum_{j=1}^m \nu_j a_j, \varepsilon_\nu = \gamma - \frac{1}{2}\sum_{j=1}^m a_j$, and

$$h_\nu(t) = t^\gamma \prod_{j=1}^m J_{\nu_j}(t^{a_j}).$$

In fact, using the local behavior of the Bessel function, we obtain

$$\left|h_\nu(t)\right| \le t^\gamma \prod_{j=1}^{m} (2\pi t^{a_j})^{\nu_j} = Ct^{\gamma+\sum_{j=1}^{m}\nu_j a_j} \tag{6.2}$$

for $t \le 1$. On the other hand, by the asymptotic behavior of the Bessel function at infinity, we obtain

$$\left|h_\nu(t)\right| \le t^\gamma \prod_{j=1}^{m} (2\pi t^{a_j})^{-\frac{1}{2}} = Ct^{\gamma-\frac{1}{2}\sum_{j=1}^{m} a_j} \tag{6.3}$$

for $t > 1$. In conclusion, we get $h_\nu \in \mathcal{B}_\nu^{(0)}$ with $\nu > 0$. Hence, the proof is concluded by Theorem 1.3 (c).

We remark here that L^p estimates of the operators $S_{\Gamma,\vec{\nu},\vec{a},\alpha}$ in Corollary 6.1 imply similar estimates for operators with general oscillatory factors. As a model result, we prove the following:

Theorem 6.2. *Suppose that $G : \mathbf{R}^n \to \mathbf{R}$ is a suitable mapping such that (6.1) holds provided that (i)-(iii) hold. Let $\nu > 0$, $\vec{a} = (a_1, ..., a_m) \in \mathbf{R}_+^m$, and let $\mu(t) = \sum_{j=1}^{m} \varepsilon_j t^{a_j}, t \ge 0$ where $\varepsilon_j = \pm 1$. Then the operator*

$$S_{G,\nu,\mu}(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y) e^{i\mu(|y|)} J_\nu(|y|) \frac{G(y)}{|y|^{n-1}} dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}$$

satisfies $\left\|S_{G,\nu,\mu}(f)\right\|_p \le C \|f\|_p$ for all $1 < p < \infty$.

The proof of Theorem 6.2 is based on the representation of the Bessel functions $J_{1/2}$ and $J_{-1/2}$ in terms of the trigonometric functions $\sin t$ and $\cos t$ respectively. The detailed proof is as follows:

Proof of Theorem 6.2. Without loose of generality, we may assume that $\varepsilon_j = 1$ for all j, i.e., $\mu(t) = \sum_{j=1}^{m} t^{a_j}$. Then by using the identities

$$J_{1/2}(t) = \sqrt{\frac{2}{\pi}} \frac{\sin t}{\sqrt{t}} \quad \text{and } J_{-1/2}(t) = \sqrt{\frac{2}{\pi}} \frac{\cos t}{\sqrt{t}},$$

we obtain

$$\begin{aligned} e^{i\mu(|y|)} &= \prod_{j=1}^{m} e^{i|y|^{a_j}} = (\frac{\pi}{2})^{\frac{m}{2}} |y|^\gamma \prod_{j=1}^{m} (J_{-1/2}(|y|^{a_j}) + iJ_{1/2}(|y|^{a_j})) \\ &= (\frac{\pi}{2})^{\frac{m}{2}} |y|^\gamma \sum_{k=0}^{2^m} c_k \prod_{j\in A_k} J_{-1/2}(|y|^{a_j}) \prod_{j\in B_k} J_{1/2}(|y|^{a_j}) \end{aligned} \tag{6.4}$$

where $A_k \cup B_k = \{1, 2, ..., m\}, A_k \cap B_k = \varnothing$, and $\gamma = \frac{1}{2}\sum_{j=1}^{m} a_j$. Therefore,

$$S_{G,\nu,\mu,\gamma}(f)(x) \le (\frac{\pi}{2})^{\frac{m}{2}} \sum_{k=0}^{2^m} |c_k| S_{G,\gamma,k}(f)(x) \tag{6.5}$$

where

$$S_{G,\gamma,k}(f)(x) = \left(\int_{-\infty}^{\infty} \left| \int_{|y|<2^t} f(x-y)P(t)\frac{G(y)}{|y|^{n-\gamma-1}}dy \right|^2 \frac{dt}{2^{2t}} \right)^{\frac{1}{2}}$$

where

$$P(t) = \left(\prod_{j\in A_k} J_{-1/2}(|y|^{a_j}) \prod_{j\in B_k} J_{1/2}(|y|^{a_j}) \right) J_\nu(|y|).$$

Now, notice that for $0 \le k \le 2^m$, we have

$$\gamma - \frac{1}{2}\sum_{j\in A_k} a_j + \frac{1}{2}\sum_{j\in B_k} a_j + \nu \quad > \quad 0,$$

$$\gamma - \frac{1}{2}\sum_{j=1}^{m} a_j - \frac{1}{2} \quad < \quad 0.$$

Thus, by assumption it follows that $\left\|S_{G,\gamma,k}(f)\right\|_p \le C\,\|f\|_p$ *for all* $1 < p < \infty$ and $0 \le k \le 2^m$. And hence the boundedness of $S_{G,\nu,\mu}$ follows by (6.5) and the aid of Minkowski's inequality. This completes the proof.

Acknowledgments

The publication of this paper is supported by Sultan Qaboos University research grant IG/SCI/DOMAS/09/04. The author thanks the referees for their careful reading of the manuscript and insightful comments.

References

[1] Y. Ding, Q. Xuea, and K. Yabuta, A remark to the L^2 boundedness of parametric Marcinkiewicz integral. J. Math. Anal. Appl. Volume 387, Issue 2, (2011), 691-697.

[2] A. Al-Salman, On the L^2 boundedness of parametric Marcinkiewicz integral operator. J. Math. Anal. Appl. Volume 375, Issue 2, (2011), 745-752.

[3] A. Al-Salman and H. Al-Qassem, On the L^p boundedness of rough parametric Marcinkiewicz functions. J. Inequal. Pure and Appl. Math, 8(4) (2007) Art. 108, 8pp.

[4] A. Al-Salman, H. Al-Qassem, L. Cheng, and Y. Pan, L^p bounds for the functions of Marcinkiewicz. Math. Res. Lett. 9: 697-700.

[5] A. Al-Salman and Y. Pan, Singular integrals with rough kernels in $Llog^+L(\mathbf{S}^{n-1})$. J. London Math. Soc. (2) 66 (2002) 153-174.

[6] A. Benedek, A. Calderón, and R. Panzone, Convolution operators on Banach space valued functions. Proc. Nat. Acad. Sci. U. S. A 48 (1962) 356-365.

[7] A. Córdoba, A note on Bochner-Riesz operators. (1977). Duke Math. J. 46: 505-511.

[8] J. Chen, D. Fan, and Y. Ying, Certain operators with rough singular kernels. Canad. J. Math. Vol. 55 (3), 2003 pp. 504-532.

[9] J. Chen, D. Fan, and Y. Ying, Singular kernel operators on function spaces. J. Math. Anal. Appl. 276 (2002) 691-708.

[10] Y. Ding, S. Lu, and K. Yabuta, A problem on rough parametric Marcinkiewicz functions. J. Austral. Math. Soc. 72 (2002), 13-21.

[11] Y. Ding, D. Fan, and Y. Pan, L^p boundedness of Marcinkiewicz integrals with Hardy space function kernel. Acta. Math. Sinica (English Series) 16 (2000), 593-600.

[12] I. S. Gradshteyn and I. M. Ryzhik, Table of integrals, series, and products. Fifth edition, Academic Press, 1994.

[13] L. Grafakos, Classical and Modern Fourier Analysis. Pearson Education, Inc. Upper Saddle River, New Jersey, 2004.

[14] Hörmander, 'Translation invariant operators'. Acta Math. 104 (1960), 93-139.

[15] S. Sato, Remarks on square functions in the Littlewood-Paley theory. Bull. Austral. Math. Soc. 58 (1998), 199-211.

[16] Z. Y. Si, L. N. Wang, and Y. S. Jiang, fractional type Marcinkiewicz integral on Hardy space. J. Math. Res. Exposition, Mar., 2011, Vol. 31, No. 2, pp 233-241.

[17] E. M. Stein, On the functions of Littlewood-Paley, Lusin and Marcinkiewicz. Trans. Amer. Math. Soc. 88 (1958), 430-466.

[18] E. M. Stein, Harmonic Analysis: Real-Variable Methods, Orthogonality and Oscillatory Integrals. Princeton University Press, Princeton, NJ, 1993.

[19] T. Walsh, On the function of Marcinkiewicz. Studia Math. 44 (1972), 203-217.