Communications in Mathematical Analysis
Volume 13, Number 2, pp. 27–55 (2012) www.math-res-pub.org/cma
ISSN 1938-9787

Existence of Positive Solutions for Some Dirichlet Problems Associated to Fractional Laplacian in Exterior Domains

Rym Chemmam[*]
Département de Mathématiques,
Faculté des Sciences de Tunis,
Campus universitaire, 2092 Tunis, TUNISIA.

Habib Mâagli [†]
Département de Mathématiques,
Faculté des Sciences de Tunis,
Campus universitaire, 2092 Tunis, TUNISIA.

(Communicated by Hitoshi Kitada)

Abstract

In this paper, we use tools of potential theory to study the existence of positive continuous solutions for some boundary value problems based on the fractional Laplacian $(-\Delta)^{\alpha/2}$, $0<\alpha<2$, in an exterior domain D in $\mathbb{R}^n$, $n\geq 3$. Our arguments use properties of an appropriate Kato class of functions $K_\alpha^\infty(D)$.

AMS Subject Classification: 31B05, 31C35, 34B27, 60J50.

Keywords: Fractional Laplacian, Green function, symmetric stable process, Dirichlet problem.

1 Introduction

For $n\geq 2$ and $0<\alpha<2$, an n-dimensional α-stable process is a Levy process $X=(X_t)_{t\geq 0}$ in $\mathbb{R}^n$ whose characteristic function has the form

$$E^x\left(e^{i\zeta(X_t-X_0)}\right)=e^{-t|\zeta|^\alpha} \text{ for } \zeta \text{ and } x\in\mathbb{R}^n,$$

where E^x is the expectation with respect to the distribution P^x of the process starting from $x\in\mathbb{R}^n$.

[*]E-mail address: Rym.Chemmam@fst.rnu.tn
[†]E-mail address: habib.maagli@fst.rnu.tn

In this paper, we always assume that D is a $C^{1,1}$ exterior domain in $\mathbb{R}^n$ $(n \geq 3)$ and we put $\tau_D := \inf\{t > 0 : X_t \notin D\}$, the first exit time of X from D. Let $X_t^D(w) = X_t(w)$ if $t < \tau_D(w)$ and set $X_t^D(w) = \partial$ if $t \geq \tau_D(w)$, where ∂ is the cemetery point. The process $X^D = \left(X_t^D\right)_{t\geq 0}$ (i.e. the process X killed upon leaving D) is called the killed symmetric α-stable process in D. The infinitesimal generator of X^D is the fractional power $(-\Delta)^{\frac{\alpha}{2}}$ of the Laplacian in D, which is a prototype of non-local operator. Considerable progress has been made recently in extending potential-theoretic properties of Brownian motion to symmetric α-stable processes on Lipschitz domains (see for instance [3 – 20, 24 – 26]).

We collect in this paper some basic facts concerning the process X^D, the Green function G_D^α, the Martin kernel M_D^α and α-harmonic functions which are direct adaptations of well known results on Brownian motion. These facts will be useful for our study. In particular, we give precise estimates on G_D^α, which enable us to introduce a functional class $K_\alpha^\infty(D)$ (see Definition 2.4) characterized by an integral condition involving G_D^α and called fractional Kato class. This class is quite rich (see Proposition 4.1) and it is a key tool in our study.

On the other hand, unlike Brownian motion, we prove that harmonic functions with respect to X^D blow up at the boundary of D. While the classical formulation of the Dirichlet problem becomes impossible, we provide an appropriate reformulated Dirichlet problem associated to $(-\Delta)^{\frac{\alpha}{2}}$ in D (see Remark 2.16). This approach allows us to study two different nonlinear Dirichlet problems associated to $(-\Delta)^{\frac{\alpha}{2}}$ in D, what generalize some existence results for nonlinear Dirichlet problems associated to $(-\Delta)$, obtained in [2] and [23].

Our results follow up those obtained in [9] for the fractional Laplacian in a bounded domain. Many well known properties of the killed symmetric α-stable process in a bounded domain are not provided in an exterior domain. This makes one of difficulties in this paper.

The content of the paper is organized as follows. In Section 2, we recapitulate some tools of potential theory pertaining to the process X^D. Namely, we discuss some properties of harmonic functions with respect to X^D and we apply these facts to reformulate Dirichlet problem associated to X^D. Also, we present in this Section our main results (see Theorems 2.17 and 2.18). In Section 3, we establish some estimates on G_D^α and we obtain some properties of potential functions. We give in Section 4 some interesting properties of the class $K_\alpha^\infty(D)$ including a careful analysis about continuity of some potential functions. Our main results are proved in Sections 5 and 6.

2 Preliminaries and Main Results

2.1 Notations and Terminology

In this paper, we always assume that D is a $C^{1,1}$ exterior domain in $\mathbb{R}^n$ $(n \geq 3)$ such that $\overline{D}^c = \bigcup_{1\leq j\leq k} D_j$, where D_j is a bounded $C^{1,1}$ domain of $\mathbb{R}^n$ and $\overline{D_i} \cap \overline{D_j} = \emptyset$, $i \neq j$. We denote by $B^+(D)$ the cone of nonnegative Borel measurable functions defined on D.

It is well known that there is a symmetric function $G_D^\alpha(x,y)$ continuous on $D\times D$ except along the diagonal, called Green function associated to X^D. We will denote $G_D(x,y)$ for the Green function associated to Brownian motion in D (i.e. $\alpha = 2$).

Remark 2.1. Let a $\in \overline{D}^c$ and $r > 0$ such that $\overline{B}(a,r) \subset \overline{D}^c$. Then we have from [12] that for each $x, y \in D$

$$G_D^\alpha(x,y) = r^{\alpha-n} G^\alpha_{\frac{D-a}{r}}\left(\frac{x-a}{r}, \frac{y-a}{r}\right).$$

Thus, without loss of generality, we may assume throughout this paper that $a = 0$ and $r = 1$, that is $\overline{B}(0,1) \subset \overline{D}^c$.

Let $x^* = \dfrac{x}{|x|^2}$ be the Kelvin transformation from D onto $D^* = \{x^* \in B(0,1) : x \in D\}$. Then, it is easy to see that for $x, y \in D$, we have

$$\left|x^* - y^*\right| = \frac{|x-y|}{|x|\,|y|}. \tag{2.1}$$

For $f \in B^+(D)$, we denote by f^* the α-order Kelvin transform of f defined in D^* by

$$f^*(x^*) = |x|^{n-\alpha} f(x).$$

We note that the 2-order Kelvin transform is the usual Kelvin transform. Our interest in Kelvin transform comes primarily from the fact that it transfers questions at the point infinity to those at the origin (see [6] for more details).

Throughout the paper $\delta_D(x)$ denotes the Euclidian distance between x and the boundary ∂D. We put for $x \in D$

$$\rho_D(x) = \frac{\delta_D(x)}{\delta_D(x)+1}$$

and

$$\lambda_D(x) = \delta_D(x)\left(\delta_D(x)+1\right).$$

By simple calculation, we have for $x \in D$,

$$1 + \delta_D(x) \approx |x| \tag{2.2}$$

and

$$\delta_{D^*}(x^*) \approx \rho_D(x) \approx \frac{\delta_D(x)}{|x|} \approx \frac{\lambda_D(x)}{|x|^2}. \tag{2.3}$$

Here for two nonnegative functions f and g defined on a set S, the notation $f(x) \approx g(x)$, $x \in S$, means that there exists $c > 0$ such that $\frac{1}{c} f(x) \le g(x) \le c f(x)$ for all $x \in S$. Also, for $s, t \in \mathbb{R}$, we denote by $\min(s,t) = s \wedge t$ and we remark that for $s, t \ge 0$ and $p \ge 0$

$$s \wedge t \approx \frac{st}{s+t}, \tag{2.4}$$

$$1 \wedge s(1+s) \approx 1 \wedge s \tag{2.5}$$

and

$$(s+t)^p \approx s^p + t^p. \tag{2.6}$$

2.2 Potential theory associated to $(-\Delta)^{\frac{\alpha}{2}}$

The following sharp estimates on $G_D^\alpha(x,y)$ are given in a recent paper of Chen and Tokle (see [14], Corollary 1.5)

$$G_D^\alpha(x,y) \approx \frac{1}{|x-y|^{n-\alpha}}\left(1 \wedge \frac{(\delta_D(y))^{\frac{\alpha}{2}}}{1 \wedge |x-y|^{\frac{\alpha}{2}}}\right)\left(1 \wedge \frac{(\delta_D(x))^{\frac{\alpha}{2}}}{1 \wedge |x-y|^{\frac{\alpha}{2}}}\right). \tag{2.7}$$

In this paper, we give other estimates on $G_D^\alpha(x,y)$, to be used in our approach. First, we remark by ([6], Theorem 2) that for $x,y \in D$

$$G_D^\alpha(x,y) = |x|^{\alpha-n}\,|y|^{\alpha-n}\,G_{D^*}^\alpha(x^*,y^*). \tag{2.8}$$

Hence, from esimates on the Green function $G_{D^*}^\alpha$ of the bounded domain D^* (see [12], Corollary 1.3) and using (2.1) and (2.3), we get the following

$$G_D^\alpha(x,y) \approx \frac{1}{|x-y|^{n-\alpha}}(1 \wedge \frac{(\lambda_D(x)\lambda_D(y))^{\frac{\alpha}{2}}}{|x-y|^\alpha}). \tag{2.9}$$

Remark 2.2. We have obviously by (2.7) and (2.9) that

$$1 \wedge \frac{(\lambda_D(x)\lambda_D(y))^{\frac{\alpha}{2}}}{|x-y|^\alpha} \approx \left(1 \wedge \frac{(\delta_D(y))^{\frac{\alpha}{2}}}{1 \wedge |x-y|^{\frac{\alpha}{2}}}\right)\left(1 \wedge \frac{(\delta_D(x))^{\frac{\alpha}{2}}}{1 \wedge |x-y|^{\frac{\alpha}{2}}}\right).$$

Note that the interesting estimates (2.9) extend those for the Green function G_D of the killed Brownian motion in D. In [2], it was shown a 3G-inequality for G_D allowing to introduce and study the Kato class of functions $K(D)$. This class was extensively used in the study of various elliptic differential equations (see [2, 23]).
Analogously, Theorem 2.3 below provides a fundamental 3G-inequality for G_D^α, as a consequence of the estimates (2.9). Its proof is a direct adaptation of the elliptic case (see [2]). So we omit it.

Theorem 2.3. *(3G Theorem) There exists a positive constant C_0 such that for all x,y and z in D we have*

$$\frac{G_D^\alpha(x,z)G_D^\alpha(z,y)}{G_D^\alpha(x,y)} \leq C_0\left[\left(\frac{\rho_D(z)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,z) + \left(\frac{\rho_D(z)}{\rho_D(y)}\right)^{\frac{\alpha}{2}} G_D^\alpha(y,z)\right]. \tag{2.10}$$

This allows us to introduce a new fractional Kato class of functions in D denoted by $K_\alpha^\infty(D)$ and defined as follows.

Definition 2.4. A Borel measurable function q in D belongs to the Kato class $K_\alpha^\infty(D)$ if q satisfies the following conditions

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{(|x-y|\leq r)\cap D}\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y)\,|q(y)|\,dy\right) = 0 \tag{2.11}$$

and

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{(|y|\geq M)\cap D}\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y)\,|q(y)|\,dy\right) = 0. \tag{2.12}$$

As a typical example of functions in $K_\alpha^\infty(D)$, we cite $q(x) = \dfrac{1}{(1+|x|)^{\mu-\lambda}(\delta_D(x))^\lambda}$, $\lambda < \alpha < \mu$.

Remark 2.5. Replacing G_D^α by G_D and putting $\alpha = 2$ in Definition 2.4 above, we find again the Kato class $K(D)$ introduced in [2].

Let us define the potential kernel G_D^α of X^D on $B^+(D)$ by

$$G_D^\alpha f(x) = \int_D G_D^\alpha(x,y) f(y) dy.$$

As in the classical case, we have the following equivalence

$$G_D^\alpha f \neq \infty \iff \int_D \frac{(\rho_D(y))^{\frac{\alpha}{2}}}{(1+|y|)^{n-\alpha}} f(y) dy < \infty. \tag{2.13}$$

On the other hand, for any $f \in B^+(D)$ such that $G_D^\alpha f \neq \infty$ and for any $\psi \in C_c^\infty(D)$, we have

$$\int_D f(x)(-\Delta)^{\frac{\alpha}{2}} \psi(x) dx = \int_D G_D^\alpha f(x) \psi(x) dx.$$

That is

$$(-\Delta)^{\frac{\alpha}{2}} G_D^\alpha f = f \text{ in } D \text{ (in the distributional sense).} \tag{2.14}$$

In what follows, we recall the definition of harmonic and superharmonic functions associated to the process X^D (see [13]).

Definition 2.6. A locally integrable function f defined on D taking values in $(-\infty, \infty]$ and satisfying the condition $\int_{(|x|>1)\cap D} |f(x)|\,|x|^{-(n+\alpha)} dx < \infty$, is said to be

(i) α-harmonic with respect to X^D if for each open set S with $\overline{S} \subset D$,

$$E^x\left[\left|f\left(X_{\tau_S}^D\right)\right|\right] < \infty \text{ and } f(x) = E^x\left[f\left(X_{\tau_S}^D\right)\right], \text{ for } x \in S.$$

(ii) α-superharmonic with respect to X^D if f is lower semicontinuous in D and for each open set S with $\overline{S} \subset D$,

$$E^x\left[f^-\left(X_{\tau_S}^D\right)\right] < \infty \text{ and } f(x) \geq E^x\left[f\left(X_{\tau_S}^D\right)\right], \text{ for } x \in S.$$

We will use $\mathcal{H}_D^\alpha$ to denote the collection of all nonnegative functions on D which are α-harmonic with respect to X^D and $\mathcal{S}_D^\alpha$ to denote the collection of all nonnegative functions on D which are α-superharmonic with respect to X^D.

Example 2.7. It is well known that for each $y \in D$, the function $x \mapsto G_D^\alpha(x,y)$ is in $\mathcal{S}_D^\alpha$ and so it is for the potential function $x \mapsto G_D^\alpha f(x)$, for any $f \in B^+(D)$.

Remark 2.8. We have from ([20], Theorem 2) that a function f belongs to $\mathcal{S}_D^\alpha$ (respectively $\mathcal{H}_D^\alpha$) if and only if the function f^* is α-superharmonic (respectively α-harmonic) with respect to X^{D^*}.

To characterize functions beloging to $\mathcal{H}_D^\alpha$, we are going to introduce the Martin kernel associated to X^D. Let $x_0 \in D$ and let for $(x,z) \in D \times \partial D \cup \{\infty\}$,

$$M_D^\alpha(x,z) := \lim_{y \to z} \frac{G_D^\alpha(x,y)}{G_D^\alpha(x_0,y)}$$

be the Martin kernel of X^D based at x_0. We shall denote by $M_D(x,z)$ the Martin kernel of the killed Brownian motion ($\alpha = 2$). It is well known from the general potential theory that for each $z \in \partial D \cup \{\infty\}$, the function $x \mapsto M_D^\alpha(x,z)$ belongs to $\mathcal{H}_D^\alpha$ and for any function u in $\mathcal{H}_D^\alpha$, there exist a unique constant $c \geq 0$ and a unique nonnegative measure ν on ∂D such that

$$u(x) = \int_{\partial D} M_D^\alpha(x,z)\,\nu(dz) + cM_D^\alpha(x,\infty).$$

The following relation between functions in $\mathcal{H}_D^\alpha$ and solutions of the equation $(-\Delta)^{\frac{\alpha}{2}} u = 0$ (in the distributional sense) is due to ([8], Theorem 3.9).

Proposition 2.9. *A function $f \in B^+(D)$ belongs to $\mathcal{H}_D^\alpha$ if and only if it is continuous in D and satisfies $(-\Delta)^{\frac{\alpha}{2}} f = 0$ in D (in the distributional sense).*

We provide in what follows estimates on the Martin kernel M_D^α which extend those of M_D. These estimates will play a crucial role in our study.

Remark 2.10. By using estimates on the Green function G_D (see [1]), we have for $x \in D$ and $z \in \partial D \cup \{\infty\}$

$$M_D(x,z) \approx \frac{\lambda_D(x)}{|x-z|^n}$$

and

$$M_D(x,\infty) \approx \frac{\delta_D(x)}{\delta_D(x)+1}.$$

On the other hand, let $M_{D^*}^\alpha$ be the Martin kernel of D^* with reference point x_0^*. Thanks to (2.8), we obtain for $x \in D$ and $z \in \partial D \cup \{\infty\}$

$$M_D^\alpha(x,z) = \frac{|x|^{\alpha-n}}{|x_0|^{\alpha-n}} M_{D^*}^\alpha(x^*,z^*). \tag{2.15}$$

This leads to the following.

Proposition 2.11. *For $x \in D$ and $z \in \partial D$, we have*

$$M_D^\alpha(x,z) \approx \frac{(\lambda_D(x))^{\frac{\alpha}{2}}}{|x-z|^n} \tag{2.16}$$

and

$$M_D^\alpha(x,\infty) \approx (\rho_D(x))^{\frac{\alpha}{2}}. \tag{2.17}$$

In particular, we have for each $z \in \partial D$

$$\lim_{|x|\to\infty} M_D^\alpha(x,z) = 0 \text{ and } \lim_{x \to z} M_D^\alpha(x,\infty) = 0.$$

Proof. From ([13], Theorem 3.9), we know that

$$M^{\alpha}_{D^*}(x^*, z^*) \approx \frac{(\delta_{D^*}(x^*))^{\frac{\alpha}{2}}}{|x^* - z^*|^n}. \tag{2.18}$$

Which together with (2.3) and (2.15) gives (2.16).
From (2.9), we deduce that for $x \in D$,

$$M^{\alpha}_{D}(x, \infty) \approx (1 \wedge \lambda_D(x))^{\frac{\alpha}{2}}.$$

Thus (2.17) holds by (2.5) and (2.4). □

Example 2.12. Let $D = \{x \in \mathbb{R}^n : |x| > 1\}$. We have on $D \times \partial D$,

$$M^{\alpha}_{D}(x, z) \approx \frac{\left(|x|^2 - 1\right)^{\frac{\alpha}{2}}}{|x - z|^n}$$

and

$$M^{\alpha}_{D}(x, \infty) \approx \left(1 - \frac{1}{|x|}\right)^{\frac{\alpha}{2}}.$$

Now, by Martin's representation theorem (see [1]), there exist a unique positive constant c_0 and a unique finite positive measure σ on ∂D such that

$$1 = M_D\sigma(x) + c_0 M_D(x, \infty), \ x \in D, \tag{2.19}$$

where

$$M_D\sigma(x) = \int_{\partial D} M_D(x, z)\sigma(dz).$$

From Remark 2.10, we note that

$$\lim_{x \to \partial D} M_D\sigma(x) = 1, \ \lim_{|x| \to \infty} M_D\sigma(x) = 0 \tag{2.20}$$

and

$$\lim_{x \to \partial D} M_D(x, \infty) = 0, \ \lim_{|x| \to \infty} c_0 M_D(x, \infty) = 1. \tag{2.21}$$

Here the notation $x \to \partial D$ means that x tends to a point $\xi \in \partial D$. For a nonnegative measurable function f on ∂D, we put

$$M^{\alpha}_{D} f(x) = \int_{\partial D} M^{\alpha}_{D}(x, z) f(z)\sigma(dz), \text{ for } x \in D.$$

Then it follows from (2.16) and Remark 2.10 that

$$M^{\alpha}_{D} 1(x) \approx (\lambda_D(x))^{\frac{\alpha}{2} - 1} M_D\sigma(x), \text{ for } x \in D. \tag{2.22}$$

Now, let us define the following function w_α on D that will be of great interest in our study. For $x \in D$, we put

$$w_\alpha(x) = M^{\alpha}_{D} 1(x) + c_0 M^{\alpha}_{D}(x, \infty). \tag{2.23}$$

Proposition 2.13. *The function w_α belongs to $\mathcal{H}_D^\alpha$ and satisfies*

$$w_\alpha(x) \approx (\rho_D(x))^{\frac{\alpha}{2}-1}, \ x \in D. \tag{2.24}$$

Moreover, we have

$$\lim_{x\to\partial D} \frac{w_\alpha(x)}{M_D^\alpha 1(x)} = \lim_{|x|\to\infty} \frac{w_\alpha(x)}{c_0 M_D^\alpha(x,\infty)} = 1. \tag{2.25}$$

Proof. Using (2.17) and (2.22), we have for $x \in D$

$$\begin{aligned} w_\alpha(x) &\approx (\delta_D(x)(\delta_D(x)+1))^{\frac{\alpha}{2}-1} M_D\sigma(x) + (\rho_D(x))^{\frac{\alpha}{2}} \\ &= (\rho_D(x))^{\frac{\alpha}{2}-1}\left[(\delta_D(x)+1)^{\alpha-2} M_D\sigma(x) + \rho_D(x)\right] \\ &=: (\rho_D(x))^{\frac{\alpha}{2}-1} m_\alpha(x). \end{aligned}$$

Thanks to (2.20), we obtain

$$\lim_{x\to\partial D\cup\{\infty\}} m_\alpha(x) = 1.$$

Now, since the function m_α is positive and continuous on the compact set $\overline{D}\cup\{\infty\}$, we deduce that $m_\alpha(x) \approx 1$ for $x \in D$. This gives (2.24). The assertion (2.25) holds by Proposition 2.11. □

Remark 2.14. Let $\lambda \geq 0$ and f be a nonnegative continuous function on ∂D. Then the function h defined in D by

$$h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty)$$

belongs to $\mathcal{H}_D^\alpha$ and satisfies $\lim\limits_{x\to z\in\partial D} \frac{h(x)}{w_\alpha(x)} = f(z)$ and $\lim\limits_{|x|\to\infty} \frac{h(x)}{w_\alpha(x)} = \lambda$.
Indeed, for $z \in \partial D$ and $x \in D$, we have

$$\left|\frac{M_D^\alpha f(x)}{M_D^\alpha 1(x)} - f(z)\right| \leq \frac{1}{M_D^\alpha 1(x)} \int_{\partial D} M_D^\alpha(x,y)|f(y)-f(z)|dy.$$

Then we prove as in the classical case (see [1]) that

$$\lim_{x\to z} \frac{M_D^\alpha f(x)}{M_D^\alpha 1(x)} = f(z). \tag{2.26}$$

Using (2.25) and Proposition 2.11, we conclude the result.

Proposition 2.15. *Let $\lambda \geq 0$ and let f be a nonnegative continuous function on ∂D. Then the function h defined in D by*

$$h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty)$$

is the unique function in $\mathcal{H}_D^\alpha$ such that

(*i*) $\lim\limits_{x\to z\in\partial D} \frac{h(x)}{w_\alpha(x)} = f(z)$,

(*ii*) $\lim\limits_{|x|\to\infty} \frac{h(x)}{w_\alpha(x)} = \lambda$.

Proof. By Remark 2.14, the function h belongs to $\mathcal{H}_D^\alpha$ and satisfies (i) and (ii).
Now, recall that for $x \in D$,

$$w_\alpha(x) = \int_{\partial D} M_D^\alpha(x,z)\sigma(dz) + c_0 M_D^\alpha(x,\infty),$$

where c_0 and σ are respectively the constant and the measure given by (2.19). Then, by α-order Kelvin transform with (2.15), we obtain for $x \in D$

$$w_\alpha^*(x^*) = |x_0|^{n-\alpha}\left(\int_{(\partial D)^*} M_{D^*}^\alpha(x^*,\xi)\sigma^*(d\xi) + c_0 M_{D^*}^\alpha(x^*,0)\right).$$

Note that $\partial D^* = (\partial D)^* \cup \{0\}$ and put $\sigma_0 = |x_0|^{n-\alpha}(\sigma^* + c_0\delta_0)$, where δ_0 is the Dirac measure concentrated at 0 and σ^* is the image measure of σ by the Kelvin transform $x \mapsto x^*$. Thus we have

$$\begin{aligned} w_\alpha^*(x^*) &= \int_{\partial D^*} M_{D^*}^\alpha(x^*,\xi)\sigma_0(d\xi) \\ &=: M_{D^*}^\alpha 1(x^*). \end{aligned}$$

Now, let $h \in \mathcal{H}_D^\alpha$ satisfying (i) and (ii). We obtain by Kelvin transform that

$$\lim_{x^*\to\xi\in\partial D^*} \frac{h^*(x^*)}{M_{D^*}^\alpha 1(x^*)} = \widetilde{f}(\xi)$$

where $\widetilde{f}$ is the function defined on ∂D^* by

$$\widetilde{f}(\xi) = \begin{cases} |\xi|^{n-\alpha} f^*(\xi), & \xi \in (\partial D)^* \\ \lambda, & \xi = 0. \end{cases}$$

Then using ([9], Theorem 6), we deduce that

$$h^*(x^*) = M_{D^*}^\alpha \widetilde{f}(x^*) = \int_{\partial D^*} M_{D^*}^\alpha(x^*,\xi)\widetilde{f}(\xi)\sigma_0(d\xi).$$

Hence, using again Kelvin transform, we deduce that

$$h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty), \ x \in D.$$

□

Remark 2.16. Proposition 2.15 provides the solvability of the following reformulated Dirichlet problem associated to $(-\Delta)^{\frac{\alpha}{2}}$. Namely, if f is a nonnegative continuous function on ∂D and λ is a nonnegative constant, then the function defined in D by

$$h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty)$$

is the unique positive continuous solution of

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = 0 \ \text{ in } D \ \text{ (in the distributional sense)} \\ \lim\limits_{x\to z\in\partial D} \dfrac{u(x)}{w_\alpha(x)} = f(z), \\ \lim\limits_{|x|\to\infty} \dfrac{u(x)}{w_\alpha(x)} = \lambda. \end{cases}$$

2.3 Main results

As it is mentioned above, the main goal of this paper is to prove two existence results for fractional equations with reformulated Dirichlet boundary conditions stated in Theorems 2.17 and 2.18 below.

Our first purpose is to study the following problem

$$(P)\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \varphi(\cdot,u) \text{ in } D \text{ (in the distributional sense)} \\ \lim\limits_{x\to\partial D} (\delta_D(x))^{1-\frac{\alpha}{2}} u(x) = 0, \\ \lim\limits_{|x|\to\infty} u(x) = 0. \end{cases}$$

In view of (2.24), we remark that the boundary conditions in (P) are equivalent to

$$\lim_{x\to\partial D\cup\{\infty\}} \frac{u(x)}{w_\alpha(x)} = 0.$$

The nonlinearity φ is required to satisfy the following assumptions

(H_1) φ is a non-trivial nonnegative measurable function in $D\times(0,\infty)$ which is continuous and nonincreasing with respect to the second variable.

(H_2) For all $c>0$, the function $x\mapsto(\rho_D(x))^{1-\frac{\alpha}{2}}\varphi(x,c(\rho_D(x))^{\frac{\alpha}{2}-1})$ belongs to $K_\alpha^\infty(D)$.

As a typical example of functions φ satisfying (H_1) and (H_2), we quote $\varphi(x,s)=k(x)s^{-\sigma}$, where $\sigma\ge 0$ and k is a nonnegative measurable function in D such that the function

$$x\mapsto k(x)(\rho_D(x))^{(1-\frac{\alpha}{2})(\sigma+1)}\in K_\alpha^\infty(D).$$

Using a fixed point theorem, we prove in Section 5 the following.

Theorem 2.17. *Assume $(H_1)-(H_2)$. Then problem (P) has a positive continuous solution u in D satisfying*

$$u(x)=G_D^\alpha(\varphi(\cdot,u))(x),\ x\in D.$$

This result extends the one of [2] in the elliptic case ($\alpha=2$). In fact the authors of [2], showed that if φ satisfies (H_1) and $\varphi(\cdot,c)\in K(D)$ for each $c>0$, then the nonlinear elliptic equation $\Delta u+\varphi(\cdot,u)=0$, has a unique positive continuous solution u in D satisfying

$$\lim_{x\to\partial D} u(x)=\lim_{|x|\to\infty} u(x)=0.$$

For our second purpose, we are interested in the following problem

$$(Q)\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u + u\varphi(\cdot,u) = 0 \text{ in } D \text{ (in the distributional sense)} \\ \lim\limits_{x\to z\in\partial D} \dfrac{u(x)}{w_\alpha(x)} = f(z), \\ \lim\limits_{|x|\to\infty} \dfrac{u(x)}{w_\alpha(x)} = \lambda, \end{cases}$$

where $\lambda\ge 0$, f is a non-trivial nonnegative continuous function on ∂D and the nonlinear term is required to satisfy the following assumptions

(H_3) φ is a non-trivial nonnegative measurable function in $D\times(0,\infty)$.

(H_4) For all $c>0$, there exists a nonnegative function $q_c\in K_\alpha^\infty(D)$ such that the function $s\mapsto s\left[q_c(x)-\varphi(x,s(\rho_D(x))^{\frac{\alpha}{2}-1})\right]$ is continuous and nondecreasing on $[0,c]$, for every $x\in D$.

To illustrate, let us present an example. Let $p>0$ and k be a nonnegative measurable function such that the function

$$x\mapsto k(x)(\rho_D(x))^{(\frac{\alpha}{2}-1)p}\in K_\alpha^\infty(D).$$

Then the function $\varphi(x,u)=k(x)u^p$ satisfies (H_3) and (H_4).
Using a potential theory approach, we establish in Section 6 the following.

Theorem 2.18. *Assume* $(H_3)-(H_4)$. *Then problem* (Q) *has a positive continuous solution u in D satisfying*

$$c\left(M_D^\alpha f(x)+\lambda c_0 M_D^\alpha(x,\infty)\right)\le u(x)\le M_D^\alpha f(x)+\lambda c_0 M_D^\alpha(x,\infty),$$

where $c\in(0,1)$.

We achieve this section by noting that solutions to problems (P) and (Q) blow up at the boundary ∂D. On the contrary, for the classical case (i.e. $\alpha=2$), solutions of elliptic nonlinear problems corresponding to (P) and (Q) are bounded (see [2, 23]).

From here on, c denotes a positive constant which may vary from line to line. Also we refer to $C(\overline{D})$ the collection of all continuous functions in $\overline{D}$ and $C_0(D)$ the subclass of $C(\overline{D})$ consisting of functions which vanish continuously on ∂D and at infinity.

3 Estimates and properties of $\mathbf{G}_D^\alpha$

We provide in this section some estimates on the Green function $G_D^\alpha(x,y)$ and some interesting properties of the potential kernel G_D^α, related to potential theory.

Proposition 3.1. *For* $x,y\in D$, *we have*

$$\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}}G_D^\alpha(x,y)\approx\frac{1}{|x-y|^{n-\alpha}}\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha}\right). \tag{3.1}$$

In particular

$$\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}}G_D^\alpha(x,y)\le\frac{c}{|x-y|^{n-\alpha}}. \tag{3.2}$$

Proof. It follows from ([22], Proposition 2.4) that

$$\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}}\left(1\wedge\frac{(\lambda_D(x)\lambda_D(y))^{\frac{\alpha}{2}}}{|x-y|^\alpha}\right)\approx\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha}\right),$$

Then using (2.9), we deduce (3.1). □

Proposition 3.2. *For each x, $y \in D$, we have*

$$c\frac{(\delta_D(x)\delta_D(y))^{\frac{\alpha}{2}}}{|x|^{n-\frac{\alpha}{2}}\,|y|^{n-\frac{\alpha}{2}}} \le G_D^\alpha(x,y). \tag{3.3}$$

Moreover, if $|x-y| \ge r$ and $|y| \le M$, then

$$G_D^\alpha(x,y) \le c\frac{(\rho_D(x)\rho_D(y))^{\frac{\alpha}{2}}}{|x-y|^{n-\alpha}}. \tag{3.4}$$

Proof. Let x, $y \in D$, then from ([9], Proposition 1), we have

$$(\delta_{D^*}(x^*)\delta_{D^*}(y^*))^{\frac{\alpha}{2}} \le cG_{D^*}^\alpha(x^*,y^*).$$

Hence by (2.8) and (2.3), we deduce (3.3).
Moreover, let $x, y \in D$ such that $|x-y| \ge r$ and $|y| \le M$. Then, since $\min(1, |x-y|^\alpha) \approx 1$, we obtain by (3.1) that

$$G_D^\alpha(x,y) \le c\left(\frac{\rho_D(x)}{\rho_D(y)}\right)^{\frac{\alpha}{2}}\frac{(\delta_D(y))^\alpha}{|x-y|^{n-\alpha}}.$$

Then using (2.3), we get

$$G_D^\alpha(x,y) \le c\frac{(\rho_D(x)\rho_D(y))^{\frac{\alpha}{2}}}{|x-y|^{n-\alpha}}.$$

□

It is the same as the case $\alpha = 2$, the potential kernel G_D^α satisfies some preliminary potential properties.

Proposition 3.3. *If f and g are in $B^+(D)$ such that $g \le f$ and the potential function $G_D^\alpha f$ is continuous in D. Then the potential function $G_D^\alpha g$ is also continuous in D.*

Proof. Let $\theta \in B^+(D)$ be such that $f = g+\theta$. So, we have $G_D^\alpha f = G_D^\alpha g + G_D^\alpha \theta$. Now since $G_D^\alpha g$ and $G_D^\alpha \theta$ are two lower semi-continuous functions in D, we deduce the result. □

Now, we note that the potential kernel G_D^α satisfies the complete maximum principle, i.e. for each $f \in B^+(D)$ and $v \in \mathcal{S}_D^\alpha$ such that $G_D^\alpha f \le v$ in $\{f > 0\}$, we have $G_D^\alpha f \le v$ in D (see [3], Chap. II, Proposition 7.1). Consequently, we deduce the following.

Proposition 3.4. *Let $h \in B^+(D)$ and $v \in \mathcal{S}_D^\alpha$. Let w be a Borel measurable function in D such that $G_D^\alpha(h|w|) < \infty$ and $v = w + G_D^\alpha(hw)$. Then w satisfies*

$$0 \le w \le v.$$

Proof. Since $G_D^\alpha(h|w|) < \infty$, then we have

$$G_D^\alpha(hw^+) \le v + G_D^\alpha(hw^-) \text{ in } \{w > 0\} = \{w^+ > 0\}.$$

Now, since the function $v+G_D^\alpha(hw^-)$ is in $\mathcal{S}_D^\alpha$, then we deduce by the complete maximum principle that

$$G_D^\alpha(hw^+) \le v + G_D^\alpha(hw^-) \text{ in } D.$$

That is

$$G_D^\alpha(hw) \le v = w + G_D^\alpha(hw).$$

This implies that

$$0 \le w \le v.$$

□

4 The Kato class $\mathbf{K}_\alpha^\infty(D)$

We look in this section at some interesting properties of functions belonging to the Kato class $K_\alpha^\infty(D)$ (see Definition 2.4). In particular, a careful analysis about equicontinuity of a family of functions is performed. First to illustrate the class $K_\alpha^\infty(D)$, let us present the following.

4.1 A subclass in $\mathbf{K}_\alpha^\infty(D)$

By (3.1), one can see that a function q in D belongs to the class $K_\alpha^\infty(D)$ if q satisfies

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{B(x,r)\cap D}\frac{1}{|x-y|^{n-\alpha}}\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha}\right)|q(y)|\,dy\right)=0 \tag{4.1}$$

and

$$\lim_{M\to \infty}\left(\sup_{x\in D}\int_{(|y|\ge M)\cap D}\frac{1}{|x-y|^{n-\alpha}}\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha}\right)|q(y)|\,dy\right)=0. \tag{4.2}$$

Proposition 4.1. *Let $p>\frac{n}{\alpha}$, then for each $\lambda<\alpha-\frac{n}{p}<\mu$ and $f\in L^p(D)$, the function defined in D by*

$$\varphi(y)=\frac{f(y)}{(1+|y|)^{\mu-\lambda}(\delta_D(y))^\lambda}$$

belongs to $K_\alpha^\infty(D)$.

Proof. We aim to show that φ satisfies (4.1). Let $x\in D$, $0<r<\frac{1}{2}$ and $\lambda^+=\max(\lambda,0)$. Put

$$I(x,r):=\int_{B(x,r)\cap D}\frac{|\varphi(y)|}{|x-y|^{n-\alpha}}(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha})dy.$$

Applying the following inequality

$$a^\alpha+b^\alpha\ge a^{\alpha-\lambda^+}b^{\lambda^+},$$

with $a = \delta_D(x)$ and $b = |x-y|$, we deduce by (2.4) and (2.2) that for $y \in D$ such that $|x-y| \le r < 1$, we have

$$\begin{aligned}
\frac{|\varphi(y)|}{|x-y|^{n-\alpha}}(1 \wedge \frac{(\delta_D(y))^{\alpha}}{1 \wedge |x-y|^{\alpha}}) &\le c\frac{(\delta_D(y))^{\alpha-\lambda}}{(|x-y|^{\alpha}+(\delta_D(y))^{\alpha})}\frac{|f(y)|}{(1+|y|)^{\mu-\lambda}|x-y|^{n-\alpha}} \\
&\le c\frac{(\delta_D(y))^{\lambda^+-\lambda}|f(y)|}{(1+|y|)^{\mu-\lambda}|x-y|^{n+\lambda^+-\alpha}} \\
&\le c\frac{|f(y)|}{|x-y|^{n+\lambda^+-\alpha}}.
\end{aligned}$$

Let $p > \frac{n}{\alpha}$ and $q \ge 1$ such that $\frac{1}{p}+\frac{1}{q} = 1$. Using Hölder inequality, we obtain

$$\begin{aligned}
\int_{B(x,r)\cap D}\frac{|\varphi(y)|}{|x-y|^{n-\alpha}}(1 \wedge \frac{(\delta_D(y))^{\alpha}}{1 \wedge |x-y|^{\alpha}})dy &\le c\|f\|_p\left(\int_{B(x,r)\cap D}\frac{dy}{|x-y|^{(n+\lambda^+-\alpha)q}}\right)^{\frac{1}{q}} \\
&\le c\|f\|_p\left(\int_0^r t^{\left(\alpha-\frac{n}{p}-\lambda^+\right)q-1}dt\right)^{\frac{1}{q}} \\
&\le c\|f\|_p\, r^{\left(\alpha-\frac{n}{p}-\lambda^+\right)}.
\end{aligned}$$

Which implies that $I(x,r)$ tends to zero as $r \to 0$, uniformly in x. This proves that φ satisfies (4.1).
Now, we intend to prove that φ satisfies (4.2). Let $x \in D$ and $M > 1$. We put

$$J(x,M) := \int_{\{|y|\ge M\}\cap D}\frac{|\varphi(y)|}{|x-y|^{n-\alpha}}(1 \wedge \frac{(\delta_D(y))^{\alpha}}{1 \wedge |x-y|^{\alpha}})dy.$$

For $y \in D$ such that $|y| \ge M$, we have $\delta_D(y) \approx |y|$. Then using the Hölder inequality, we obtain

$$\begin{aligned}
J(x,M) &\le c\|f\|_p\left(\int_{\{|y|\ge M\}\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy\right)^{\frac{1}{q}} \\
&=: c\|f\|_p(A(x,M))^{\frac{1}{q}}.
\end{aligned}$$

Also, we have

$$\begin{aligned}
A(x,M) &\le c\left(\int_{(M\le|y|\le|x-y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy + \int_{\{|y|\ge M\}\cap(|x-y|\le|y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy\right) \\
&\le c\left(\int_{(M\le|y|\le|x-y|)\cap D}\frac{1}{|y|^{(\mu+n-\alpha)q}}dy + \int_{\{|y|\ge M\}\cap(|x-y|\le|y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy\right) \\
&\le c\left(\frac{1}{M^{(\mu+\frac{n}{p}-\alpha)q}} + \int_{(M\le|x-y|\le|y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy + \int_{(|x-y|\le M\le|y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}\frac{1}{|y|^{\mu q}}dy\right) \\
&\le c\left(\frac{1}{M^{(\mu+\frac{n}{p}-\alpha)q}} + \int_{(M\le|x-y|\le|y|)\cap D}\frac{1}{|x-y|^{(n+\mu-\alpha)q}}dy + \frac{1}{M^{\mu q}}\int_{(|x-y|\le M\le|y|)\cap D}\frac{1}{|x-y|^{(n-\alpha)q}}dy\right) \\
&\le c\left(\frac{1}{M^{(\mu+\frac{n}{p}-\alpha)q}} + \int_M^{\infty}\frac{1}{t^{(\mu+\frac{n}{p}-\alpha)q+1}}dt + \frac{1}{M^{\mu q}}\int_0^M t^{\left(\alpha-\frac{n}{p}\right)q-1}dt\right) \\
&\le \frac{c}{M^{(\mu+\frac{n}{p}-\alpha)q}}.
\end{aligned}$$

Hence, φ satisfies (4.2) and the proof is achieved. □

4.2 Properties of functions in $\mathbf{K}_\alpha^\infty(D)$

Proposition 4.2. *Let q be a function satisfying (2.11). Then for $M>0$, we have*

$$\int_{(|y|\leq M)\cap D} (\delta_D(y))^\alpha \, |q(y)|\, dy < \infty.$$

Proof. Since q satisfies (2.11), there exists $r>0$ such that for each $x\in D$, we have

$$\int_{B(x,r)\cap D} \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y)\,|q(y)|\,dy \leq 1.$$

Let $x_1, x_2, ..., x_p$ in $D\cap B(0,M)$ be such that $D\cap B(0,M)\subset \bigcup_{i=1}^{p} B(x_i,r)$, then by (3.3) and (2.3), there exists $c>0$ such that for each $y\in D\cap B(0,M)$ and $i\in\{1,2,...,p\}$, we have

$$(\delta_D(y))^\alpha \leq c\left(\frac{\rho_D(y)}{\rho_D(x_i)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x_i,y).$$

Hence, we have

$$\begin{aligned}\int_{(|y|\leq M)\cap D} (\delta_D(y))^\alpha \, |q(y)|\, dy &\leq c\sum_{i=1}^{p}\int_{B(x_i,r)\cap D}\left(\frac{\rho_D(y)}{\rho_D(x_i)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x_i,y)\,|q(y)|\,dy\\ &\leq pc<\infty.\end{aligned}$$

This completes the proof. □

In the sequel, we use the notations

$$\|q\|_D := \sup_{x\in D}\int_D \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y)\,|q(y)|\,dy$$

and

$$a_\alpha(q) := \sup_{x,z\in D}\int_D \frac{G_D^\alpha(x,y)G_D^\alpha(y,z)}{G_D^\alpha(x,z)}\,|q(y)|\,dy. \tag{4.3}$$

Proposition 4.3. *Let q be a function in $K_\alpha^\infty(D)$. Then*

$$a_\alpha(q)\leq 2C_0\,\|q\|_D<\infty,$$

where C_0 is the constant given in Theorem 2.3.

Proof. By using Theorem 2.3, we have immediately that

$$a_\alpha(q)\leq 2C_0\,\|q\|_D.$$

Next, we will prove that $\|q\|_D$ is finite. By (4.1) and (4.2), there exist $r\in(0,1)$ and $M>1$ such that for each $x\in D$, we have

$$\int_{B(x,r)\cap D}\frac{1}{|x-y|^{n-\alpha}}\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha}\right)|q(y)|\,dy\leq 1$$

and

$$\int_{(|y|\geq M)\cap D} \frac{1}{|x-y|^{n-\alpha}}(1 \wedge \frac{(\delta_D(y))^\alpha}{1 \wedge |x-y|^\alpha})\,|q(y)|\,dy \leq 1.$$

On the other hand, put $\Omega = B^c(x,r)\cap(|y|\leq M)\cap D$. Then by (3.1), we have for $x \in D$

$$\begin{aligned}
\int_D \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y)\,|q(y)|\,dy &\leq c\int_D \frac{1}{|x-y|^{n-\alpha}}(1\wedge \frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha})\,|q(y)|\,dy \\
&\leq c(2+\int_\Omega \frac{1}{|x-y|^{n-\alpha}}(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|x-y|^\alpha})\,|q(y)|\,dy), \\
&\leq c(2+\frac{1}{r^n}\int_{(|y|\leq M)\cap D}(\delta_D(y))^\alpha\,|q(y)|\,dy).
\end{aligned}$$

Thus the result follows by Proposition 4.2. □

Corollary 4.4. *Let q be a function in $K_\alpha^\infty(D)$. Then the function*

$$y \mapsto \frac{(\rho_D(y))^\alpha}{|y|^{n-\alpha}}q(y)$$

is in $L^1(D)$.

Proof. Let $x_0 \in D$. Using (3.3), we have for $y \in D$

$$\frac{(\rho_D(y))^\alpha}{|y|^{n-\alpha}} \leq c\left(\frac{\rho_D(y)}{\rho_D(x_0)}\right)^{\frac{\alpha}{2}} |x_0|^{n-\alpha}\, G_D^\alpha(x_0,y).$$

Hence, the result follows from Proposition 4.3. □

Proposition 4.5. *Let q be a function in $K_\alpha^\infty(D)$. Then for any h in $\mathcal{S}_D^\alpha$ and $x \in D$, we have*

$$\int_D G_D^\alpha(x,y)h(y)\,|q(y)|\,dy \leq a_\alpha(q)\,h(x). \tag{4.4}$$

Moreover, we have for $x_0 \in \overline{D}$

$$\lim_{r\to 0}\left(\sup_{x\in D}\frac{1}{h(x)}\int_{B(x_0,r)\cap D} G_D^\alpha(x,y)h(y)\,|q(y)|\,dy\right) = 0 \tag{4.5}$$

and

$$\lim_{M\to\infty}\left(\sup_{x\in D}\frac{1}{h(x)}\int_{(|y|\geq M)\cap D} G_D^\alpha(x,y)h(y)\,|q(y)|\,dy\right) = 0. \tag{4.6}$$

Proof. Let h be a function in $\mathcal{S}_D^\alpha$. Then by ([3], Chap. II, Proposition 3.11), there exists a sequence $(f_k)_k \subset B^+(D)$ such that for all $y \in D$

$$h(y) = \sup_k \int_D G_D^\alpha(y,z) f_k(z) dz.$$

Hence, it is enough to prove (4.4), (4.5) and (4.6) for $h(y) = G_D^\alpha(y,z)$ uniformly in $z \in D$.
For each $x, z \in D$, we have

$$\begin{aligned}\int_D G_D^\alpha(x,y) G_D^\alpha(y,z)\,|q(y)|\,dy &\leq G_D^\alpha(x,z)\int_D \frac{G_D^\alpha(x,y)G_D^\alpha(y,z)}{G_D^\alpha(x,z)}\,|q(y)|\,dy\\ &\leq a_\alpha(q) G_D^\alpha(x,z).\end{aligned}$$

Then (4.4) holds. Now, we shall prove (4.5). Let $\varepsilon > 0$, then by (2.11) and (2.12), there exist $r_1 \in (0,1)$ and $M > 1$ such that

$$\sup_{\xi\in D}\int_{(|\xi-y|\leq r_1)\cap D}\left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy \leq \frac{\varepsilon}{2} \tag{4.7}$$

and

$$\sup_{\xi\in D}\int_{(|y|\geq M)\cap D}\left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy \leq \frac{\varepsilon}{2}. \tag{4.8}$$

Let $r > 0$. Then using Theorem 2.3, we have for all $x, z \in D$

$$\begin{aligned}\frac{1}{G_D^\alpha(x,z)}\int_{D\cap B(x_0,r)} G_D^\alpha(x,y)G_D^\alpha(y,z)\,|q(y)|\,dy &\leq C_0\int_{D\cap B(x_0,r)}\left[\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}} G_D^\alpha(x,y) + \left(\frac{\rho_D(y)}{\rho_D(z)}\right)^{\frac{\alpha}{2}} G_D^\alpha(z,y)\right]|q(y)|\\ &\leq 2C_0\sup_{\xi\in D}\int_{D\cap B(x_0,r)}\left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy.\end{aligned}$$

On the other hand, it follows from (4.7) and (4.8) that

$$\int_{D\cap B(x_0,r)}\left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy \leq \varepsilon + \int_\Omega \left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy,$$

where $\Omega = B(x_0,r)\cap B^c(\xi,r_1)\cap(1\leq|y|\leq M)\cap D$. So, we obtain by (3.1) that

$$\begin{aligned}\int_{D\cap B(x_0,r)}\left(\frac{\rho_D(y)}{\rho_D(\xi)}\right)^{\frac{\alpha}{2}} G_D^\alpha(\xi,y)\,|q(y)|\,dy &\leq \varepsilon + c\int_\Omega \frac{1}{|\xi-y|^{n-\alpha}}\left(1\wedge\frac{(\delta_D(y))^\alpha}{1\wedge|\xi-y|^\alpha}\right)|q(y)|\,dy\\ &\leq \varepsilon + \frac{c}{r_1^n}\int_{D\cap B(x_0,r)\cap(1\leq|y|\leq M)} (\delta_D(y))^\alpha\,|q(y)|\,dy.\end{aligned}$$

Hence, by letting $r \to 0$, we reach (4.5) from Proposition 4.2.
The assertion (4.6) follows immediately from Theorem 2.3 and (4.8). □

Corollary 4.6. *Let q be a nonnegative function in $K_\alpha^\infty(D)$. Then we have*

$$\left\|G_D^\alpha(q)\right\|_\infty = \sup_{x\in D}\int_D G_D^\alpha(x,y)q(y)dy < \infty.$$

Proof. Put $h = 1$ in (4.4) and using Proposition 4.3, we obtain the result. □

Corollary 4.7. *There exists $c > 0$ such that for each $q \in K_\alpha^\infty(D)$,*

$$\sup_{x\in D}\int_D \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,|q(y)|\,dy \le c a_\alpha(q). \tag{4.9}$$

Moreover, if $x_0 \in \overline{D}$ we have

$$\lim_{r\to 0}\left(\sup_{x\in D}\int_{B(x_0,r)\cap D}\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,|q(y)|\,dy\right) = 0 \tag{4.10}$$

and

$$\lim_{M\to\infty}\left(\sup_{x\in D}\int_{(|y|\ge M)\cap D}\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,|q(y)|\,dy\right) = 0. \tag{4.11}$$

Proof. By Proposition 2.13, the function w_α is in $\mathcal{H}_D^\alpha$ and satisfies $w_\alpha(x) \approx (\rho_D(x))^{\frac{\alpha}{2}-1}$. Hence, we deduce the result by applying Proposition 4.5 for $h = w_\alpha$. □

Corollary 4.8. *Let q be a function in $K_\alpha^\infty(D)$. Then the function*

$$y \mapsto \frac{(\rho_D(y))^{\alpha-1}}{|y|^{n-\alpha}} q(y) \in L^1(D).$$

In particular, the function $y \mapsto (\delta_D(y))^{\alpha-1} q(y)$ is in $L^1_{loc}(D)$.

Proof. Let $x_0 \in D$, we have by (3.3) that for $y \in D$

$$\frac{(\rho_D(y))^{\alpha-1}}{|y|^{n-\alpha}} \le c\frac{|x_0|^{n-\alpha}}{\rho_D(x_0)} G_D^\alpha(x_0,y)\left(\frac{\rho_D(y)}{\rho_D(x_0)}\right)^{\frac{\alpha}{2}-1}.$$

Hence the result follows by (4.9). □

4.3 Modulus of continuity

In order to prove our existence results, we need the following theorem. The idea of the proof follows closely from the properties of functions in $K_\alpha^\infty(D)$.

Theorem 4.9. *Let q be a nonnegative function in $K_\alpha^\infty(D)$. Then the family of functions defined in D by*

$$\Lambda_q = \left\{x \mapsto J(f)(x) := \int_D \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y) f(y)\,dy : f \in K_\alpha^\infty(D),\ |f| \le q\right\}$$

is uniformly bounded and equicontinuous in $\overline{D}\cup\{\infty\}$. Consequently Λ_q is relatively compact in $C_0(D)$.

Proof. Let q be a nonnegative function in $K_\alpha^\infty(D)$ and f be a function in $K_\alpha^\infty(D)$ such that $|f| \leq q$ in D. By (4.9), we have

$$\sup_{x\in D}|J(f)(x)| \leq \sup_{x\in D}\int_D \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,q(y)\,dy < \infty.$$

Hence Λ_q is uniformly bounded. Let us prove the equicontinuity. Let $x_0 \in \overline{D}$ and $\varepsilon > 0$. Then by (4.10) and (4.11), there exist $r > 0$ and $M > 1$ such that

$$\sup_{z\in D}\int_{D\cap B(x_0,2r)} \left(\frac{\rho_D(y)}{\rho_D(z)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(z,y)\,q(y)dy \leq \frac{\varepsilon}{2} \tag{4.12}$$

and

$$\sup_{z\in D}\int_{(|y|\geq M)\cap D} \left(\frac{\rho_D(y)}{\rho_D(z)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(z,y)\,q(y)dy \leq \frac{\varepsilon}{2}. \tag{4.13}$$

Now, if $x_0 \in D$ and $x, x' \in B(x_0,r)\cap D$, we have

$$\begin{aligned}\left|J(f)(x) - J(f)(x')\right| &\leq \varepsilon + \int_\Omega \left|\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y) - \left(\frac{\rho_D(y)}{\rho_D(x')}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x',y)\right| q(y)dy \\ &=: \varepsilon + I(x,x')\end{aligned}$$

where $\Omega = B^c(x_0,2r)\cap(|y|\leq M)\cap D$.
On the other hand, since $|x-x_0| \leq r$ and $|x'-x_0| \leq r$, then for $y \in B^c(x_0,2r)$, we have $|x-y| \geq r$ and $|x'-y| \geq r$. Hence, it follows from (3.4) that

$$\left|\frac{G_D^\alpha(x,y)}{(\rho_D(x))^{\frac{\alpha}{2}-1}} - \frac{G_D^\alpha(x',y)}{(\rho_D(x'))^{\frac{\alpha}{2}-1}}\right| (\rho_D(y))^{\frac{\alpha}{2}-1} \leq c\,(\delta_D(y))^{\alpha-1}.$$

Now, since for each $y \in \Omega$, the function $x \mapsto \dfrac{G_D^\alpha(x,y)}{(\rho_D(x))^{\frac{\alpha}{2}-1}}$ is continuous in $B(x_0,r)$, we deduce by Corollary 4.8 and the dominated convergence theorem that $I(x,x')$ tends to zero as $|x-x'| \to 0$.
Next, if $x_0 \in \partial D$ and $x \in B(x_0,r)\cap D$, then we have by (4.12) and (4.13), that

$$\begin{aligned}|J(f)(x)| &\leq \varepsilon + \int_\Omega \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,|q(y)|\,dy \\ &=: \varepsilon + I(x).\end{aligned}$$

Now, for $y \in \Omega$, we have by (3.4) that $\dfrac{G_D^\alpha(x,y)}{(\rho_D(x))^{\frac{\alpha}{2}-1}} \to 0$ as $|x-x_0| \to 0$. So by a same argument as for $I(x,x')$, we prove that $I(x)$ tends to zero as $|x-x_0| \to 0$ and then $J(f)(x) \to 0$ as $x \to x_0$ uniformly in f.
Finally, let $x \in D$ such that $|x| \geq M+1$, then we have by (4.13), that

$$|J(f)(x)| \leq \varepsilon + \int_{(|y|\leq M)\cap D} \left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)\,|q(y)|\,dy.$$

Since $|y| \leq M$, then $|x-y| \geq 1$ and by (3.4) we deduce that

$$\begin{aligned} |J(f)(x)| &\leq \varepsilon + c\int_{(|y|\leq M)\cap D} \frac{(\rho_D(y))^{\alpha-1}}{|x-y|^{n-\alpha}}|q(y)|dy \\ &\leq \varepsilon + \frac{c}{(|x|-M)^{n-\alpha}}\int_{(|y|\leq M)\cap D}(\delta_D(y))^{\alpha-1}|q(y)|dy. \end{aligned}$$

Using Corollary 4.8, we deduce that $J(f)(x) \to 0$ as $|x| \to \infty$ uniformly in f. Consequently, by Ascoli's theorem, we deduce that Λ_q is relatively compact in $C_0(D)$. □

5 First existence result

In this section, we aim at proving the existence of a positive continuous solution to the following nonlinear elliptic problem

$$(P_\lambda)\begin{cases} (-\Delta)^{\frac{\alpha}{2}}u = \varphi(\cdot,u) \text{ in } D \text{ (in the distributional sense)} \\ \lim\limits_{x\to\partial D}\dfrac{u(x)}{w_\alpha(x)} = \lambda, \\ \lim\limits_{|x|\to\infty}\dfrac{u(x)}{w_\alpha(x)} = \lambda, \end{cases}$$

where λ is a nonnegative constant.

Remark 5.1. (i) For $\lambda > 0$, we shall prove the uniqueness of the solution of problem (P_λ).
(ii) We remark that problem (P_0) is equivalent to problem (P).

In order to reach our purpose, we need the following lemma.

Lemma 5.2. *Let φ be a function satisfying (H_1) and (H_2) and u be a positive continuous function in D such that*

$$\lim_{x\to\partial D\cup\{\infty\}}\frac{u(x)}{w_\alpha(x)} = \lambda > 0. \tag{5.1}$$

Then the function $x \mapsto \dfrac{G_D^\alpha(\varphi(\cdot,u))(x)}{w_\alpha(x)}$ belongs to $C_0(D)$.

Proof. Since the function $x \mapsto \dfrac{u(x)}{w_\alpha(x)}$ is positive and continuous in D and satisfies (5.1), it follows that $u(x) \approx w_\alpha(x)$, for $x \in D$ and so by (2.24), we deduce that $u(x) \approx (\rho_D(x))^{\frac{\alpha}{2}-1}$. Then we conclude by the monotonicity of φ that there exists $c > 0$ such that

$$\varphi(x,u(x)) \leq \varphi(x, c(\rho_D(x))^{\frac{\alpha}{2}-1}),\ x \in D. \tag{5.2}$$

Put $\theta(x) := \varphi(x, c(\rho_D(x))^{\frac{\alpha}{2}-1})$, for $x \in D$. Then we have

$$\begin{aligned} G_D^\alpha(\theta)(x) &= \int_D G_D^\alpha(x,y)\theta(y)dy \\ &= (\rho_D(x))^{\frac{\alpha}{2}-1}\int_D\left(\frac{\rho_D(y)}{\rho_D(x)}\right)^{\frac{\alpha}{2}-1} G_D^\alpha(x,y)(\rho_D(y))^{1-\frac{\alpha}{2}}\theta(y)dy. \end{aligned}$$

Since φ satisfies hypothesis (H_2), then it follows from Theorem 4.9 that the function

$$x \mapsto (\rho_D(x))^{1-\frac{\alpha}{2}} G_D^\alpha(\theta)(x) \in C_0(D). \tag{5.3}$$

This implies by (5.2) and Proposition 3.3, that the function $x \mapsto (\rho_D(x))^{1-\frac{\alpha}{2}} G_D^\alpha(\varphi(\cdot,u))(x)$ belongs to $C_0(D)$. The result is deduced by (2.24). □

Remark 5.3. Let $\lambda > 0$ and put $u = \lambda w_\alpha$ in Lemma 5.2, we obtain that the function

$$x \mapsto \frac{G_D^\alpha(\varphi(\cdot,\lambda w_\alpha))(x)}{w_\alpha(x)} \in C_0(D). \tag{5.4}$$

Lemma 5.4. *Let $\lambda > 0$ and φ be a function satisfying (H_1) and (H_2). Let u be a positive continuous function in D. Then u is a solution of problem (P_λ) if and only if u satisfies the integral equation*

$$u(x) = \lambda w_\alpha(x) + \int_D G_D^\alpha(x,y)\varphi(y,u(y))dy, \ x \in D. \tag{5.5}$$

Proof. Suppose that u satisfies (5.5). Since φ is nonincreasing with respect to the second variable, we have obviously

$$G_D^\alpha(\varphi(\cdot,u)) \leq G_D^\alpha(\varphi(\cdot,\lambda w_\alpha)).$$

This together with (5.4) implies that $\lim\limits_{x\to\partial D\cup\{\infty\}} \dfrac{u(x)}{w_\alpha(x)} = \lambda > 0$. Now, since u is continuous, we apply $(-\Delta)^{\frac{\alpha}{2}}$ on both sides of (5.5) and we conclude that u is a positive continuous solution of problem (P_λ) by Propositions 2.9 and 2.13.
Conversely, suppose that u is a positive continuous solution of problem (P_λ), then u satisfies (5.1). It follows by Lemma 5.2 that the function $v = u - G_D^\alpha(\varphi(\cdot,u))$ satisfies

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v = 0 \ \text{ in } D, \\ \lim\limits_{x\to\partial D\cup\{\infty\}} \dfrac{v(x)}{w_\alpha(x)} = \lambda. \end{cases}$$

Thus we deduce by Proposition 2.15, that $v = \lambda w_\alpha$. This ends the proof. □

Proposition 5.5. *Let φ be a function satisfying (H_1) and (H_2) and let $0 < \mu \leq \lambda$. Then we have*

$$0 \leq u_\lambda - u_\mu \leq (\lambda-\mu)w_\alpha \ \text{ in } D,$$

where u_λ and u_μ are respectively solutions of problems (P_λ) and (P_μ).

Proof. Let h be the function defined on D by

$$h(x) = \begin{cases} \frac{\varphi(x,u_\lambda(x))-\varphi(x,u_\mu(x))}{u_\mu(x)-u_\lambda(x)} & \text{if} \quad u_\mu(x) \neq u_\lambda(x) \\ 0 & \text{if} \quad u_\mu(x) = u_\lambda(x). \end{cases}$$

Then $h \in B^+(D)$. Using Lemma 5.4, we deduce

$$\left(u_\lambda - u_\mu\right)(x) + G_D^\alpha\left(h\left(u_\lambda - u_\mu\right)\right)(x) = (\lambda-\mu)\, w_\alpha(x).$$

Furthermore, by (5.4), we conclude that

$$\begin{aligned} G_D^\alpha\left(h\left|u_\lambda - u_\mu\right|\right) &\leq G_D^\alpha\varphi(\cdot,u_\lambda) + G_D^\alpha\varphi(\cdot,u_\mu) \\ &\leq G_D^\alpha\varphi(\cdot,\lambda w_\alpha) + G_D^\alpha\varphi(\cdot,\mu w_\alpha) < \infty \end{aligned}$$

Now, the result holds by Proposition 3.4. □

Theorem 5.6. *Assume $(H_1)-(H_2)$. Then for each $\lambda > 0$, problem (P_λ) has a unique positive continuous solution u_λ in D satisfying*

$$\lambda w_\alpha(x) \leq u_\lambda(x) \leq \gamma w_\alpha(x), \textit{ for } x \in D,$$

where γ is a constant strictly larger than λ.

Proof. In view of (5.4), the constant

$$\gamma := \lambda + \sup_{x\in D}\frac{1}{w_\alpha(x)}G_D^\alpha(\varphi(\cdot,\lambda w_\alpha))(x)$$

is finite.

In order to apply a fixed point argument, we consider the convex set given by

$$\Lambda = \left\{v \in C(\overline{D}\cup\{\infty\}) : \lambda \leq v \leq \gamma\right\}.$$

We define the integral operator T on Λ by

$$Tv(x) := \lambda + \frac{1}{w_\alpha(x)}\int_D G_D^\alpha(x,y)\varphi(y,w_\alpha(y)v(y))dy,\ x \in D.$$

First, we aim to prove that the operator T maps Λ into itself. Let $v \in \Lambda$, we have clearly $\lambda \leq Tv \leq \gamma$. By (5.4) with Proposition 3.3, we see $Tv \in C(\overline{D}\cup\{\infty\})$. More strongly, we can show that $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$ as in the proof of Theorem 4.9. In particular, $T\Lambda \subset \Lambda$.

So it remains to prove the continuity of T in Λ. Consider a sequence $(v_k)_k$ in Λ which converges uniformly to a function v in Λ. Then we obtain

$$|Tv_k(x) - Tv(x)| \leq c\int_D \frac{w_\alpha(y)}{w_\alpha(x)}G_D^\alpha(x,y)\frac{1}{w_\alpha(y)}\left|\varphi(y,w_\alpha(y)v_k(y)) - \varphi(y,w_\alpha(y)v(y))\right|dy.$$

Using the monotonicity of φ, we deduce that

$$\frac{1}{w_\alpha(y)}\left|\varphi(y,w_\alpha(y)v_k(y)) - \varphi(y,w_\alpha(y)v(y))\right| \leq 2\theta(y),$$

where $\theta(y) := \dfrac{\varphi(y,\lambda w_\alpha(y))}{w_\alpha(y)}$. By (2.24) and hypothesis (H_2), the function $\theta \in K_\alpha^\infty(D)$ and so since φ is continuous with respect to the second variable, we deduce by (2.24), (4.9) and the dominated convergence theorem that

$$\forall x \in D,\ Tv_k(x) \to Tv(x), \text{ as } k \to \infty.$$

Since $T\Lambda$ is relatively compact in $C(\overline{D}\cup\{\infty\})$, we have the uniform convergence, namely,

$$\|Tv_k - Tv\|_\infty \to 0 \text{ as } k \to \infty.$$

Thus we have proved that T is a compact mapping from Λ to itself. Hence, by the Schauder fixed-point theorem, T has a fixed point $v_\lambda \in \Lambda$. Put $u_\lambda(x) = w_\alpha(x)v_\lambda(x)$, for $x \in D$. Then u_λ is a continuous function in D and satisfies

$$u_\lambda(x) = \lambda w_\alpha(x) + \int_D G_D^\alpha(x,y)\varphi(y,u_\lambda(y))dy$$

and

$$\lambda w_\alpha(x) \le u_\lambda(x) \le \gamma w_\alpha(x),\ x \in D.$$

By Lemma 5.4, we conclude that u_λ is a solution of problem (P_λ). The uniqueness follows by Proposition 5.5. □

Proof of Theorem 2. Let (λ_k) be a sequence of positive real numbers, nonincreasing to zero. For each $k \in \mathbb{N}$, put

$$\gamma_k = \lambda_k + \sup_{x\in D}\frac{1}{w_\alpha(x)}G_D^\alpha(\varphi(\cdot,\lambda_k w_\alpha))(x)$$

and denote by u_k the unique solution of problem (P_{λ_k}). By Proposition 5.5, the sequence (u_k) decreases to a function u and so the sequence $(u_k - \lambda_k w_\alpha)$ increases to u. Moreover, we have for each $x \in D$

$$\begin{aligned} u(x) &\ge u_k(x) - \lambda_k w_\alpha(x) \\ &= \int_D G_D^\alpha(x,y)\varphi(y,u_k(y))\,dy \\ &\ge \int_D G_D^\alpha(x,y)\varphi(y,\gamma_k w_\alpha(y))\,dy > 0. \end{aligned}$$

Hence, applying the monotone convergence theorem and using the continuity of φ with respect to the second variable, we get

$$u(x) = \int_D G_D^\alpha(x,y)\varphi(y,u(y))\,dy,\ x \in D. \tag{5.6}$$

Let us prove that u is a positive continuous solution of (P). It is clear that u is continuous on D. Indeed, we have

$$u = \sup_k (u_k - \lambda_k w_\alpha) = \inf_k u_k$$

and u_k and w_α are continuous functions in D. So applying $(-\Delta)^{\frac{\alpha}{2}}$ on both sides of the equation (5.6), we conclude that u is a positive continuous solution of

$$(-\Delta)^{\frac{\alpha}{2}} u = \varphi(\cdot,u) \text{ (in the distributional sense).}$$

Furthermore, since $0 < u(x) \leq u_k(x)$, for each $x \in D$ and $k \in \mathbb{N}$, we deduce that $\lim\limits_{x\to\partial D\cup\{\infty\}} \frac{u(x)}{w_\alpha(x)} = 0$ by applying Lemma 5.2 to $\gamma_k w_\alpha$. Then by (2.24), we have

$$\lim_{x\to\partial D} (\delta_D(x))^{1-\frac{\alpha}{2}} u(x) = 0 \text{ and } \lim_{|x|\to\infty} u(x) = 0.$$

This proves that u is a positive continuous solution of (P).

Corollary 5.7. *Let φ be a function satisfying (H_1) and (H_2). Then for each $\lambda \geq 0$ and for each nonnegative continuous function f on ∂D, the following nonlinear problem*

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} u = \varphi(\cdot, u) \text{ in } D \text{ (in the distributional sense)} \\ \lim\limits_{x\to z\in\partial D} \frac{u(x)}{w_\alpha(x)} = f(z), \\ \lim\limits_{|x|\to\infty} \frac{u(x)}{w_\alpha(x)} = \lambda, \end{cases} \tag{5.7}$$

has a positive continuous solution u in D satisfying

$$u(x) = G_D^\alpha(\varphi(\cdot, u))(x) + M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x, \infty), \ x \in D.$$

Proof. Put $h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x, \infty)$ and let Ψ be the function defined on $D \times (0, \infty)$ by

$$\Psi(x, t) = \varphi(x, t + h(x)).$$

Then Ψ satisfies (H_1) and (H_2). Hence, by Theorem 2.17, the following problem

$$\begin{cases} (-\Delta)^{\frac{\alpha}{2}} v = \Psi(\cdot, v) \quad \text{in } D \text{ (in the distributional sense)} \\ \lim\limits_{x\to\partial D} \frac{v(x)}{w_\alpha(x)} = 0 \\ \lim\limits_{|x|\to\infty} v(x) = 0, \end{cases}$$

has a positive continuous solution v satisfying $v = G_D^\alpha(\Psi(\cdot, v))$ in D.
Then, the function

$$\begin{aligned} u(x) &= h(x) + v(x) \\ &= h(x) + G_D^\alpha(\Psi(\cdot, v))(x) \\ &= h(x) + G_D^\alpha(\varphi(\cdot, u))(x) \end{aligned}$$

is a positive continuous solution of problem (5.7). This completes the proof. □

6 Second existence result

Before giving the proof of Theorem 2.18, some tools of potential theory are needed. We are going to recall them in this paragraph and we refer to [15] or [21] for more details. For $q \in B^+(D)$, we define the potential kernel V_q on $B^+(D)$ by

$$V_q f(x) := \int_0^\infty E^x(e^{-\int_0^t q(X_t^D)ds} f(X_t^D))dt. \tag{6.1}$$

Note that $V := V_0 = G_D^\alpha$.
Furthermore if q satisfies $Vq < \infty$, then the kernel V_q satisfies the following resolvent equation

$$V = V_q + V_q(qV) = V_q + V(qV_q). \tag{6.2}$$

In particular, if $u \in B^+(D)$ is such that $V(qu) < \infty$, then we have

$$(I - V_q(q.))(I + V(q.))u = (I + V(q.))(I - V_q(q.)) = u. \tag{6.3}$$

The following lemma plays a key role.

Lemma 6.1. *Let q be a nonnegative function in $K_\alpha^\infty(D)$ and v be a positive finite function in $\mathcal{S}_D^\alpha$. Then for all $x \in D$, we have*

$$\exp(-a_\alpha(q))\, v(x) \leq v(x) - V_q(qv)(x) \leq v(x).$$

Proof. Since $v \in \mathcal{S}_D^\alpha$ then by ([3], Chap. II, Proposition 3.11), there exists a sequence $(f_k)_k$ in $B^+(D)$ such that $v = \sup_k V f_k$.

Let $x \in D$ and $k \in \mathbb{N}$ such that $0 < Vf_k(x) < \infty$. Consider $\gamma(t) = V_{tq} f_k(x)$, for $t \geq 0$. Then by (6.1), the function γ is completely monotone on $[0,\infty)$. So we deduce from ([27], Theorem 12a) and the Schwarz inequality that $\log\gamma$ is convex on $[0,\infty)$. This implies that

$$\log\gamma(1) \geq \log\gamma(0) + \frac{\gamma'(0)}{\gamma(0)}.$$

That is

$$\gamma(0) \leq \gamma(1)\exp(-\frac{\gamma'(0)}{\gamma(0)}).$$

Which implies that

$$Vf_k(x) \leq V_q f_k(x) \exp\left(\frac{V(qVf_k)(x)}{Vf_k(x)}\right).$$

Since Vf_k is in $\mathcal{S}_D^\alpha$, it follows from (4.4) that

$$Vf_k(x) \leq \exp(a_\alpha(q))\, V_q f_k(x).$$

Hence by (6.2), we obtain

$$\exp(-a_\alpha(q))\, Vf_k(x) \leq V_q f_k(x) = Vf_k(x) - V_q(qVf_k)(x) \leq Vf_k(x).$$

The result holds by letting $k \to \infty$. □

Proof of Theorem 3. We shall convert problem (Q) into a suitable integral equation. So, we aim to show an existence result for the equation

$$u + V(u\varphi(\cdot,u)) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty), \tag{6.4}$$

where f is a non-trivial nonnegative continuous function on ∂D and λ is a nonnegative constant.
Put

$$h(x) = M_D^\alpha f(x) + \lambda c_0 M_D^\alpha(x,\infty).$$

First, we remark by (2.24) that

$$h(x) \le \max(\lambda c_0, \|f\|_\infty) w_\alpha(x) \le c\,(\rho_D(x))^{\frac{\alpha}{2}-1}. \tag{6.5}$$

Put $q := q_c$ be the function in $K_\alpha^\infty(D)$ given by (H_4).
Let Λ be the closed convex set given by

$$\Lambda = \{u \ \in B^+(D) : \exp(-a_\alpha(q))h \le u \le h\}$$

and let T be the operator defined on Λ by

$$Tu = h - V_q(qh) + V_q((q - \varphi(\cdot,u))u).$$

We claim that Λ is invariant under T. Indeed by (H_4), we have for any $u \in \Lambda$

$$0 \le \varphi(\cdot,u) \le q. \tag{6.6}$$

Then, it follows by Lemma 6.1, that for $u \in \Lambda$, we have

$$Tu \ge h - V_q(qh) \ge \exp(-a_\alpha(q))h.$$

Moreover, since for $u \in \Lambda$, we have $u \le h$ and consequently

$$Tu \le h - V_q(qh) + V_q(qu) \le h.$$

This shows that $T\Lambda \subset \Lambda$.
Let u and v be two functions in Λ such that $u \le v$. Then from (H_4), we have

$$Tv - Tu = V_q\left[(q - \varphi(\cdot,v))v - (q - \varphi(\cdot,u))u\right] \ge 0.$$

Thus, T is nondecreasing on Λ. Now, let (u_k) be the sequence defined by

$$u_0 = \exp(-a_\alpha(q))h \text{ and } u_{k+1} = Tu_k, \text{ for } k \in \mathbb{N}.$$

Since $T\Lambda \subset \Lambda$ and from the monotonicity of T, we obtain

$$u_0 \le u_1 \le ..... \le u_{n+1} \le h.$$

Hence by (H_4) and the dominated convergence theorem, we conclude that the sequence (u_k) converges to a function $u \in \Lambda$ satisfying

$$u = h - V_q(qh) + V_q(u(q - \varphi(\cdot,u))).$$

That is

$$(I - V_q(q.))u + V_q(u\varphi(\cdot,u)) = (I - V_q(q.))h.$$

Applying the operator $(I+V_q(q.))$ on both sides of the above equality and using (6.2) and (6.3), we deduce that u satisfies (6.4).
It remains to prove that u is a positive continuous solution of problem (Q). Since q is in $K_\alpha^\infty(D)$, then we have by (6.6) that for $x \in D$

$$0 \leq u(x)\varphi(x,u(x)) \leq u(x)q(x) \leq q(x)h(x). \tag{6.7}$$

It follows from (6.5) that

$$0 \leq u(x)\varphi(x,u(x)) \leq cq(x)(\rho_D(x))^{\frac{\alpha}{2}-1}. \tag{6.8}$$

So by Theorem 4.9, we conclude that the function $x \mapsto (\rho_D(x))^{1-\frac{\alpha}{2}} V(u\varphi(\cdot,u))(x)$ is in $C_0(D)$. According to (6.4) we deduce that u is continuous in D. Now, going back to (6.4) and applying $(-\Delta)^{\frac{\alpha}{2}}$, we deduce that u satisfies

$$(-\Delta)^{\frac{\alpha}{2}} u + u\varphi(\cdot,u) = 0 \text{ (in the distributional sense).}$$

On the other hand, by (6.4) and Remark 2.14, we conclude that

$$\lim_{x\to z\in\partial D} \frac{u(x)}{w_\alpha(x)} = f(z) \text{ and } \lim_{|x|\to\infty} \frac{u(x)}{w_\alpha(x)} = \lambda.$$

This completes the proof. □

Acknowledgments

The authors thank the referee for his/her careful reading of the first version of the paper and for insightful comments.

References

[1] D.H. Armitage, S.J. Gardiner, Classical Potential Theory. Springer-Verlag, 2001.

[2] I. Bachar, H. Maâgli, N. Zeddini, Estimates on the Green function and existence of positive solutions of nonlinear singular elliptic equations. Comm. in Contemporary Math. (5) (2003), pp 401-433.

[3] J. Bliedtner and W. Hansen, Potential Theory, An Analytic and Probabilistic Approach to Balayage. Springer-Verlag, Berlin/New York, 1986.

[4] R.M. Blumenthal, R.K Getoor, D.B. Ray, On the distribution of first hits for the symmetric stable processes. Trans. Amer. Math. Soc. 99 (1961), pp 540-554.

[5] K. Bogdan, The boundary Harnack principle for the fractional Laplacian. Studia Math. 123 (1) (1997), 43-80.

[6] K. Bogdan, T. Zak, On Kelvin transformation. J. Theor. Prob. 19 (1) (2006), pp 89-120.

[7] K. Bogdan, Representation of α-harmonic functions in Lipschitz domains. Hirochima Math. J. 29 (1999), pp 227-243.

[8] K. Bogdan, T. Byczkowski, Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains. Studia Math. 133 (1999), pp 53-92.

[9] R. Chemmam, H. Mâagli, S. Masmoudi, On a new Kato class and positive solutions of Dirichlet problems for the fractional Laplacian in bounded domains. Nonlinear Anal. 74 (2011), pp 1555-1576.

[10] Z.Q. Chen and T. Kumagai, Heat kernel estimates for stable-like processes on d-sets. Stoch. Proc.Appl. 108 (2003), pp 27-62.

[11] Z.Q. Chen, P. Kim, R. Song, Heat kernel estimates for Dirichlet fractional Laplacian. J. Euro. Math. Soc. 12 (2010), pp 1307-1329.

[12] Z.Q Chen, R. Song, Estimates on Green functions and poisson kernels for symmetric stable processes. Math. Ann. 312 (1998), pp 465-501.

[13] Z.Q. Chen, R. Song, Martin boundary and integral representation for harmonic functions of symmetric processes. J. Funct. Anal. 159 (1998), pp 267-294.

[14] Z.Q. Chen, J. Tokle, Global heat kernel estimates for fractional Laplacians in unbounded open sets. Probab. Theory Relat. Feilds 149 (2011), pp 373-395.

[15] K.L. Chung, Z. Zhao, From Brownian motion to Schrödinger's equation. Springer-Verglag, Berlin 1995.

[16] N. Jacob, Pseudo differential operators and Markov processes. Vol.I, II, III. Imperial College Press, London, 2001.

[17] A. Janicki, A. Weron, Simulation and chaotic behavior of α-stable processes. Dekker, (1994).

[18] P. Kim, Relative Fatou's theorem for $(-\Delta)^{\frac{\alpha}{2}}$-harmonic function in bounded k-fat open sets. J. Funct. Anal. 234 (2006), pp 70-105.

[19] T. Kulczcki, Properties of Green function of symmetric stable processes. Probab. Math. Statist.17 (1997), pp 384-406.

[20] H. Mâagli, Invariance des fonctions α-harmoniques par les transformations de Möbius. Expo. Math. 13 (1995), pp 469-480.

[21] H. Mâagli, Perturbation semi-linéaire des résolvantes et des semi-groupes. Potential Anal. 3 (1994), pp 61-87.

[22] H. Mâagli, N. Zeddini, Global solutions for a nonlinear absorption parabolic problem in exterior domains. Nonlinear Anal. 72 (2010), pp 1423-1437.

[23] H. Mâagli, S. Masmoudi, Positive solutions of some nonlinear elliptic problems in unbounded domain. Anal. Acad. Sc. Fennicae. 29 (2004), pp 151-166.

[24] H.P. Mckean, Sample functions of stable processes. Ann. Math. 61 (1995), pp 564-579.

[25] H. Kunita, T. Watanabe, Markov processes and Martin boundaries. Illinois J. Math. 9 (1965), pp 485-526.

[26] R. Song, J. Wu, Boundary Harnack principle for symmetric stable processes. J. Funct. Anal. 168 (1999), pp 403-427.

[27] D.V. Widder, The Laplace Transform. Princeton Univ. Press, Princeton, NJ, 1941.