On Sums of Zeros of Infinity Order Entire Functions

Michael Gil'[*]

Department of Mathematics

XY Ben Gurion University of the Negev

P.0. Box 653, Beer-Sheva 84105, Israel

(Communicated by Palle Jorgensen)

Abstract

We consider an infinite order entire functions $f(z)$, whose zeros $z_1(f), z_2(f), ...$ are enumerated in the increasing order. For a nondecreasing sequence $\{p_k\}$ of positive numbers, a bound for the sums

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|^{p_k}} \quad (j = 1, 2, ...)$$

is suggested. That bound gives us conditions providing the convergence of the corresponding series.

AMS Subject Classification: 30D20

Keywords: entire function, infinite order, sum of zeros

1   Introduction and statement of the main result

Consider the entire function

$$f(\lambda) = \sum_{k=0}^{\infty} b_k \lambda^k \quad (b_0 = 1; \; \lambda \in \mathbb{C})$$

with complex coefficients. Let $z_1(f), z_2(f), ...$ be the zeros of f counted with their multiplicities and enumerated in the increasing order: $|z_{k+1}(f)| \geq |z_k(f)|$. If f has a finite number l of zeros, we put $1/z_k(f) = 0$ $(k = l+1, l+2, ...)$.

In the paper [9] (see also [10, Chapter 5]), bounds for the sums

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|^p} \quad (j = 1, 2, ...; \; p = const \geq 1)$$

---

[*]E-mail address: gilmi@bezeqint.net

are established, provided f has finite order.

Sums containing the zeros of entire functions arise in various applications. In particular, N. Anghel in very the interesting paper [2] considered the following problem: when is an entire function of finite order solution to a complex 2nd order homogeneous linear differential equation with polynomial coefficients? In that paper Anghel gives two (equivalent) answers to this question. The starting point of both answers is the Hadamard product representation of a given entire function of finite order. While the first answer involves certain Stieltjes-like relations associated to the function, the second one requires the vanishing of all but finitely many suitable expressions constructed via the sums of the zeros of the function established in [10, Chapter 5]. Applications of these results are also given, most notably to the spectral theory of one-dimensional Schrödinger operators with polynomial potentials. Moreover, the classical Stieltjes - Calogero relations involving the zeros of the Hermite polynomials found over the years counterpart associated with virtually all the important polynomials appearing in mathematical physics. The standard method of deriving them typically rests on two different ways of looking at the Laurent expansion of the logarithmic derivative of a polynomial about a singular point, that is, a zero of the polynomial. While the first way is a straightforward formalism, the second proves to be cumbersome and is usually handled on a case by case basis, often based on Painleve transcendent techniques, if the logarithmic derivative of the polynomial in question is solution to a suitable nonlinear differential equation. In the remarkable paper [1], the just pointed problem is considered completely and in full generality via an analysis based on the relations involving the sums of zeros of an entire function of finite order established in [10]. Another application of the mentioned sums is connected with the bounds for the sums of the zeros of solutions of differential equations. The first results in this direction have been established in the paper [11] which deals with second order equations with polynomial coefficients. The results from [11] were generalized in the interesting paper [5].

The aim of this paper is to generalize the main result from [9] to infinite order entire functions.

Introduce the notations. Let $\psi_1 = 1$ and ψ_k ($k = 2, 3, ...$) be positive numbers, such that the sequence

$$m_1 = 1, \; m_j := \frac{\psi_j}{\psi_{j-1}} \; (j = 2, 3, ...)$$

is nonincreasing and tends to zero. So

$$\psi_j := \prod_{k=1}^{j} m_k \; (j = 1, 2, ...).$$

Put $a_k = b_k / \psi_k$. Then the considered entire function takes the form

$$f(\lambda) = 1 + \sum_{k=1}^{\infty} a_k \psi_k \lambda^k \; (\lambda \in \mathbb{C}). \tag{1.1}$$

It is supposed that

$$\theta(f) := [\sum_{k=1}^{\infty} |a_k|^2]^{1/2} < \infty. \tag{1.2}$$

We will call (1.1) the ψ-representation of f. Since $a_k \to 0$ and $\psi_{k+1}/\psi_k = m_{k+1} \to 0$, f is really an entire function. In addition, since $m_k \to 0$ monotonically, for all sufficiently large v we have

$$\theta(f) + \sum_{k=1}^{v} m_{k+1} \le v. \tag{1.3}$$

Below we take an arbitrary v satisfying (1.3). For a fixed v, let $\{p_k\}_{k=v}^{\infty}$ be a nondecreasing sequence of numbers $p_k > 1$ $(k \ge v)$. In addition, we take

$$p_1 = p_2 = ... = p_{v-1} = p \tag{1.4}$$

for an arbitrary $1 < p \le p_v$. If $v = 1$, then condition (1.4) is not required. Denote

$$\pi_v = \{\underbrace{p, \cdots, p}_{v-1}, p_v, p_{v+1}, ...\} = \{p_k\}_{k=1}^{\infty}.$$

In the sequel we put

$$\sum_{j=2}^{1} = 0.$$

Now we are in a position to formulate the main result of this paper.

Theorem 1.1. *Let function f be represented by (1.1), and conditions (1.2) and (1.3) be fulfilled for some integer $v \ge 1$. Then*

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|^{p_k}} \le \frac{1}{1-c} \left[\frac{(\theta(f) + m_2)^{p_1}}{c^{p_1} p_1} + \sum_{k=2}^{j} \frac{m_{k+1}^{p_k}}{c^{p_k} p_k} \right] \quad (j = 1, 2, ...) \tag{1.5}$$

for any $c \in (0, 1)$ and $p_k \in \pi_v$ $(k = 1, 2, ...)$.

This theorem is proved in the next section.

Corollary 1.2. *Under the hypotheses of Theorem 1.1, let*

$$J(c) := \sum_{k=2}^{\infty} \frac{m_{k+1}^{p_k}}{c^{p_k} p_k} < \infty \tag{1.6}$$

for a $c \in (0, 1)$. Then

$$\sum_{k=1}^{\infty} \frac{1}{|z_k(f)|^{p_k}} \le \frac{1}{1-c} \left[\frac{(\theta(f) + m_2)^{p_1}}{c^{p_1} p_1} + J(c) \right] < \infty. \tag{1.7}$$

Theorem 1.1 and its corollary supplement the very interesting recent investigations of infinite order entire functions, cf. papers [3, 4, 6, 7, 13, 14] and references therein. Note also that in [8], inequalities of the type

$$\sum_{k=1}^{j} \frac{t_k}{|z_k(f)|} < \theta(f) + \sum_{k=1}^{j} t_k m_{k+1} \quad (j = 1, 2, ...).$$

were established for a sequence $\{t_k\}$ of nonincreasing numbers. But they are not convenient for the investigation of the convergence of the series of the zeros.

To illustrate Theorem 1.1, consider the function

$$f(\lambda) = 1 + \sum_{k=1}^{\infty} \frac{\lambda^k}{4^k \ln^{k-1}(k+1)}. \tag{1.8}$$

We take $a_1 = 1/4, \psi_1 = 1$,

$$a_k = 1/2^k, \psi_k = \frac{1}{2^k \ln^{k-1}(k+1)}, k \geq 2.$$

So $m_2 = \psi_2 = \frac{1}{4\ln 3}$, and

$$m_k = \frac{\ln^{k-2} k}{2 \ln^{k-1}(k+1)} \leq \frac{1}{2\ln(1+k)} \quad (k \geq 3),$$

and

$$\theta^2(f) = \frac{1}{4^2} + \sum_{k=2}^{\infty} \frac{1}{4^k} = \frac{7}{48}.$$

Clearly,

$$\tau_1 := m_2 + \theta(f) = \frac{1}{4\ln 3} + \frac{\sqrt{7}}{4\sqrt{3}} < 1.$$

So one can take $\nu = 1$. Put

$$p_1 = p_2 = \ln 3 \text{ and } p_k = \ln k \ (k \geq 3), \text{ and } c = 1/2.$$

By Theorem 1.1, for the function defined by (1.8), we have

$$\sum_{k=1}^{j} \frac{1}{|z_k(f)|^{p_k}} < \frac{2^{p_1+1}\tau_1^{p_1}}{p_1} + 2\sum_{k=2}^{j} \frac{1}{p_k \ln^{p_k}(2+k)}.$$

But $\ln^{\ln k}(2+k) \geq k^{\ln \ln k} > k^3$ for large enough k. So condition (1.6) holds, and thus

$$\sum_{k=1}^{\infty} \frac{1}{|z_k(f)|^{p_k}} \leq \frac{2^{p_1+1}\tau_1^{p_1}}{p_1} + 2\sum_{k=2}^{\infty} \frac{1}{p_k \ln^{p_k}(2+k)} < \infty.$$

2 Proof of Theorem 1.1

For an $n \times n$-matrix A, $\lambda_k(A)$ denote the eigenvalues and $s_k(A) = \sqrt{\lambda_k(A^*A)}$ $(k = 1, 2, ..., n)$ are the singular values taken with their multiplicities and ordered in the decreasing way: $|\lambda_k(A)| \geq |\lambda_{k+1}(A)|$, $s_k(A) \geq s_{k+1}(A)$.

Lemma 2.1. *Let A be an $n \times n$ matrix, such that*

$$\sum_{j=1}^{v} s_j(A) \le v \qquad (2.1)$$

for an integer $1 \le v < n$. Then for any constant $c \in (0,1)$ and $p_k \in \pi_v$ ($k = 1, ..., n$), we have

$$\sum_{k=1}^{j} |\lambda_k(A)|^{p_k} \le \frac{1}{1-c} \sum_{k=1}^{j} \frac{s_k^{p_k}(A/c)}{p_k} \quad (j = 1, ..., n).$$

Proof. Put $\lambda_j(A) = \lambda_j, s_j(A) = s_j$. According to (2.1), $s_v \le 1$ and

$$\sum_{k=1}^{v} |\lambda_k| \le \sum_{k=1}^{v} s_k \le v.$$

Therefore $|\lambda_v| \le 1$. Thanks to the Weyl inequalities [12, Lemma II.3.4] we have

$$\sum_{k=1}^{j} t_k |\lambda_k| \le \sum_{k=1}^{j} t_k s_k$$

for any nonincreasing sequence t_k. Take $t_k = |\lambda_k|^{p_k - 1}$. Then by (1.4), $\{t_k\}$ really nonincreases and since $p_k > 1$, by the Young inequality

$$ab \le \frac{a^p}{p} + \frac{b^q}{q} \quad (a, b > 0; \ \frac{1}{q} + \frac{1}{p} = 1),$$

we obtain

$$\sum_{k=1}^{j} |\lambda_k|^{p_k - 1} s_k \le \sum_{k=1}^{j} \left(\frac{c^{q_k} |\lambda_k|^{q_k(p_k-1)}}{q_k} + \frac{(s_k/c)^{p_k}}{p_k} \right)$$

with $1/q_k + 1/p_k = 1$. So $q_k(p_k - 1) = p_k$, and $c^{q_k} \le c$, and

$$\sum_{k=1}^{j} |\lambda_k|^{p_k - 1} s_k \le \sum_{k=1}^{j} \left(\frac{c|\lambda_k|^{p_k}}{q_k} + \frac{(s_k/c)^{p_k}}{p_k} \right) \le \sum_{k=1}^{j} \left(c|\lambda_k|^{p_k} + \frac{(s_k/c)^{p_k}}{p_k} \right).$$

Hence,

$$(1-c) \sum_{k=1}^{j} |\lambda_k|^{p_k} \le \sum_{k=1}^{j} \frac{(s_k/c)^{p_k}}{p_k}.$$

The lemma is proved. □

Furthermore, for a v satisfying (1.3) and an $n > v$ consider the polynomial

$$P(\lambda) = \sum_{k=0}^{n} a_k \psi_k \lambda^{n-k} \quad (a_0 = \psi_0 = 1).$$

Denote

$$\theta(P) := \left[\sum_{k=1}^{n} |a_k|^2 \right]^{1/2}.$$

Lemma 2.2. *Let*

$$\theta(P) + \sum_{k=1}^{v} m_{k+1} \le v \tag{2.2}$$

for an integer $1 \le v < n$. Then the zeros of polynomial P satisfy the inequalities

$$\sum_{k=1}^{j} |z_k(P)|^{p_k} \le \frac{1}{1-c}\left[\frac{(\theta(P)+m_2)^{p_1}}{c^{p_1}p_1} + \sum_{k=2}^{j}\frac{m_{k+1}^{p_k}}{c^{p_k}p_k}\right] \quad (j = 1,...,n)$$

for any constant $c \in (0,1)$ and $p_k \in \pi_v$ $(k = 1,...,n)$.

Proof. Introduce the $n \times n$-matrix

$$A_P = \begin{pmatrix} -a_1 & -a_2 & ... & -a_{n-1} & -a_n \\ m_2 & 0 & ... & 0 & 0 \\ 0 & m_3 & ... & 0 & 0 \\ . & . & ... & . & . \\ 0 & 0 & ... & m_n & 0 \end{pmatrix}.$$

As it is proved in [10, Section 5.2],

$$\lambda_k(A_P) = z_k(P) \quad (k = 1,...,n). \tag{2.3}$$

We have $A_P = M + C$, where

$$C = \begin{pmatrix} -a_1 & -a_2 & -a_3 & ... & -a_n \\ 0 & 0 & 0 & ... & 0 \\ . & . & . & ... & . \\ 0 & 0 & 0 & ... & 0 \end{pmatrix} \text{ and } M = \begin{pmatrix} 0 & 0 & 0 & ... & 0 & 0 \\ m_2 & 0 & 0 & ... & 0 & 0 \\ 0 & m_3 & 0 & ... & 0 & 0 \\ . & . & . & ... & . & . \\ 0 & 0 & 0 & ... & m_n & 0 \end{pmatrix}.$$

Matrices MM^* and CC^* are diagonal. Moreover, $s_1(C^*) = \theta(P)$ and $s_k(C^*) = 0$, $k > 1$. In addition, $s_k(M^*) = m_{k+1}$ $(k < n)$; $s_n(M^*) = 0$. By Lemma II.4.2 [12],

$$\sum_{k=1}^{j} s_k(A_P^*) \le \sum_{k=1}^{j} s_k(M^*) + s_k(C^*) = \theta(P) + m_2 + \sum_{k=2}^{j} m_{k+1}.$$

Repeating the arguments of the previous lemma, we get

$$(1-c)\sum_{k=1}^{j} |\lambda_k(A_P)|^{p_k} \le \frac{(\theta(P)+m_2)^{p_1}}{c^{p_1}p_1} + \sum_{k=2}^{j}\frac{m_{k+1}^{p_k}}{c^{p_k}p_k}.$$

This and (2.3) prove the lemma. $\qquad\square$

Proof of Theorem 1.1: Consider the polynomial

$$f_n(\lambda) = 1 + \sum_{k=1}^{n} a_k\psi_k\lambda^k.$$

Clearly, $\lambda^n f_n(1/\lambda) = P(\lambda)$. So $z_k(P) = 1/z_k(f_n)$. Take into account that the zeros continuously depend on coefficients due to the Hurwitz theorem, cf. [10, p. 60].

We thus have the required result, letting in Lemma 2.2 $n \to \infty$. $\qquad\square$

References

[1] N. Anghel, Stieltjes - Calogero - Gil' relations associated to entire functions of finite order, *Journal of Mathematical Physics*, **51**, No. 5 (2010), pp 251-262.

[2] N. Anghel, Entire functions of finite order as solutions to certain complex differential equations *Proceedings of the American Mathematical Society* **140**, No 7, (2012), pp 2319 -2332.

[3] W. Bergweiler, Canonical products of infinite order. *J. Reine Angew. Math.* **430**, (1992), pp 85-107.

[4] W. Bergweiler, W., A. Eremenko and J. K. Langley, Real entire functions of infinite order and a conjecture of Wiman. *Geom. Funct. Anal.*, **13**, No. 5, (2003), pp 975-991.

[5] Cao, Ting-Bin, Liu, Kai and Xu, Hong-Yan, Bounds for the sums of zeros of solutions of $du/dz = P(z)u$, where P is a polynomial. *Electron. J. Qual. Theory Differ. Equ.* **60**, (2011), pp 1-10.

[6] W. Foster and I. Krasikov, Inequalities for real-root polynomials and entire functions. *Adv. Appl. Math.* **29**, No.1, (2002), pp 102-114.

[7] R. Gardner and N.K. Govil, Some inequalities for entire functions of exponential type. *Proc. Am. Math. Soc.* **123**, No.9, (1995), pp 2757-2761.

[8] M.I. Gil'. On sums of roots of infinite order entire functions. *Complex Variables, Theory Appl.* **50**, No. 1, (2005), pp 27-33.

[9] M.I. Gil'. On inequalities for zeros of entire functions, *Proceedings of the American Mathematical Society* **133**, No 1, (2005), pp 97-101.

[10] M.I. Gil'. *Localization and Perturbation of Zeros of Entire Functions*, CRC Press, Taylor and Francis Group, New York, 2010.

[11] M.I. Gil', Bounds for zeros of solutions of second order differential equations with polynomial coefficients, *Results Math.*, **59** (2011), pp 115-124.

[12] I.C. Gohberg and M.G. Krein, *Introduction to the Theory of Linear Nonselfadjoint Operators*, Trans. Mathem. Monographs, v. 18, Amer. Math. Soc., Providence, R. I., 1969.

[13] J.K. Langley. Non-real zeros of higher derivatives of real entire functions of infinite order, *J. Anal. Math.* **97**, (2006), pp 357-396.

[14] Yang Weifeng and Deng Jin, Newton's method of entire functions with infinite order. *Far East J. Math. Sci.* **34**, No. 3, (2009), pp 317-327.