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On Strong Oscillation Criteria for Bounded Solutions for Some Quasilinear Second Order Elliptic Equations

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Abstract

In this work, some boundedness conditions on the coefficients of some general half-linear quasilinear elliptic equations are used to establish some oscillation criteria for those equations. Emphasis is put on strongly oscillatory conditions. This is obtained by using some comparison methods and Picone-type formulas.

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1 Introduction: Preliminaries

In the sequel, when we write $\mathbb{R}^n$, it will be assumed that $n \geq 3$.

Definition 1.1. Let $h \in C(E)$ where E denotes $\mathbb{R}$ or $\mathbb{R}^n$. h will be said to be
(i) (weakly) Oscillatory in E if h has a zero in any $\Omega_T := \{x \in E\ ;\ |x| > T\}$;
(ii) strongly oscillatory if it has a nodal set in any Ω_T, $\forall T > 0$, where a nodal set is any non trivial connected and bounded component of support $D(h)$ of h.
(iii) A differential equation will be said to be oscillatory if any of its non trivial and bounded solution is oscillatory.
(iv) Therefore a function w will be said not to be oscillatory if either there are $\mu, R > 0$ such that $|w| > \mu$ in Ω_R or $\liminf_{t\nearrow\infty} |w(t)| = 0$.

As we will be focussing on strongly oscillatory solutions, we will be using some comparison methods: in fact if a function w is known to be strongly oscillatory and another function v satisfies for some large $T > 0$,

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(v) v has a zero in any nodal set $D(w) \subset \Omega_T$; and

(vi) w has a zero in any nodal set $D(v) \subset \Omega_T$;

then v is also strongly oscillatory. Moreover if only (v) holds, v would be also strongly oscillatory unless it has compact support (i.e. there is $\rho > 0$ such that $v \equiv 0$ in Ω_ρ).

We will be dealing with equations of the type

$$\begin{cases} P(y) := \Big\{a(t)\phi(y')\Big\}' + q(t)\phi(y) + f(t,y,y') = 0, \quad t \in \mathbb{R} \quad \text{and} \\ Q(u) := \quad \nabla \cdot \Big\{A(x)\Phi(\nabla u)\Big\} + C(x)\phi(u) + F(x,u,\nabla u) = 0, \quad x \in \mathbb{R}^n; \end{cases} \tag{1.1}$$

where $a \in C^1(\mathbb{R}, (0, \infty))$, $A \in C^1(\mathbb{R}^n, (0, \infty))$, $q \in C(\mathbb{R}, \mathbb{R})$, $C \in C(\mathbb{R}^n, \mathbb{R})$ and for some $\alpha > 0$, $t \in \mathbb{R}$, $\zeta \in \mathbb{R}^n$, $\phi(t) := \phi_\alpha(t) = |t|^{\alpha-1}t$; $\Phi(\zeta) := \Phi_\alpha(\zeta) = |\zeta|^{\alpha-1}\zeta$ with the properties that $t\phi(t) = |t|^{\alpha+1}$; $\zeta \cdot \Phi(\zeta) = |\zeta|^{\alpha+1}$; $\phi(t)\Phi(\zeta) = \Phi(t\zeta)$ and $t\phi'(t) = \alpha\phi(t)$. The functions $f(t,y,y')$ and $F(x,u,\nabla u)$ are the perturbations terms added to the respective half-linear equations. General hypotheses will be on the coefficients of the half-linear parts of the equations mainly

(H): the numerical functions a and A are continuously differentiable in their respective domains ; C and q are continuous in their respective arguments and eventually positive (i.e. $\exists T > 0$; $C, q > 0$ in Ω_T).

By means of some comparison results based on some Picone-type formulas, we will deduce the strong oscillatory criteria of some more general equations from that of simple known ones.

We recall that a function y for P (u for Q) will be said to be a solution for P (Q for u) if it satisfies the corresponding equation and $y(.)$ and $a(.)\phi(y'(.))$ (respectively $u(.)$ and $A(.)\Phi(\nabla u(.))$) are continuously differentiable. Also we will say that a solution is bounded in Ω, say, if it is bounded in $C^1(\Omega)$.

Remark 1.2. R1) The equations we will be dealing with are of the form

$$\begin{cases} \{a(t)|y'|^{\alpha-1}y'\}' + q(t)|y|^{\alpha-1}y + f(t,y,y') = 0 \; ; \; t > 0 \\ \text{and for multi-dimensional cases} \\ \nabla \cdot \{A(x)|\nabla u|^{\alpha-1}\nabla u\} + c(x)|u|^{\alpha-1}u + F(x,u,\nabla u) = 0 \end{cases} \tag{1.2}$$

with regular coefficients (e.g. a and A are continuously differentiable) and c, q continuous. Also we assume that considering these equations in any bounded regular domain D, there are generic constants $C(D) > 0$ such that for small $|y'|$ ($|\nabla u|$ for multi-dimensional cases)

$$\begin{gathered} |f(t,y,y')| \le C(D)|y'|^\alpha + k(u) \qquad (|F(x,u,\nabla u)| \le C(D)|\nabla u|^\alpha + k(u)) \\ \text{where for } K(t) = \int_0^t k(s)ds, \quad \int_0^1 K(t)^{\frac{-1}{\alpha+1}} dt = \infty. \end{gathered} \tag{1.3}$$

Therefore no non-trivial solution of the equations has compact support . Also any such a continuous solution which is bounded in $C(\Omega_R)$ is also bounded in $C^1(\Omega_R$), (see [2] ,[6]).

R2) Under those conditions, such solutions need only to be oscillatory to be strongly oscillatory. In the sequel, unless indicated otherwise, the perturbation terms containing y' (damping term) will be understood to satisfy (1.3) locally in Ω_T. This is obviously true for half-linear equations. Also in the sequel, bounded solution will mean bounded in $C^1(\Omega_T)$ for some $T > 0$.

Here is a theorem whose proof is a mere application of the definition of an oscillatory solution:

Theorem 1.3. *Suppose that for some $T > 0$ there is a continuous function c such that either $c(t) \geq \mu > 0$ in Ω_T or $\int_T^r c(s)ds$ diverges to $+\infty$.*
Then, $\forall \alpha > 0$ and $\beta \geq \alpha$ any non trivial and bounded solution of

$$\Big(\phi_\alpha(y')\Big)' + c(t)\phi_\beta(y) = 0 \tag{1.4}$$

is strongly oscillatory. The same conclusion holds for

$$\Big(a(t)\phi_\alpha(y')\Big)' + c(t)\phi_\beta(y) = 0 \tag{1.5}$$

if for some Ω_T and $a_0 > 0$, $\quad a \in C^1(\Omega_T, (a_0, \infty))$, $a' > 0$ and decreases to 0.

Proof. A) Those equations are oscillatory.

Assume that there is such a solution which is strictly positive in Ω_T, say. From the equation (1.4) and $t > t_1 > T$

$\phi_\alpha(y'(t)) - \phi_\alpha(y'(t_1)) = -\int_{t_1}^t c(s)\phi_\beta(y(s))ds < 0$ implying that y' keeps the same sign in some Ω_{T_1}, $T_1 \geq T$. But

$\Big(\phi_\alpha(y')\Big)' = \dfrac{y''}{y'} y'\phi'_\alpha(y') = \alpha\dfrac{y''}{y'}\phi_\alpha(y') = -c(t)\phi_\beta(y) < 0$ therefore $y'' < 0$ and y being bounded, y' then decreases to 0 at ∞. y' is then positive and decreases to 0. So, the function $W(t) := \dfrac{\phi_\alpha(y')}{\phi_\beta(y)}$ is non negative and satisfies

$W'(t) = -\Big\{c(t) + \beta|\dfrac{y'}{y}|^{\alpha+1}\dfrac{|y|^{\alpha-1}}{|y|^{\beta-1}}\Big\} \leq -c(t)$ for $t > T_1$. We then have

$W(t) \leq W(T_1) - \int_{T_1}^t c(s)ds$ which is a contradiction for large t. In fact the second member diverges to $-\infty$ while the first member is non negative. Therefore any such a solution of (1.4) has to have zeros in any Ω_T.

For the second part we proceed as above; in fact if y is a solution of (1.5) which is supposed strictly positive in Ω_T,

$(a(t)\phi_\alpha(y'))' = a'(t)\phi_\alpha(y') + \alpha a(t)\dfrac{y''}{y'}\phi_\alpha(y') \leq -c(t)\phi_\beta(y) < 0.$

From the hypotheses on a, for a large $t > T$, $\quad a'(t)\phi_\alpha(y') + \alpha a(t)\dfrac{y''}{y'}\phi_\alpha(y')$ has eventually the sign of y'' , therefore $y' > 0$ and decreases to 0. The non negative function $K(t) := \dfrac{a(t)\phi_\alpha(y')}{\phi_\beta(y)}$ satisfies also $K'(t) \leq -c(t)$ and the proof is completed as before.

B) If any of those equations is oscillatory, it is strongly oscillatory.

It is enough to show that if the equation is simply oscillatory, its zeros are simple. Suppose that the equation (1.5) has an oscillatory solution y, say, such that $\exists \tau > 0$; $y(\tau) = y'(\tau) = 0$ and $y' \neq 0$ in some $(t_0, \tau)\bigcup(\tau, t_1) := I_0 \bigcup I_1$.

B1) Assume that $y > 0$ in I_0. From the equation,

$$-a(t_0)\phi_\alpha(y'(t_0)) = -\int_{t_0}^{\tau} c(s)\phi_\beta(y(s))ds \qquad (*)$$

which is strictly negative implying that $y' > 0$ in I_0. This conflicts with $y > 0$ in I_0.

B2) Assume that $y < 0$ in I_0. Using $(*)$ we find that $y' < 0$ in I_0, conflicting with $y < 0$ in I_0. Therefore $y' \neq 0$ cannot hold in I_0 if such a τ exists.

Similarly we get that $y' \neq 0$ cannot hold in I_1 if such a τ exists. This completes the proof. □

Consider for some $\alpha, \beta > 0$ the equation

$$\Big\{\phi_\alpha(u')\Big\}' + g(t,u)\phi_\beta(u) = 0. \tag{1.6}$$

Theorem 1.4. *Assume that* $\forall w \in \mathbb{R}\setminus\{0\}$ *and some* $T > 0$

$$g(t,w) > 0 \quad \textit{in } \Omega_T \textit{ and } \quad \int_T^t g(s,w)ds \quad \textit{is unbounded.} \tag{1.7}$$

Then any non-trivial and bounded solution u of the equation (1.6) is oscillatory.

Proof. If we suppose that u is such a solution positive in say, Ω_T, then as in the proof of Theorem 1.3, $u' > 0$ and tends to 0 at infinity. This implies that u is increasing and bounded above.

The function $W(t) = \dfrac{\phi_\alpha(u')}{\phi_\alpha(u)}$ is non-negative and

$W'(t) = -\Big[g(t,u)u^{\beta-\alpha} + \alpha\Big|\dfrac{u'}{u}\Big|^{\alpha+1}\Big] \le -g(t,u)u^{\beta-\alpha}$ and

$W(t) \le W(T) - \int_T^t g(s,u)u^{\beta-\alpha}ds$ leading to a contradiction as the right hand side is eventually negative. Therefore no such a solution can be eventually non zero. □

For $c \in C(\mathbb{R}, \mathbb{R})$, $a \in C^1(\mathbb{R}, \mathbb{R})$ with $a, c > 0$ in Ω_T say, simple calculation shows that the equation

$$\begin{cases} (i) \quad \Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) = 0 \\ \text{has the same oscillatory character in } \Omega_T \text{ as} \\ (ii) \quad \Big(\phi(u')\Big)' + \dfrac{c(t)}{a(t)}\phi(u) + \dfrac{a'(t)}{a(t)}\phi(u') = 0. \\ \text{If in addition } |\dfrac{a'}{c}| \text{ is bounded in } \Omega_T \text{, the same conclusion holds for} \\ (iii) \quad \Big(\dfrac{a}{c}\phi(z')\Big)' - \dfrac{ac'}{c^2}\phi(z') + \phi(z) = 0. \end{cases} \tag{1.8}$$

The equations (1.8)(ii) and (iii) are obtained from (1.8)(i) by simple division and for (iii) it is enough to notice that $-\dfrac{ac'}{c^2} = -\dfrac{a'}{c} + (\dfrac{a}{c})'$.

We then have the following result:

Theorem 1.5. *Consider the equation*

$$\Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) = 0 \tag{1.9}$$

where $c \in C(\mathbb{R}, \mathbb{R})$, $a \in C^1(\mathbb{R}, \mathbb{R})$ *with* $a', c, a > 0$ *in some* Ω_T. *If any of the following conditions*

$$\begin{cases} (a) \quad \dfrac{c}{a} > \mu > 0 \ \text{ in } \Omega_T \ \text{ or } \displaystyle\int_{\Omega_T} \frac{c(s)}{a(s)} ds = \infty \\ (b) \quad \exists \theta > 0; \quad 0 < \dfrac{a}{c} < \theta \ \text{ and } |\dfrac{a'}{c}| \text{ bounded in } \Omega_T; \\ (c) \quad \exists k \in \mathbb{R}\setminus\{0\}; \quad 0 < \dfrac{a}{c} < e^{-k\alpha t} := a_1(t) \ \text{ and} \\ |k|^{\alpha+1}\alpha e^{kt} < 1 \quad \text{in} \quad \Omega_T \end{cases} \tag{1.10}$$

holds, then any non-trivial and bounded solution of (1.9) is strongly oscillatory.

Proof. Under the hypotheses in (a), $\Big(\phi(u')\Big)' + \dfrac{c(t)}{a(t)}\phi(u) = 0$ is strongly oscillatory by Theorem 1.3, the equation (1.8)(ii) is then strongly oscillatory by the Theorem 3.4 of [9].

Also from the hypotheses in (b), $\Big(\dfrac{a}{c}\phi(z')\Big)' + \phi(z) = 0$ is strongly oscillatory from the comparison with a strongly oscillatory solution of $\Big(\theta\phi(u')\Big)' + \phi(u) = 0$ and (1.8)(iii) is strongly oscillatory by the Theorem 3.4 of [9].

For the case (c), let u be a strongly oscillatory solution of $\Big\{a_1(t)\phi(u')\Big\}' + \phi(u) = 0$ as in (2.4)(i). If y is a bounded and non-trivial solution of $\Big\{\dfrac{a}{c}\phi(y')\Big\}' + \phi(y) = 0$,

$$\Big\{a_1(t)u\phi(u') - u\phi(\frac{u}{y})\frac{a}{c}\phi(y')\Big\}' = (a_1 - \frac{a}{c})|u'|^{\alpha+1} + \frac{a}{c}\zeta_\alpha(u, y).$$

If we suppose that $y \neq 0$ in Ω_R, the integration of the equation above over any $D(u) \subset \Omega_{(R+T)}$ leads to a contradiction. Therefore y cannot be eventually non zero.

□

Remark 1.6. R3) It is useful to note that the function $g(t,u)\phi_\beta(u)$ of the Theorem 1.4 can be replaced by any function $\psi(t,u,u')$ provided that

$u\psi(t,u,u') > 0 \quad \forall u \neq 0$ and $|\psi(t,u,u')| = O(|u'|^\alpha) + O(|u|^\alpha)$ for small $|u'| + |u|$ (see Remark 1.2).

R4) Also from Remak 1.2, for the bounded and non-trivial solution of Theorem 1.3 to be strongly oscillatory, we need to have $\beta \geq \alpha$.

R5) For the solution of Theorem 1.4 to be strongly oscillatory we need to have $|g(t,u)| = O(|u|^{\alpha-\beta})$ for small $|u|$; otherwise solutions so obtained for these equations are not bounded and non-trivial, unless they have compact supports.

2 Some comparison results via Picone's formulae

For ease in writing, the equation $P_i(.)$ will denote the equation $P(.)$ in (1.1) in which the coefficients a, q and the function f carry the index i. Similarly is defined $Q_i(.)$.

2.1 Some Picone's formulae

Let y_1 and y_2 be respectively solutions of $P_i(y_i)$, $i = 1,2$. Then wherever y_2 is non zero, a version of Picone's identity reads

$$\begin{aligned}&\Big\{y_1 a_1(t)\phi(y_1') - y_1\phi(\frac{y_1}{y_2})a_2(t)\phi(y_2')\Big\}' = \\ &= a_2(t)\zeta_\alpha(y_1,y_2) + \Big[a_1(t)-a_2(t)\Big]|y_1'|^{\alpha+1} + \Big[q_2(t)-q_1(t)\Big]|y_1|^{\alpha+1} \\ &+ |y_1|^{\alpha+1}\Big[\frac{f_2(t,y_2,y_2')}{\phi(y_2)} - \frac{f_1(t,y_1,y_1')}{\phi(y_1)}\Big]\end{aligned} \tag{2.1}$$

where, $\forall\gamma > 0$, the two-form function ζ_γ is defined $\forall u,v \in C^1(\mathbb{R},\, \mathbb{R})$ by

$$\textbf{(Z1)}: \qquad \zeta_\gamma(u,v)\begin{cases} = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma v'\frac{u}{v}\phi_\gamma(\frac{u}{v}v') \\ = |u'|^{\gamma+1} - (\gamma+1)u'\phi_\gamma(\frac{u}{v}v') + \gamma|\frac{u}{v}v'|^{\gamma+1}\end{cases}$$

is strictly positive for non null $u \neq v$ and null only if $u = \lambda v$ for some $\lambda \in \mathbb{R}$. Similarly, if u_1 and u_2 are respectively solutions of Qu_i, $i = 1,2$, then wherever u_2 is non zero, a version of Picone's identity reads

$$\begin{aligned}&\nabla\cdot\Big\{u_1 A_1(x)\Phi(\nabla u_1) - u_1\phi(\frac{u_1}{u_2})A_2(r)\Phi(\nabla u_2)\Big\} = A_2(r)Z_\alpha(u_1,u_2) \\ &+\Big(A_1(x)-A_2(r)\Big)|\nabla u_1|^{\alpha+1} + \Big(C_2(r)-C_1(x)\Big)|u_1|^{\alpha+1} \\ &+|u_1|^{\alpha+1}\Big[\frac{F_2(x,u_2,\nabla u_2)}{\phi(u_2)} - \frac{F_1(x,u_1,\nabla u_1)}{\phi(u_1)}\Big] \\ &\text{where} \quad \forall\gamma>0, \quad \forall u,v\in C^1(\mathbb{R}^n) \\ &\textbf{(Z2)}: \quad Z_\gamma(u,v) := |\nabla u|^{\gamma+1} - (\gamma+1)\Phi_\gamma(\frac{u}{v}\nabla v)\cdot\nabla u + \gamma|\frac{u}{v}\nabla v|^{\gamma+1} \\ &= |\nabla u|^{\gamma+1} - (\gamma+1)|\frac{u}{v}\nabla v|^{\gamma-1}\,\frac{u}{v}\nabla v\cdot\nabla u + \gamma|\frac{u}{v}\nabla v|^{\gamma+1}.\end{aligned} \tag{2.2}$$

We recall that $\forall\gamma > 0$ the two-form $Z_\gamma(u,v) \geq 0$ and is null only if $\exists k \in \mathbb{R};\ u = kv$. (see e.g. [5], [7, 8]).

We note that Z_γ is associated to the two-form Ψ_γ defined on $\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}$ by

$$\textbf{(}\boldsymbol{\Psi}\textbf{)}: \qquad \Psi_\gamma(X,Y;\mu) := |X|^{\gamma+1} - (\gamma+1)|\mu Y|^{\gamma-1}\mu Y\cdot X + |\mu Y|^{\gamma+1}$$

which is non-negative and null only if $X = 0$ or $X = \mu Y$ for some $\mu \in \mathbb{R}$.

2.2 Some basic strongly oscillatory equations

As example of strongly oscillatory functions in $\mathbb{R}$ we have the following: $\forall\alpha > 0$ the solutions of

$$\Big\{\phi_\alpha(u')\Big\}' + \alpha\phi_\alpha(u) = 0 \tag{2.3}$$

whose solutions are the generalized sine functions $S := S_\alpha$ with the following properties:

$$\begin{cases} |S_\alpha(t)|^{\alpha+1} + |S'_\alpha(t)|^{\alpha+1} = 1\ ; \quad S_\alpha(t+\pi_\alpha) = -S_\alpha(t); \\ \text{where} \quad \pi_\alpha = \dfrac{2\pi}{(\alpha+1)\sin\{\dfrac{\pi}{\alpha+1}\}}. \\ \text{Easy calculations show that } \forall k \in \mathbb{R}\setminus\{0\}, \quad W(t) := S_\alpha(e^{kt}) \text{ satisfies} \\ (i) \quad \Big\{e^{-k\alpha t}\phi_\alpha(W')\Big\}' + |k|^{\alpha+1}\alpha e^{kt}\phi_\alpha(W) = 0; \\ \text{and for} \quad Y(t) := S_\alpha(t^k); \quad t \geq 0, \\ (ii) \quad \Big\{t^{(1-k)\alpha}\ \phi_\alpha(Y')\Big\}' + |k|^{\alpha+1}t^{k-1}\ \alpha\phi_\alpha(Y) = 0. \end{cases} \tag{2.4}$$

When $\alpha = 1$ then those functions are the usual trigonometric functions. For $k \in \mathbb{R}\setminus\{0\}$, let $\sigma_k(t) := S_\alpha(kt)$. Then $\Big(\phi_\alpha(\sigma'(t)\Big)' + \alpha|k|^{\alpha+1}\phi_\alpha(\sigma_k(t)) = 0$. Because k is arbitrary we then have the following results:

$$\begin{cases} (i) \quad \forall \alpha, b > 0, \qquad \Big\{\phi_\alpha(u')\Big\}' + b\phi_\alpha(u) = 0 \quad \text{is strongly oscillatory} \\ (ii) \quad \text{and from (2.4)(ii), } \forall\alpha > 0,\ \forall k \in \mathbb{R}\setminus\{0\} \\ \Big(\phi_\alpha(y')\Big)' + |k|^{\alpha+1}t^{(k-1)(1+\alpha)}\phi_\alpha(y) + \alpha(1-k)t^{-1}\phi_\alpha(y') = 0 \\ \qquad \text{is also strongly oscillatory.} \end{cases} \tag{2.5}$$

Theorem 2.1. *For some* $\nu, T > 0$*, assume that a non-decreasing* $a \in C^1(\Omega_T, (\nu, \infty))$ *and* $c \in C(\Omega_T, (\nu, \infty))$ *are such that*

$$\Big\{a(t)\phi(u')\Big\}' + c(t)\phi(u) = 0 \tag{2.6}$$

is oscillatory . Then for any $M \in \mathbb{R}$ *large enough,*

$$\Big\{a(t)\phi(z')\Big\}' + c(t)\phi(z) + a(t)M = 0 \tag{2.7}$$

will be strongly oscillatory .

Proof. Let such a solution z of (2.7) and u that of (2.6) be given. We have the following Picone-type formulas:

$$\begin{aligned} &(i) \qquad \Big\{a(t)u\phi(u') - a(t)u\phi(\frac{u}{z})\phi(z')\Big\}' = a(t)\Big\{\zeta_\alpha(u,z) + |u|^{\alpha+1}\frac{M}{\phi(z)}\Big\}; \\ &(ii) \qquad \Big\{a(t)z\phi(z') - a(t)z\phi(\frac{z}{u})\phi(u')\Big\}' = a(t)\Big\{\zeta_\alpha(z,u) - zM\Big\}. \end{aligned} \tag{2.8}$$

The functions u and z being bounded in Ω_T, the second members of (2.8) (i) and (2.8)(ii) are non zero and have the sign of M or $-M$ provided that M is large enough. Therefore if

z is non zero over a nodal set $D(u) \subset \Omega_T$, the integration of both sides of (2.8)(i) over $D(u)$ leads to $0 = \int_{D(u)} a(t)\Big\{\zeta_\alpha(u,z) + |u|^{\alpha+1}\dfrac{M}{\phi(z)}\Big\}dt$ which would be absurd for large enough M as the second member is non zero. Consequently z has a zero in any such $D(u)$. Similarly if u is non zero on such nodal set $D(z) \subset \Omega_T$ of z, the integration over that $D(z)$ of (2.8)(ii) leads to a contradiction; also u has a zero in any such a $D(z) \subset \Omega_T$. □

Remark 2.2. R6) When a solution u of (2.6) is oscillatory, it has to have nodal sets in any Ω_T. In fact if u has a zero r_1 in Ω_T , it has to have a zero in Ω_{2r_1} and obviously, from the half-linear equation, any second zero has to be simple unless u has a compact support.

R7) Theorem 2.1 shows that if M is large enough, when any of the equations (2.6) and (2.7) is oscillatory of any type, so does the other; also in Ω_T, no nodal set of one is strictly contained in a nodal set of the other.

Theorem 2.3. *Let $a \in C^1(\mathbb{R},\mathbb{R})$ and $c \in C(\mathbb{R},\mathbb{R})$ be such that for some $\mu, T > 0$,*
a', $c > \mu$ and $\gamma := \dfrac{c}{a} \geq \nu > 0$ in Ω_T. Then

$$\Big\{a(t)\phi_\nu(u')\Big\}' + c(t)\phi_\nu(u) = 0 \tag{2.9}$$

is strongly oscillatory.

Proof. From the hypotheses, (2.9) can be written as

$$\Big\{\phi_\nu(u')\Big\}' + \gamma(t)\phi_\nu(u) + [\log a(t)]'\phi_\nu(u') = 0$$

and this equation is strongly oscillatory by the Theorems 3.1 and 3.4 of [9]. □

Consider the equations

$$\begin{aligned} &(i) \quad \Big\{a(t)\phi(y')\Big\}' + c(t)\phi(y) = 0 \quad \text{and} \\ &(ii) \quad \Big\{a(t)\phi(z')\Big\}' + c(t)\phi(z) + h(t,z,z') = 0 \end{aligned} \tag{2.10}$$

where $h \in C(\mathbb{R}^3, \mathbb{R})$ and satisfies (1.3).

Theorem 2.4. *Assume that for some $T, k > 0$,*
either $h(t,s,z) > 0$ or $\dfrac{h(t,s,z)}{\phi(s)}$ is bounded $\forall (t,S,Z) \in \Omega_T \times (-k, k)^2$. Then if (2.10)(i) is oscillatory, so is (2.10)(ii).

Proof. 1) Assume that $h(t,z,z') > 0$ in Ω_T. If such a solution z is strictly positive in Ω_T,
$\Big\{ya(t)\phi(y') - y\phi(\dfrac{y}{z})a(t)\phi(z')\Big\}' = a(t)\zeta_\alpha(y,z) + |y|^{\alpha+1}\dfrac{h(t,z,z')}{\phi(z)}$
leading to a contradiction if we integrate the formula over a nodal set $D(y^+) \subset \Omega_T$. In fact the left hand side would be zero and the right strictly positive.

2) If $\dfrac{h(t,s,z)}{\phi(s)}$ is bounded, taking the strongly oscillatory solution u of

$\Big\{a(t)\phi(u')\Big\}' + c(t)\phi(u) - M = 0$ for a large enough M,

$\Big\{ua(t)\phi(u') - u\phi(\frac{u}{z})a(t)\phi(z')\Big\}' = a(t)\zeta_\alpha(u,z) + |u|^{\alpha+1}\Big(\frac{h(t,z,z')}{\phi(z)} + \frac{M}{\phi(u)}\Big)$

and the integration of the formula over a nodal set $D(u) \subset \Omega_T$ leads to a contradiction as this time, if M is large enough, the right hand side would be non zero while the left is zero. □

3 Application to multi-dimensional cases

Define for $x := (x_1, x_2, ..., x_n) \in \mathbb{R}^n$ the position-vector

$$X := \frac{1}{r} \quad {}^t(x_1, x_2, ..., x_n)$$

where $r := \sqrt{x_1^2 + x_2^2 + ... + x_n^2} \equiv |X|$. If the function $V = V(r)$ depends only on r then $\nabla V = XV'(r)$ and $\nabla g(V) = XV'g'(V)$. So, if the unknown function u and the other functions are radially symmetric e.g. $u(x) := U(r)$. The equation

$$Q(u) := \quad \nabla \cdot \Big\{A(x)\Phi(\nabla u)\Big\} + C(x)\phi(u) + F(x,u,\nabla u) = 0; \quad x \in \mathbb{R}^n \tag{3.1}$$

in terms of U , $A(x) := \tilde{a}(r)$, $C(x) := \tilde{c}(r)$ and $F(x,u,\nabla u) := \tilde{f}(r,U,U')$ becomes

$$\tilde{Q}(U) = \Big\{r^{n-1}\tilde{a}(r)\phi(U')\Big\}' + r^{n-1}\tilde{c}(r)\phi(U) + r^{n-1}\tilde{f}(r,U,U') = 0 \quad \text{in } \mathbb{R}^+ \tag{3.2}$$

since $\nabla \cdot X = \frac{n-1}{r}$ and $\nabla \cdot \Big\{A(r)\Phi(\nabla V)\Big\} = \frac{1}{r^{n-1}}\Big\{r^{n-1}A(r)\phi(V')\Big\}'$. (see [9], [5])

For any function $w \in C(\mathbb{R}^n)$, define the functions

$$\begin{cases} w^+(r) := r^{n-1} \max\limits_{|x|=r} w(x) \text{ and } w^-(r) := r^{n-1} \min\limits_{|x|=r} w(x) \quad \text{and} \quad \text{for} \quad g(x,v,\nabla v), \\ g^+(r,V,V') \;(\; g^-(r,V,V')\;) := \max(\min)_{|x|=r}\, g(x,v,\nabla v). \end{cases} \tag{3.3}$$

The aim of this paragraph is to find sufficient conditions under which the oscillation criteria seen earlier apply to some multi-dimensional equations.

To the equation (3.1), we associate the equation

$$\Big\{A^-(r)\phi(v')\Big\}' + C^+(r)\phi(v) + r^{n-1}\tilde{f}(r,v,v') = 0 \quad \text{in } \mathbb{R}^+. \tag{3.4}$$

Lemma 3.1. *Assume that*

a) $\tilde{f}(r,v,v')$ *satisfies (1.3)*

b) and there are $M,T > 0$ *such that whenever* $|S| + |V| < M$,

$t \mapsto \dfrac{\tilde{f}(r,S,V)}{\phi(S)}$ *is bounded or positive in* Ω_T*. Then*

if $\Big\{A^-(r)\phi(z')\Big\}' + C^+(r)\phi(z) = 0$ *is strongly oscillatory, any bounded and non-trivial solution of* $\Big\{A^-(r)\phi(v')\Big\}' + C^+(r)\phi(v) + r^{n-1}\tilde{f}(r,v,v') = 0$ *is strongly oscillatory.*

Proof. Consider for a large $M \in \mathbb{R}$ the strongly oscillatory and bounded solution y of $\{A^-(r)\phi(y')\}' + C^+(r)\phi(y) + r^{n-1}M = 0$. Whenever $v \neq 0$

$$
\begin{aligned}
&(i) \qquad \Big\{yA^-(r)\phi(y') - y\phi(\frac{y}{v})A^-(r)\phi(v')\Big\}' \\
&= A^-(r)\zeta_\alpha(y,v) - yr^{n-1}M + |y|^{\alpha+1}\frac{r^{n-1}\tilde{f}(r,v,v')}{\phi(v)} \\
&= A^-(r)\zeta_\alpha(y,v) - |y|^{\alpha+1}r^{n-1}\Big(\frac{M}{\phi(y)} - \frac{\tilde{f}(r,v,v')}{\phi(v)}\Big) \\
&= r^{n-1}\Big[\min_{|x|=r} A(x)\,\zeta_\alpha(y,v) - |y|^{\alpha+1}\Big(\frac{M}{\phi(y)} - \frac{\tilde{f}(r,v,v')}{\phi(v)}\Big)\Big] \qquad (3.5) \\
&(ii) \quad \text{and} \qquad \Big\{vA^-(r)\phi(v') - v\phi(\frac{v}{y})A^-(r)\phi(y')\Big\}' \\
&= A^-(r)\zeta_\alpha(v,y) + |v|^{\alpha+1}r^{n-1}\Big(\frac{M}{\phi(y)} - \frac{\tilde{f}(r,v,v')}{\phi(v)}\Big) \\
&= r^{n-1}\Big[\min_{|x|=r} A(x)\,\zeta_\alpha(v,y) + |v|^{\alpha+1}\Big(\frac{M}{\phi(y)} - \frac{\tilde{f}(r,v,v')}{\phi(v)}\Big)\Big]
\end{aligned}
$$

wherever $y \neq 0$.When M is large enough, the right hand side of (i) is of the sign of $-M$ and that of (ii) the sign of M. The integration over any nodal set $D(y) \subset \Omega_T$ the equation (i) and the equation (ii) over any nodal set $D(v) \subset \Omega_T$, we get that v has a zero in any such a $D(y)$ and y has a zero in any such a $D(v)$. □

For the radial functions $U, V \in C^1(\mathbb{R})$ and $u \in C^1(\mathbb{R}^n)$, consider the half-linear equations

$$
\begin{aligned}
&(i) \quad \nabla\cdot\Big\{a(x)\Phi(\nabla u)\Big\} + c(x)\phi(u) = 0, \quad x \in \mathbb{R}^n; \\
&(ii) \quad \Big\{a^+(r)\phi(U')\Big\}' + c^-(r)\phi(U) = 0, \quad r > 0 \qquad (3.6) \\
&(iii) \quad \Big\{a^-(r)\phi(V')\Big\}' + c^+(r)\phi(V) = 0, \quad r > 0
\end{aligned}
$$

where the coefficients in (3.6)(ii) are related to those in (3.6)(i) as in (3.3).

The next theorem follows from Theorem 1.3.

Theorem 3.2. *Assume that there are some $\mu, T > 0$ such that*

a) $a \in C^1(\Omega_T, (0,\infty))$ *and* $c \in C(\Omega_T, (0,\infty))$

b) and either

$$
\begin{aligned}
&(i) \quad \gamma(r) := \frac{\min_{|x|=r} c(x)}{\max_{|x|=r} a(x)} > \mu \text{ or } \int_{\Omega_T}^{r}\tilde{\gamma}(s)ds \quad \text{diverges to } \infty \\
&(ii) \quad \text{or} \quad \exists\theta > 0;\ 0 < \frac{1}{\gamma(r)} < \theta \quad \text{and} \quad \Big|\frac{[\max_{|x|=r} a(x)]'}{\min_{|x|=r} c(x)}\Big| \qquad (3.7) \\
&\text{is bounded in } \Omega_T \\
&(iii) \quad \text{or } a \in C^1(\Omega_T, (0,\infty)) \text{ is increasing and } c \in C(\Omega_T, (0,\infty)) \\
&\text{is strictly positive.}
\end{aligned}
$$

Then any bounded and non-trivial solution of

$$\nabla\cdot\Big\{a(x)\Phi(\nabla u)\Big\}+c(x)\phi(u)=0,\quad x\in\mathbb{R}^n$$

is strongly oscillatory.

Proof. Under the hypotheses, any bounded and non-trivial solution U of (3.6)(ii) is strongly oscillatory by Theorem 1.3 if (i) holds (see (1.8)ii). For the condition (ii), we refer to (1.10)(b) for U to be strongly oscillatory. In the case of the hypothesis (iii), Theorem 3.1 of [9] shows that U is strongly oscillatory.

If (3.6)(i) has a non-trivial and bounded solutioun u, say, a Picone formula for those functions reads

$$\begin{aligned}&\nabla\cdot\Big\{Ua\Phi(\nabla U)-U\phi(\frac{U}{u})a^+\Phi(\nabla u)\Big\}=(a^+-a)|\nabla U|^{\alpha+1}\\&+a^+(r)Z_\alpha(U,u)+(c-c^-)|U|^{\alpha+1}\quad\text{wherever } u\neq 0\ .\end{aligned}\tag{3.8}$$

If we assume that u is strictly positive in some Ω_R then the integration of (3.8) over any nodal set $D(U)\subset\Omega_R$ leads to a contradiction as the right hand side of the integral will be strictly positive and the left zero. Thus u has a zero in any such a nodal set and such a solution cannot be compactly supported. □

From the Lemma 3.1, we have the following result:

Theorem 3.3. *Under the hypotheses of Theorem 3.2, if* $f\in C(\mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n,\ \mathbb{R})$ *satisfies*

(i) $\tilde f(r,v,v')$ *satisfies (1.3) and either*

(ii) $\forall M>0,\ \exists R,M_1>0;\quad |V|+|V'|<M\Longrightarrow\dfrac{|f^-(r,V,V')|+|f^+(r,V,V')|}{\phi(|V|)}<M_1$ *in* Ω_R

(iii) *or* $\dfrac{f^-(r,V,V')}{\phi(V)}$ *keeps the same sign in some* Ω_T.

then any bounded and non-trivial solution of

$$\nabla\cdot\Big\{a(x)\Phi(\nabla v)\Big\}+c(x)\phi(v)+f(x,v,\nabla v)=0,\quad x\in\mathbb{R}^n\tag{3.9}$$

is strongly oscillatory.

3.1 Concluding remarks

(a) The oscillation criteria displayed in Theorem 3.2 are selected for their novelty and most of known criteria can be used instead.

(b) From [6] the condition (1.3) allows us to confidently know wether or not an oscillation criterion leads to strongly or weakly oscillatory solution. For example if $\beta\in(0,\ \alpha)$ in Theorem 1.3, the corresponding solution would be weakly oscillatory and here it will have a compact support.

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