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Parabolic Square Functions and Caloric Measure

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Abstract

We prove estimates for certain square functions of solutions to divergence form linear parabolic equations. The estimates are related to singularity and mutual absolute continuity of caloric measure with respect to surface measure of non-cylindrical domains. The square functions and the results included here are adapted from works on elliptic equations.

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1 Caloric measure, square functions and statement of the main theorems

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open bounded set, $n \geq 3$. Consider operators of the form

$$Lu = \operatorname{div}(A(X,t)\nabla u(X,t)) - \frac{\partial u}{\partial t}(X,t) \tag{1.1}$$

where $(X,t) \in \mathbb{R}^n \times \mathbb{R}$, and $A(X,t) = (a_{i,j}(X,t))$ is a symmetric matrix of real-valued functions that satisfies a standard ellipticity condition of the form

$$\lambda_1 |\xi|^2 < \sum_{i,j} a_{ij}(X,t)\xi_i \xi_j < \lambda_2 |\xi|^2 \tag{1.2}$$

for certain $0 < \lambda_1 < \lambda_2$ and every $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$. Also, here and throughout this paper ∇u denotes the gradient of u with respect to *space variables X* only.

Solutions to $Lu = 0$ are understood in the weak sense, but we assume that the coefficients are of class C^∞. Still, we obtain estimates depending at most on the *ellipticity constants*

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λ_1, λ_2, dimension n, and the size or other geometric constants of Ω, and hence standard limit arguments may be applied to conclude estimates that hold when the coefficients are measurable and bounded.

The *parabolic distance* between (X,t), $(Y,s) \in \mathbb{R}^{n+1}$ is defined by

$$\delta(X,t;Y,s) \equiv \operatorname{dist}(X,t;Y,s) = |X-Y| + |t-s|^{1/2},$$

and by extension we define the parabolic distance between subsets of $\mathbb{R}^{n+1}$ as

$$\delta(A;B) = \inf\{\delta(X,t;Y,s) : (X,t) \in A,\ (Y,s) \in B\}.$$

We will use as framework a type of time-varying domain Ω that we call *starlike parabolic Lipschitz cylinder*. This is properly defined later in the bulk of the paper, and at this point we introduce this terminology to continue with a basic description of the contents of this paper.

A *bounded starlike parabolic Lipschitz cylinder* Ω_T will have the property that for each $0 < t < T$ the level domains $\Omega(t) = \Omega \cap \{(Y,s) : s = t\} \subset \mathbb{R}^n$ are *starlike Lipschitz domains*. Moreover, the boundary of Ω_T can be roughly described as a *bottom part* ($\Omega(t)$ with $t = 0$), a lateral part S_T (locally given by *patches of* sufficiently regular graphs), and a top part ($\Omega(t)$ for $t = T$, $T > 0$).

It is then clear that Ω_T is regular in the Perron-Wiener-Brelot sense, and so it makes sense to define the *parabolic measure associated to* L on S_T, denoted by $\omega_L^{(X,t)}$ for $(X,t) \in \Omega$, as the unique Borel measure supported on S_T such that

$$u_f(X,t) = \int_{\partial_p\Omega} f(Y,s)\,d\omega_L^{(X,t)}(Y,s) \tag{1.3}$$

is the Perron-Wiener-Brelot solution of the Dirichlet problem $Lu = 0$ on Ω_T, $u|_{S_T} = f$ for f continuous and supported on S_T. Observe that in particular $u(X,0) = 0$ for every X in the bottom of Ω_T.

This definition makes sense for the particular case of the heat operator $\Delta - \partial/\partial t$, obtained by taking the identity matrix in (1.1). In this case it is customary to refer to the corresponding parabolic measure as the *caloric measure*.

A basic property of parabolic measure is that for any Borel set $E \subset S_T$, the measure $\omega^{(X,t)}(E)$, as a function of (X,t) can be viewed as a solution to $Lu = 0$ with boundary data χ_E. Hence a Harnack principle can be applied, and assuming that there is a point $X_0 \in \mathbb{R}^n$ such that $(X_0,t) \in \Omega_T$ for every $0 < t < T$, $\omega^{(X,t)}$ is absolutely continuous with respect to $\omega \equiv \omega^{(X_0,3T/2)}$.

In other words, the Radon-Nikodým derivative $K(X,t;Y,s) = (d\omega^{(X,t)}/d\omega)(Y,s)$, for $(X,t) \in \Omega_T$, also called the *kernel function*, is a well defined element in $L^1(S_T, d\omega)$ as a function of the variable $(Y,s) \in S_T$, and we can rewrite (1.3) as

$$u_f(X,t) = \int_{\partial_p\Omega} K(X,t;Y,s) f(Y,s)\,d\omega(Y,s). \tag{1.4}$$

Denote by σ the *surface measure*, defined for any Borel set $F \subset \mathbb{R}^{n+1}$ by

$$\sigma(F) = \int_F d\sigma_t\,dt,$$

where σ_t is the $(n-1)$-dimensional Hausdorff measure of $F_t \equiv F \cap \mathbb{R}^n \times \{t\}$, and dt denotes integration with respect to 1-dimensional Hausdorff measure.

The question of determining whether the parabolic measure is absolute continuity with respect to σ on S_T, even for the case of the heat operator is far from trivial (see e.g. [8, 7] and references therein). And it turns out that a special mutual absolute continuity arises when studying initial L^p Dirichlet problems for the equation $Lu = 0$, and it takes the form of an A_∞ weight property (see [10]), as we now recall briefly.

Let $E \subset \mathbb{R}^{n+1}$ be a Borel set such that σ restricted to E is locally finite. For instance, E could be the graph of a continuous function of $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, or $E = S_T$. We say that the parabolic measure ω is in the class $A_\infty(E,\sigma)$ if for every $\epsilon > 0$ there exists $\delta > 0$ such that

$$\frac{\sigma(F)}{\sigma(\Delta)} < \delta \quad \text{implies} \quad \frac{\omega_L(F)}{\omega_L(\Delta)} < \epsilon$$

for every Borel set $F \subset \Delta$, and every surface cube (defined in the bulk of the paper) $\Delta \subset E$, with $\overline{\Delta} \subset E$ (to shorten notation we write $\Delta \Subset E$ in this case). By the well-known general theory of Muckenhoupt weights, it turns out that this defines a *uniform mutual absolute continuity* between ω_H and σ on E.

Now we state the results, grouping them according to the square function they use.

The result using the Area Integral operator

A first theorem we prove in this paper, is an appropriate adaptation of a recent result that we extract from [3]. This is a criterion to verify that the A_∞ property between caloric and surface measure holds. Actually, it is related to solvability of an initial Dirichlet problem, where the datum belongs to $BMO(S_T, d\sigma)$.

In those results for elliptic equations a Carleson measure condition related to the area integral is used. In our case we start defining for $(Q,s) \in S$ and a suitable function $F : \Omega \to \mathbb{R}$ the *parabolic area integral*

$$\mathcal{S}F(Q,s) = \left(\int_{\Gamma(Q,s)} |\nabla F(X,t)|^2 \delta(X,t)^{-n} dXdt \right)^{1/2}.$$

Here, for $(Q,s) \in \partial\Omega$ and $\alpha > 0$ the *non-tangential approach region* as a cone-like set with axis contained in $\Omega(s)$, pointing towards the interior of Ω, and denoted by $\Gamma_\alpha(Q,s)$. The precise definition is given in the bulk of the paper. Also we use the notation $\delta(X,t) = \mathrm{dist}(X,t;\partial\Omega)$, for $(X,t) \in \Omega$.

We continue introducing more notation required in this and subsequent sections. A point $(Q,s) \in S_T$ may be viewed as $(Q,s) = (q_0,q,s)$ and q_0 is recalled as the graph coordinate depending $(q,s) \in \mathbb{R}^{n-1} \times \mathbb{R}$. Given a surface cube $\Delta \equiv \Delta_r(Q,s)$, whose closure is contained in S_T, we define the *Carleson regions*

$$\Psi(\Delta) = \Psi_r(Q,s) \equiv \{(x_0,x,t) \in \Omega : |x-q| + |t-s|^{1/2} < r, |x_0 - q_0| < N_0)\}.$$

Here $N_0 > 0$ is a fixed constant that may depend on the constants of Ω and dimension $n \geq 3$.

For a locally integrable function $f : S \to \mathbb{R}$, and $\Delta \Subset S_T$, set $f_\Delta = \int_\Delta f\, d\sigma / \sigma(\Delta)$ and define

$$\|f\|_* \equiv \sup_{\Delta \subseteq S_T} \frac{1}{\sigma(\Delta)} \int_\Delta |f - f_\Delta|^2 d\sigma.$$

Then we denote by $BMO(S, d\sigma)$ the class of locally integrable functions (modulo constants) for which $\|f\|_* < \infty$.

For any Borel measure μ definde on S the weighted BMO space on S, denoted by $BMO(S, d\mu)$, is defined through the property

$$\|f\|_{*,\mu} \equiv \sup_{\Delta \subseteq S_T} \frac{1}{\mu(\Delta)} \int_\Delta |f - f_\Delta|^2 d\mu < \infty.$$

Theorem 1.1. *Let L be an operator as in* (1.1) *as described above. Suppose that there exists a constant $B > 0$, depending at most on the size and geometric constants of Ω, the ellipticity constants of L, and dimension, such that for every continuous function $f : S_T \to \mathbb{R}$ one has*

$$\sup_{\Delta \subseteq S_T} \left[\frac{1}{\sigma(\Delta)} \int_{\Psi(\Delta)} |\nabla u(X,t)|^2 \delta(X,t)\, dXdt + \sup_{\Psi(\Delta)} |u|^2 \right] \leq B \|f\|_* \tag{1.5}$$

where u is the solution to $Lu = 0$ with $u = f$ on S_T. Then $\omega \in A_\infty(d\sigma)$.

We note that it is well known that the integral in the expression in the left-hand side of (1.5) is related to the integral $\int_\Delta [\mathcal{S}u]^2 d\sigma$ by Fubini's Theorem.

When trying to adapt the argument for elliptic equations some issues particularly related to parabolic equations as (1.1) were evident, and so we included details of this adaptation in Section 3.

The result using the Multiplicative Square Function

Based on techniques and results for harmonic functions in [6], where the multiplicative square function is originally introduced, we prove parabolic versions of two of their results.

The *non-tangential maximal function* of $F : \Omega \to \mathbb{R}$ defined for $(Q, s) \in \partial\Omega$ is defined as

$$N_\alpha F(Q, s) = \sup\{|F(X,t)| : (X,t) \in \Gamma_\alpha(Q, s)\}. \tag{1.6}$$

Given u a positive bounded solution to $Lu = 0$ on a starlike parabolic Lipschitz cylinder Ω, define for $(Q, s) \in \partial\Omega$ the *parabolic multiplicative square function of u* as

$$\mathcal{M}u(Q, s) = \left(\int_{\Gamma(Q,s)} \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta(X,t)^{-n} dXdt \right)^{1/2}, \tag{1.7}$$

where as before, for $(X,t) \in \Omega$ we set $\delta(X,t) = \operatorname{dist}(X,t; \partial\Omega)$. Observe that for positive bounded solutions to $Lu = 0$ one has $\mathcal{M}u = \mathcal{S} \log u$.

This definition is a straightforward adaptation for parabolic setting of a definition from [6] where it is introduced for harmonic functions.

Theorem 1.2. *Let ν be a positive Borel measure defined on S_T, and define for $(X,t) \in \Omega_T$ the caloric extension of ν as*

$$u(X,t) = \int_{S_T} K(X,t; Y,s) d\nu(Y,s),$$

where $K(X,t;Y,s)$ is the kernel function in (1.4)*. Then ν is singular with respect to ω if and only if $\mathcal{M}u(Q,s) = \infty$ for ω almost every $(Q,s) \in S_T$.*

Theorem 1.3. *With the definitions introduced in Theorem 1.2 above, and assuming that $\mathcal{M}u \in L^{p_0}(S\,; d\omega)$ for some $p_0 > 1$, there exists a contant $C = C(n,\lambda) > 0$ such that* $\exp(C\mathcal{M}u) \in L^1(S_T, d\omega)$ *implies that ω and ν are mutually absolutely continuous.*

The arguments proving these theorems are in Section 4. An interesting feature of the proof of Theorem 1.3 is the use of distributional (good-λ) inequalities relating $\mathcal{M}u$ and the non-tangential maximal function $N \log u$ for solutions u to $Lu = 0$. This feature is originally present in the case of harmonic functions in [6], and it takes a good part of this paper. The proof of of the aforementioned distributional inequality is in Section 5.

2 Basic background definitions

In this and subsequent sections we retain notations and definitions already introduced in the previous section. We now continue introducing notions related to the so called *parabolic homogeneity* of the euclidean space $\mathbb{R}^{n+1}$. If the time variable t is irrelevant in the argumentation, we use the notation $\mathbf{X}$, $\mathbf{Y}$, etc. to denote points in $\mathbb{R}^{n+1}$. Otherwise we adopt the notation $(X,t) \in \mathbb{R}^n \times \mathbb{R}$.

The *parabolic distance* between (X,t), $(Y,s) \in \mathbb{R}^{n+1}$ has been already defined by

$$\delta(X,t;Y,s) = |X-Y| + |t-s|^{1/2} \equiv \|X-Y, t-s\|.$$

This last expression defines what we call the *parabolic norm* of points in $\mathbb{R}^{n+1}$, and it may also be applied to points $(x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}$.

Given $(X,t) \in \mathbb{R}^n \times \mathbb{R} \equiv \mathbb{R}^{n+1}$, denote by $C_r(X,t)$ the *cylinder* $\{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r, |t-s| < r^2\}$. The *parabolic cylinder* of radius $r > 0$ and centered at (X,t) is defined as $C_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : |X-Y| < r, 0 < t-s < r^2\}$. The *parabolic ball* of radius $r > 0$ centered at (X,t) is $Q_r(X,t) = \{(Y,s) \in \mathbb{R}^{n+1} : \delta(X,t;Y,s) < r\}$.

The *parabolic boundary* of an open connected set $\Omega \in \mathbb{R}^{n+1}$, denoted by $\partial_p\Omega$, consists of points $(Q,s) \in \partial\Omega$ such that for every $r > 0$ one has $C_r(Q,s) \setminus \Omega \neq \emptyset$.

We now describe the "good graphs" considered in previous works [8, 7], and adopt the convention that points in $\mathbb{R}^{n+1}$ may be denoted by $(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}$, to stress that in graph coordinates x_0 is the variable depending on (x,t).

A function $\psi : \mathbb{R}^n \longrightarrow \mathbb{R}$ is a *Lip(1,1/2) function with constant* $A_0 > 0$ if for $(x,t),(x,s) \in \mathbb{R}^n$, $|\psi(x,t) - \psi(y,s)| \le A_0\|x-y, t-s\|$. The function ψ is called a *parabolic Lipschitz function with constant* A_1 if it satisfies the following two conditions:

- ψ satisfies a *Lipschitz condition in the space variable*

$$|\psi(x,t) - \psi(y,t)| \le A_1|x-y| \quad \text{uniformly on } t \in \mathbb{R}, \tag{2.1}$$

- For every interval $I \subseteq \mathbb{R}$, every $x \in \mathbb{R}^n$,

$$\frac{1}{|I|}\int_I\int_I \frac{|\psi(x,t) - \psi(x,s)|^2}{|s-t|^2}\, dtds \le A_1 < \infty. \tag{2.2}$$

This last condition can be recalled as a *BMO-Sobolev scale* in the t-variable. It roughly states that a half order derivative of $\psi(x,t)$ with respect to t variable is in BMO. See [7] for details.

A *basic parabolic Lipschitz domain* is a domain of the form

$$\Omega(\psi) = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : x_0 > \psi(x,t)\}$$

for some parabolic Lipschitz function ψ.

Let $\Omega \subset \mathbb{R}^{n+1}$ be a region such that $\partial_p \Omega = \partial\Omega$ and let $A_1, r_0 > 0$. Define

$$\mathcal{Z} = \{(x_0, x, t) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R} : |x_i| < r_0,\ i = 1,2,\ldots,n-1,\ |x_0| < 2nA_1 r_0,\ t \in \mathbb{R}\}.$$

Here, $x \in \mathbb{R}^{n-1}$ is viewed as the $(n-1)$-tuple $x = (x_1, x_2, \ldots, x_{n-1})$. We denote by $2\mathcal{Z}$ the concentric double of $\mathcal{Z}$, and set $\operatorname{diam}\Omega = \sup_{s\in\mathbb{R}} \operatorname{diam}\Omega(s)$, where $\Omega(s) = \{(X,t) \in \Omega : t = s\}$.

We say that Ω is an *infinite starlike parabolic Lipschitz cylinder with constants* A_1, r_0, if there exist $\{\mathcal{Z}_i : i = 1,2,\ldots,N\}$ which are obtained from $\mathcal{Z}$ through rigid motions in the space variables, and parabolic Lipschitz functions $\{\psi_i : i = 1,2,\ldots,N\}$ with constant A_1, defined on the transformation of $\mathbb{R}^n$ through the same rigid motion defining $\mathcal{Z}_i$, and such that the following conditions hold:

- $2\mathcal{Z}_i \cap \partial\Omega = \{(x_0,x,t) : x_0 = \psi_i(x,t)\} \cap 2\mathcal{Z}_i$, $i = 1,2,\ldots,N$;
- $2\mathcal{Z}_i \cap \Omega = \{(x_0,x,t) : x_0 > \psi_i(x,t)\} \cap 2\mathcal{Z}_i$, $i = 1,2,\ldots,N$;
- $\partial\Omega$ is covered by $\bigcup_{i=1}^{N} \mathcal{Z}_i$;
- Any $\mathcal{Z}_i$ intersects only two of the other $\mathcal{Z}_j$;
- $\Omega(t)$ is a bounded starlike Lipschitz domain for every $t \in \mathbb{R}$;
- There exists $X_0 \in \mathbb{R}^n$ and $\rho_0 > 0$ such that $B_{\rho_0}(X_0) \times \mathbb{R} \subset \Omega$.

Each pair $(\mathcal{Z}_i, \psi_i)$ is referred to as a *local coordinate cylinder* of Ω.

Define the *surface cubes* $\Delta_\rho(Q,s) \equiv C_\rho(Q,s) \cap S$, for $\rho > 0$ and $(Q,s) \in S$. For shortness sake, given $(X,t) \in \Omega$ we write $\delta(X,t) = \delta(X,t;\partial\Omega)$. Now set

$$\widetilde{\Gamma}_\alpha(Q,s) = \{(X,t) \in \Omega : \delta(X,t;Q,s) \le (1+\alpha)\delta(X,t)\}$$

and denote by $\Gamma_\alpha(Q,s)$ the *non-tangential region* defined as the truncation of $\widetilde{\Gamma}_\alpha(Q,s)$ at height $\rho(Q) \equiv |Q - X_0| + \rho_0/2$. That is

$$\Gamma_\alpha(Q,s) = \widetilde{\Gamma}_\alpha(Q,s) \cap C_{\rho(Q)}(Q,s).$$

In the definitions of the square functions $\mathcal{M}$ and $\mathcal{S}$ we may include the aperture of the cone used in their definition as a subscript. Also there is some truncated versions of these functionals obtained by substituting $\Gamma_\alpha(Q,s)$ by $\Gamma^r_\alpha(Q,s) = \Gamma(Q,s) \cap C_r(Q,s)$, $r > 0$. We write a superscript to take this truncation into account. This way we may write $\mathcal{M}^r_\alpha u$ or $\mathcal{S}^r_\alpha u$ and both expressions have now a precise meaning.

If Ω is an infinite starlike parabolic cylinder, for $T > 0$ we define the *bounded parabolic cylinder of height* T as $\Omega_T = \{(X,t) \in \Omega : 0 < t < T\}$. We denote the *lateral boundary of* Ω_T by $S_T \equiv \partial_p \Omega_T \cap \partial\Omega$. Given a surface cube $\Delta \equiv \Delta_r(Q,s)$, with $(Q,s) = (q_0,q,s) \in S_T$ and $\Delta \Subset S_T$, the *Carleson region above* Δ is

$$\Psi(\Delta) = \Psi_r(Q,s) \equiv \{(x_0,x,t) \in \Omega : |x-q| + |t-s|^{1/2} < r,\ |x_0 - q_0| < N_0)\},$$

with $N_0 = 2nA_1$. And with this choice of N_0 we also set

$$\overline{\mathcal{A}}(\Delta) = \overline{\mathcal{A}}_r(Q,s) \equiv (q_0 + 2N_0 r, q, s + 2r^2), \quad \underline{\mathcal{A}}(\Delta) = \underline{\mathcal{A}}_r(Q,s) \equiv (q_0 + 2N_0 r, q, s - 2r^2).$$

Note that N_0 has been chosen so that both $\overline{\mathcal{A}}(\Delta)$ and $\underline{\mathcal{A}}(\Delta)$ are in Ω, and in fact $\partial\Psi(\Delta) \setminus \partial\Omega \subset \Omega$.

For Ω_T as described above the *parabolic measure associated to* L, denoted by $\omega_L^{(X,t)}(\cdot)$ for $(X,t) \in \Omega$, is the unique Borel measure supported on S_T such that

$$u_f(X,t) = \int_{\partial_p \Omega} f(Y,s)\, d\omega_L^{(X,t)}(Y,s)$$

is the solution, in the Perron-Wiener-Brelot sense, of the Dirichlet problem $Lu = 0$ on Ω_T, $u|_{S_T} = f$ for f continuous and supported on S_T. Observe that in particular $u(X,0) = 0$ for every $X \in \Omega(0)$. We denote by ω the parabolic measure $\omega_L(\cdot, \Xi_T)$, where $\Xi_T \equiv (X_0, 3T/2)$.

Throughout this work, at several stages, we make use of some basic properties of solutions and parabolic measure for the operator L. Some of these properties are in [10] and references therein.

Also, we adopt a standard notation that the constants in a sequence of inequalities may change from line to line, as long as they do not interfere with the main idea.

We also write $A \lesssim B$ if $A \le CB$ with a constant $C > 0$ that may depend on dimension n, the ellipticity constants λ_1, λ_2, the constants of the infinite starlike parabolic Lipschitz cylinder A_1, r_0, and $\operatorname{diam}\Omega$. Similarly $A \approx B$ means $A \lesssim B$ and $B \lesssim A$ hold simultaneously. We may retain the notation if the dependance on a constant is different, and it will be explicitly stated when needed.

3 Proof of Theorem 1.1

For completeness, we include a sketch of the adaptations of the argument in [3]. Fix two surface cubes $\Delta \equiv \Delta_r(Q_0, s_0)$ and $\Delta' \equiv \Delta_r(Q_0', s_0')$, $\Delta, \Delta' \Subset S_T$, with $\operatorname{dist}(\Delta, \Delta') \approx r$, $0 < r < r_0$. By assumption, if f is continuous on S_T, and $Lu = 0$ on Ω_T satisfies $u = f$ on S_T then

$$\int_{\Psi(\Delta')} |\nabla u(X,t)|^2 \delta(X,t)\, dXdt + \sigma(\Delta') \sup_{\Psi(\Delta')} |u|^2 \le B\|f\|_* \sigma(\Delta').$$

Suppose then that $f \ge 0$ is continuous and supported in Δ.

The key step is now to prove the following lemma. Once this is established, the A_∞ property of parabolic measure can be obtained by adapting the argument in [3, p. 85-86]. We recall this argument after proving the following

Lemma 3.1. *Retain notations and definitions from the previous paragraphs. There exists a constant $c > 0$ such that for every $(Q, s) \in \Delta'$*

$$\frac{1}{\omega(\Delta)} \int_{\Delta} f d\omega \leq c\left(\mathcal{S}^r u(Q,s) + \sup_{\Psi(\Delta')} |u|^2\right). \tag{3.1}$$

Proof. Let $(Q, s) \in \Delta'$ and set $\Gamma_j \equiv \Gamma_j(Q,s) \equiv \Gamma(Q,s) \cap C_{r/2^j}(Q,s) \setminus C_{r/2^{j+1}}(Q,s)$ Let $G(X,t)$ denote the Green's function of L over Ω_T with pole at Ξ_T. More generally, $G(X,t;Y,s)$ denotes Green's function for L on Ω_T.

Claim (Poincare-type inequality). Let Γ_j^* denote the concentric dilation of Γ_j by a factor $3/4$. Then

$$\frac{1}{(2^{-j}r)^{n+2}} \int_{\Gamma_j^*} |u(X,t)|^2 dXdt \lesssim \int_{\Gamma_j} |\nabla u(X,t)|^2 \delta(X,t)^{-n} dXdt + \frac{1}{2^{j\alpha}}\left[\sup_{\Psi(\Delta')} |u|^2\right]. \tag{3.2}$$

Proof of (3.2). The estimate one needs is

$$|u(X,t)|^2 \lesssim \int_{\Gamma_j} |\nabla u(Z,\tau)|^2 \delta(Z,\tau)^{-n} dZd\tau + \frac{1}{2^{j\alpha}} \sup_{\Psi(\Delta')} |u|^2 \quad \text{for } (X,t) \in \Gamma_j^*. \tag{3.3}$$

By Green's identity

$$u^2(X,t) \lesssim \int_{\partial_p \Gamma_j} u^2(Z,\tau) d\omega_j^{(X,t)}(Z,\tau) - \int_{\Gamma_j} g_j(X,t;Y,s) L(u^2)(Y,s) dYds$$

where g_j denotes Green's function for L on Γ_j, and ω_j is the parabolic measure on $\partial\Gamma_j$. Since $Lu^2 = 2uLu - 2\langle A\nabla u, \nabla u\rangle$ then by boundary Hölder continuity of u on $\partial_p \Omega_T \setminus \Delta$ (see e.g. [5] and references therein) and maximum princple

$$|u(X,t)|^2 \lesssim \int_{\Gamma_j} G(X,t;Y,s) |\nabla u(Y,s)|^2 dYds + \left(\frac{\delta(X,t)}{r}\right)^{\alpha} \sup_{\Psi(\Delta')} |u|^2.$$

Using the comparison between Green's function and parabolic measure (see e.g. [10, Lemmata 2.8 and 2.9]) we obtain the Claim, since $\delta(X,t) \approx 2^{-j} r$. □

Let $\widetilde{\Gamma}_j$ denote the image of Γ_j under the following transformation:

$$\mathfrak{A}(x_0, x, t) = \left(x_0 + 2N_0(6r), x, t + 2(6r)^2\right).$$

Now observe that the comparison principle [10, Lemma 3.4] implies that

$$\frac{u(\overline{\mathcal{A}}(\Delta))}{G(\overline{\mathcal{A}}(\Delta))} \lesssim \frac{u(X,t)}{G(X,t)} \quad \text{for every } (X,t) \in \widetilde{\Gamma}_j. \tag{3.4}$$

Note that by (3.2)

$$\sup_{\Psi(\Delta')} |u|^2 + \sum_j \int_{\Gamma_j} |\nabla u(X,t)|^2 \delta(X,t)^{-n} dXdt \gtrsim \sum_j \frac{1}{(2^{-j}r)^{n+2}} \int_{\Gamma_j^*} |u(X,t)|^2 dXdt.$$

Applying Carleson type estimate [10, Lemma 2.5] and (3.4) for $(X,t) \in \Gamma_j$

$$u(X,t) \gtrsim \frac{u(\overline{\mathcal{A}}(\Delta))}{G(\overline{\mathcal{A}}(\Delta))} G(X,t) \gtrsim \frac{u(\overline{\mathcal{A}}(\Delta))}{G(\overline{\mathcal{A}}(\Delta'))} G(X,t)$$

by backward Harnack princple. Hence

$$\sum_j \frac{1}{(2^{-j}r)^{n+2}} \int_{\Gamma_j^*} |u(X,t)|^2 dXdt \gtrsim \sum_j \frac{1}{(2^{-j}r)^{n+2}} \frac{u(\overline{\mathcal{A}}(\Delta))}{G(\overline{\mathcal{A}}(\Delta'))} \int_{\Gamma_j^*} |G(X,t)|^2 dXdt.$$

Now we apply the backward Harnack principle as follows: choose $A_j \in \Gamma_j$ such that $\|A_j - (Q,s)\| \approx 2^{-j}r$ and we get $G(X,t) \approx G(A_j)$ for every $(X,t) \in \Gamma_j$. This way we obtain

$$\sup_{\Psi(\Delta)} |u|^2 + \int_{\Gamma^r(Q,s)} |\nabla u(X,t)|^2 \delta(X,t)^{-n} dXdt \gtrsim \frac{u^2(\overline{\mathcal{A}}(\Delta))}{G^2(\overline{\mathcal{A}}(\Delta'))} \sum_j G^2(A_j).$$

Applying again the backward Harnack principle we obtain $G(A_{j-1}) < C_0 G(A_j)$ for certain $C_0 > 1$. Therefore $G(\overline{\mathcal{A}}(\Delta')) \le C_0^j G(A_j)$, and as a consequence

$$u^2(\overline{\mathcal{A}}(\Delta)) \lesssim \int_{\Gamma^r(Q,s)} |\nabla u(X,t)|^2 \delta(X,t)^{-n} dXdt + \sup_{\Psi(\Delta)} |u|^2. \tag{3.5}$$

On the other hand

$$u(\overline{\mathcal{A}}(\Delta)) = \int_\Delta f(Z,\tau)\, d\omega^{\overline{\mathcal{A}}(\Delta)}(Z,\tau) = \int_\Delta f(Y,s)\, K(\overline{\mathcal{A}}(\Delta); Z,\tau)\, d\omega(Z,\tau).$$

Since

$$K(\overline{\mathcal{A}}(\Delta); Z,\tau) = \lim_{\epsilon\to 0} \frac{\omega^{\overline{\mathcal{A}}(\Delta)}(\Delta_\epsilon(Z,\tau))}{\omega(\Delta_\epsilon(Z,\tau))} \quad \text{and} \quad \frac{\omega^{\overline{\mathcal{A}}(\Delta)}(\Delta_\epsilon(Z,\tau))}{\omega(\Delta_\epsilon(Z,\tau))} \gtrsim \frac{1}{\omega(\Delta)}$$

we obtain

$$u(\overline{\mathcal{A}}(\Delta)) \gtrsim \frac{1}{\omega(\Delta)} \int_\Delta f d\omega. \tag{3.6}$$

Putting together (3.5) and (3.6) we obtain

$$\sup_{\Psi(\Delta)} |u|^2 + \int_{\Gamma^r(Q,s)} |\nabla u(Y,s)|^2 \delta(Y,s)^{-n} dYds \gtrsim \frac{1}{\omega(\Delta)} \int_\Delta f d\omega$$

and the lemma follows. □

For completeness we now recall the argumentation that leads from Lemma 3.1 to the A_∞ property of parabolic measure with respect to surface measure.

First of all, observe that Lemma 3.1 implies

$$\left(\frac{1}{\omega(\Delta)} \int_\Delta f d\omega \right)^2 \le C_0^2 \|f\|_*^2 \tag{3.7}$$

with an appropriate constant $C_0 > 0$, and where Δ is a surface cube or radius $r > 0$. Let $E \subset$ be an open set and let $\epsilon > 0$ be a given constant.

Assume that $\sigma(E)/\sigma(\Delta) < \eta$ for a constant $\eta > 0$ to be determined. Let $h = \chi_E$ and consider Mh denote the Hardy-Littlewood maximal function of h, with respect to σ, and considering surface cubes with parabolic homogeneity. Note that in particular, for $(X,t) \notin 2\Delta$ on has $Mh(X,t) < \sigma(E)/\sigma(\Delta) < \eta$.

Then, for a $\delta > 0$ to be soon determined, the function $f = \max\{0, 1 + \delta Mh\}$ satisfies the following properties:

$$f \geq 0, \quad \|f\|_* < \delta, \quad f \equiv 1 \text{ on } E.$$

For any δ we can now choose an $\eta > 0$ such that $f \equiv 0$ on $\partial_p\Omega_T \setminus 2\Delta$. Using an approximation of identity, one can find a family $\{f_a\}_{a>0}$ such that

- For each $a > 0$, f_a is continuous supported in 3Δ and it satisfies $\|f_a\|_* \leq C_1\|f\|_*$;
- f_a converges to f in L^p.

Since $f \geq 1$ then by (3.7)

$$\frac{\omega(E)}{\omega(3\Delta)} \leq \frac{1}{\omega(3\Delta)} \lim_{a\to 0^+} \int_{3\Delta} f_a d\omega \leq C_0\|f\|_*$$

independently from η.

So we choose $\delta > 0$ so that $2C_0C_1\delta < \epsilon$ and use doubling property of parabolic measure to conclude that $\omega(E)/\omega(\Delta) < C_2\epsilon$. The theorem now follows.

4 Proof of the results using the Multiplicative Square Function

All throughout this and subsequent sections we assume that Ω is an infinite starlike parabolic Lipschitz cylinder with constants A_1 and r_0. Recall that given a Borel measure ν on S_T, we defined its *caloric extension* by

$$u(X,t) = \int_{S_T} K(X,t;Q,s)d\nu(Q,s),$$

where the *kernel function* $K(X,t;Q,s)$ is the Radon-Nikodým derivative $(d\omega^{(X,t)}/d\omega)(Q,s)$, $(Q,s) \in S_T$.

Proof of Theorem 1.2. Let

$$A = \left\{(Q,s) \in S_T : \lim_{\substack{(X,t)\to(Q,s)\\ NT}} u(X,t) > 0\right\}, \quad B = \{(Q,s) \in S_T : \mathcal{M}_\alpha u(Q,s) < \infty\}.$$

Cover S_T with a finite family of surface cubes $C_i = C_{r_0/2}(Z_i,\tau_i) \cap \partial\Omega$, $(Z_i,\tau_i) \in \partial\Omega$, $i = 1,2,\dots N$, with finite overlapping, and set $A_i = A \cap C_i$, $B_i = B \cap C_i$.

Lemma 4.1. *For every $1 \leq i \leq N$, A_i and B_i only differ in a set of ω zero measure.*

Note that this lemma implies Theorem 1.2 at once. □

Before proceeding with the proof of Lemma 4.1, we recall some constructions and results related to the parabolic version of the "Main Lemma" from [2] (see [1, p. 572] and [10, p. 207]).

Proposition 4.2. *Given a surface cube $\Delta \equiv \Delta_r(Q_0, s_0) \subset S_T$, with $r < r_0/2$, and a non-empty closed set $E \subset \Delta$, define for $a, b > 0$*

$$\Phi \equiv \Phi(\Delta, E, \alpha) = \Gamma_\alpha(E) \cap C_r(a,b),$$
$$\text{where } \Gamma(E) \equiv \bigcup_{(Q,s)\in E} \Gamma_\alpha(Q,s) \text{ and } C_r(a,b) = \left\{(X,t) : |X - Q_0| < ar,\ |t - s_0| < br^2\right\}.$$

Then there is a choice of a and b such that

(i) $\Phi = \widetilde{\Omega} \cap \{(X,t) : |t - s_0| < br^2\}$*, where* $\widetilde{\Omega}$ *is a parabolic Lipschitz cylinder with constants* $\widetilde{A}_1$ *and* $\tilde{r}_0$ *depending on* A_1 *and* α*.*

(ii) There exists $\Xi_\Delta = (\xi_\Delta, s_0 + br^2)$ *such that* $\delta(\Xi_\Delta; \partial_p\Phi) \geq cr$ *for certain uniform constant* $c > 0$*.*

(iii) If we write $(Q,s) = (q_0, q, s)$*, with* $q = (q_1, q_2, \dots, q_{n-1})$*, the set*

$$\partial\Phi \cap \{(x_0, x, t) : |t - s_0| < (2r)^2,\ |x_i - q_i^0| < 2r,\ i = 1, 2, \dots, n-1,\ |x_0 - q_0| < 4\sqrt{n}(A_1 + \widetilde{A}_1)r)\}$$

coincides with the set $\{(P,t) + \widetilde{\varphi}(P,t)\hat{e}_0 : (P,t) \in \Delta_{2r}(Q_0, s_0)\}$*, where* $\hat{e}_0$ *is the canonical* $(n+1)$*-dimensional vector in the direction of* x_0*, and* $\widetilde{\varphi} : S_T \to \mathbb{R}$ *is a parabolic Lipschitz function.*

We define $S_\Phi = \partial\widetilde{\Omega} \cap \{(X,t) : |t - s_0| < br^2\}$ *and recall it as the lateral boundary of* Φ*.*

Proof. This result is essentially proved in [1, p. 572], except that the parabolic Lipschitz cylinder in (i) is only a Lip(1,1/2) cylinder, and the parabolic Lipschitz function $\widetilde{\varphi}$ in (iii) is only a Lip(1,1/2) function. So, we sketch the idea to get the parabolic Lipschitz property in both (i) and (iii) as consequence of [11, Proposition 3.4] and the fact that parabolic Lipschitz functions are Lip(1,1/2) functions.

We first observe that one may recall $\widetilde{\Omega}$ roughly as Ω itself, except that there is a change in the local coordinate cylinder containing (Q_0, s_0). Denoting by $(\mathcal{Z}_0, \psi_0)$ this coordinate cylinder, one must substitute $\psi_0(x,t)$ by $\widetilde{\psi_0}(x,t) \equiv \psi_0(x,t) + c_0(\alpha)\delta((\psi_0(x,t), x, t); E)$, where $c_0(\alpha)$ is a constant depending only on the aperture α.

Now we observe that the proof of [11, Proposition 3.4] is flexible enough to imply that $\delta((\psi_0(x,t), x, t); E)$ is a parabolic Lipschitz function, exploiting the fact that E is locally given by a parabolic Lipschitz function, that $\delta((\psi_0(x,t), x, t); E)$ is Lip(1,1/2), and that parabolic Lipschitz functions are Lip(1,1/2) functions.

The assertion in (iii) follows by taking $\widetilde{\varphi} = \widetilde{\psi_0}$. □

Proposition 4.3. *Let* $\Delta \equiv \Delta_r(Q,s) \subset S \cap \left\{(X,t) : T_0 < t < T_0 + R_0^2\right\}$*, for certain* $T_0, R_0 > 0$*, and set* $\omega \equiv \omega_\Omega^{(X_0, T_0 + 2R_0^2)}$*. Let* $\Phi = \Phi(\Delta, E, \alpha)$ *be as defined above, for some closed set* $E \subset \Delta$*, and set* $\nu = \omega_\Phi^{\Xi_\Delta}$*. Then there exists constants* c_1*,* c_2 *and* κ *such that for* $F \subset S \cap \partial\Phi$ *one has*

$$c_1(\nu(F))^{1/\kappa} \leq \frac{\omega(F)}{\omega(\Delta)} \leq c_2(\nu(F))^\kappa.$$

This is proved for the case of the heat equation in [1, Lemma 2.10], and as observed in [10], the proof therein also is applicable to the more general parabolic equation $Lu = 0$ and the parabolic measure associated to it.

In the course of the proof, a measure $\tilde{\nu}$ is defined for $F \subset \Delta_{3r}(Q_0, s_0)$ as

$$\tilde{\nu}(F) = \nu(F \cap E) + \sum_{j=1}^{\infty} \frac{\omega(F \cap \Delta_j)}{\omega(\Delta_j)} \nu(\widetilde{\Delta}_j) \tag{4.1}$$

where $\{\Delta_j\}$ is a Whitney decomposition of $\Delta_{3r}(Q_0, s_0) \setminus (S \cap \partial\Phi)$ in surface cubes satisfying $\delta(\Delta_j; E) \approx r_j$, $\Delta_j \cap \Delta_k = \emptyset$ if $j \neq k$; also $\widetilde{\Delta}$ is the canonical projection of $\Delta \subset S$ onto S_Φ. It has also been observed in [1, Lemma 2.10] that $\tilde{\nu}$ coincides with ν on Borel subsets of E.

Proof of Lemma 4.1. We first prove that ω-almost every point in A_i is in B_i.

To shorten notations, we set $\Delta = C_i$, and let $\Delta_0 = C_{3r_i/4}(Z_i, \tau_i)$ be a dilation of Δ. Let $E \subset A_i$ be a closed set for which

$$\frac{1}{\epsilon} > \lim_{\substack{(X,t)\to(Q,s) \\ NT}} u(X,t) > \epsilon, \quad \text{for certain } \epsilon > 0, \quad \text{and every } (Q,s) \in E.$$

Let $D = \Phi(\Delta_0, E, \alpha)$ and let ω_D denote the parabolic measure on D with pole at Ξ_Δ. Observe that the pole of this measure is not on Ξ_{Δ_0}. Note that Proposition 4.3 implies that if $F \subset \Delta \cap \partial D$ is such that $\omega_D(F) = 0$ then $\omega(F) = 0$. In particular this property holds for $F \subset E$. Hence it suffices to prove that ω_D almost every element in A_i is in B_i.

For $\beta > \alpha$, we now observe that by Harnack's inequality and maximum principle, one can find positive constants C_1 and C_2 depending on β and ϵ such that

$$\inf\{u(X,t) : (X,t) \in \Gamma_\beta(Q,s)\} > C_1, \qquad \sup\{u(X,t) : (X,t) \in \Gamma_\beta(Q,s)\} < C_2,$$

for every $(Q,s) \in E$, and where $\Gamma_\beta(Q,s)$ denotes the non-tangential region of aperture β. Renormalizing u we can therefore assume $\epsilon < u(X,t) < 1/\epsilon$ for every $(X,t) \in \Gamma_\alpha(E)$.

By Fubini's theorem

$$\int_E [\mathcal{M}_\alpha u]^2 \, d\omega_D = \int_E \int_{\Gamma(Q,s)} \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta^{-n}(X,t) dX dt d\omega_D(Q,s) \tag{4.2}$$

$$\lesssim \int_D \Psi(X,t) \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta^{-n}(X,t) dX dt, \tag{4.3}$$

where $\Psi(X,t) = \omega_D(\Delta_{\alpha\delta(X,t)}(\widetilde{X}, \tilde{t}))$, and $(\widetilde{X}, \tilde{t})$ is the canonical projection of $(X,t) \in D$ onto S_Φ.

Now using an estimate between caloric measure and Green's function in [10, Lemma 2.9] we obtain

$$\Psi(X,t) \lesssim \delta(X,t)^n G_D(X, t; \underline{\mathcal{A}}(\widetilde{X}, \tilde{t})).$$

Here G_D denotes the Green's function on D. Therefore by Harnack's inequality

$$\int_E [\mathcal{M}_\alpha u]^2 \, d\omega_D \lesssim \frac{1}{\epsilon^2} \int_D G_D(X, t; \Xi_\Delta) |\nabla u(X,t)|^2 dX dt.$$

Since $L[(u-k)^2] = 2\langle A\nabla u, \nabla u\rangle$ for any constant k, then ellipticity, Green's identity and Harnack's principle yield

$$\int_E [\mathcal{M}_\alpha u]^2\, d\omega_D \lesssim \frac{1}{\epsilon^2} \sup\left\{|u(X,t) - u(\Xi_\Delta)|^2 : (X,t) \in \partial D \setminus E\right\} \lesssim \frac{1}{\epsilon^4}.$$

Therefore $\mathcal{M}u(Q,s) < \infty$ for ω_D-almost every $(Q,s) \in E$.

Now we prove that ω-almost every point in B_i is in A_i arguing by contradiction. This time we let $E \subset B_i$ be a closed set where u is non-tangentially bounded, and where both of the next conditions hold:

$$\mathcal{M}u(Q,s) \le 1 \quad \text{and} \quad \lim_{\substack{(X,t)\to(Q,s)\\ NT}} u(X,t) = 0, \quad \text{for every } (Q,s) \in E.$$

Again by Proposition 4.3 we can assume $\omega_D(E) > 0$. Suppose that $\log u$ is non-tangentially bounded on D from above. This may be done for instance by considering a subset of E if necessary.

Now we apply Green's identity to obtain

$$\log u(\mathbf{X_0}) + \int_D G_D(\mathbf{X_0}; Y, s) L[\log u(Y,s)]\, dY ds = \int_{\partial_p D} \log u\, d\omega_D,$$

where $\mathbf{X_0} \in D$ is to be chosen. Since $\omega_D(E) > 0$ and we have assumed that $\log u$ is non-tangentially bounded from above, the term in the right-hand-side is not bounded. Hence we can focus on proving that the integral in the left-hand-side is finite, and so get the contradiction.

Choose $\mathbf{X_0} = \overline{\mathcal{A}}(\Delta)$. A direct computation proves that $|L \log u| \lesssim |\nabla u|^2/u^2$, and using the estimate between Green's function and caloric measure in [10, Lemma 2.8] we have

$$\begin{aligned}
\int_D G_D(\mathbf{X_0}; X,t) L[\log u(X,t)]\, dXdt &\lesssim \int_D G_D(\overline{\mathcal{A}}(\Delta); X,t) \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} dXdt \\
&\lesssim \int_D \Psi(X,t) \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta^{-n}(X,t) dXdt \\
&\approx \int_E \int_{\Gamma(Q,s)} \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta^{-n}(X,t) dXdt d\omega_D(Q,s) = \int_E [M_\alpha u]^2\, d\omega_D \le 1
\end{aligned}$$

where in the second to last estimate we use Harnack's inequality and Fubini's theorem. □

In order to prove Theorem 1.3 we need a distributional inequality relating $\mathcal{M}u$ and the non-tangential maximal function of $\log u$. First, recalling the definition of the BMO spaces in page 66, we state a version of [9, Lemma 2.1] needed for the proof of such a distributional inequality.

Lemma 4.4. *If $f \in L^1_{loc}(S, d\omega)$ and $\|f\|_{*,\omega} \lesssim 1$ then for $\gamma > 1$*

$$\omega\{(Q,s) \in S : f(Q,s) > \gamma\} \le Ce^{-c\gamma} \omega\{(Q,s) \in S : f(Q,s) > 1\}.$$

The same distributional inequality is obtained under the assumption that $f \in L^q_{loc}(S, d\omega)$ for some $q > 0$. This is a consequence of John-Nirenberg's inequality.

Theorem 4.5. *Let u be a solution of $Lu = 0$ in Ω, and let $0 < \alpha < \beta$. Assume that β is chosen so that $\Xi_T \in \Gamma_\beta(Q,s)$ for all $(Q,s) \in S_T$. Then there are constants $C, c > 0$ such that for $\gamma > C$ and $t > 0$*

$$\omega\Big\{(Q;s) \in S_T : N_\alpha \log u(Q,s) > \gamma t,\ [\mathcal{M}_\beta u(Q,s)]^2 \le t,\ M\chi_{G_t}(Q,s) \le 1/2\Big\} \le \\ \le Ce^{-\gamma c}\omega\{(Q,s) \in S_T : N_\alpha \log u(Q,s)\} \tag{4.4}$$

where $G_t = \{(Q,s) \in S : \mathcal{M}_\beta u(Q,s) > t\}$ and $M\chi_{G_t}$ is the Hardy-Littlewood maximal function of χ_{G_t}.

We postpone the proof of this theorem to Section 5. Assuming momentarily this result we can proceed with the

Proof of Theorem 1.3. Recall that ν is a Borel measure on $\partial\Omega$ and that its caloric extension is

$$u(X,t) = \int_{S_T} K(X,t;Q,s)d\nu(Q,s).$$

Now define for $(Q,s) \in \partial\Omega$

$$\mathfrak{M}(Q,s) = \sup\left\{\frac{\nu(\Delta)}{\omega(\Delta)}, \frac{\omega(\Delta)}{\nu(\Delta)}\right\}, \quad \mathfrak{N}(Q,s) = \sup\left|\log k + \log\frac{\nu(\Delta)}{\omega(\Delta)}\right|,$$

where $k \in \mathbb{R}$ is a constant to be chosen, and the supremum in either case is taken over surface balls $\Delta \Subset \partial\Omega$ centered at (Q,s). Hence it is enough to prove that $\mathfrak{M} \in L^1(S_T, d\omega)$, and since $e^{\mathfrak{N}} \approx \mathfrak{M}$, then we focus on proving $e^{\mathfrak{N}} \in L^1(S_T, d\omega)$

Using standard estimates for the kernel function, contained for instance in [10, p. 216], we obtain

$$u(\mathbf{Q}_\Delta) \ge C_0\frac{\nu(\Delta)}{\omega(\Delta)} \tag{4.5}$$

for certain constant $C_0 > 0$, and where $\mathbf{Q}_\Delta \in \Omega_T$ is taken in $\Gamma(Q,s)$, and is such that its distance to S_T and (Q,s) are both proportional to the radius of Δ. Note that (4.5) implies $\mathfrak{N} \le N\log u$ pointwise by taking $k = C_0$ and so

$$\int_{S_T}(e^{\mathfrak{N}} - 1)\,d\omega = \gamma\int_0^\infty e^{\gamma\lambda}\omega\{\mathbf{Q} \in S_T : \mathfrak{N}(\mathbf{Q}) > \gamma\lambda\}d\lambda \le \\ \le \gamma\int_0^\infty e^{\gamma\lambda}\omega\{\mathbf{Q} \in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\}d\lambda \equiv A.$$

Now by Theorem 4.5, with $\epsilon > 0$ to be chosen,

$$A \le \gamma\int_0^\infty e^{\gamma\lambda}\omega\{\mathbf{Q} \in S_T : N\log u(\mathbf{Q}) > \gamma\lambda,\ [\mathcal{M}u(\mathbf{Q})]^2 \le \epsilon\lambda\}d\lambda + \\ + \gamma\int_0^\infty e^{\gamma\lambda}\omega\{\mathbf{Q} \in S_T : [\mathcal{M}u(\mathbf{Q})]^2 > \epsilon\lambda\}d\lambda \tag{4.6} \\ \le C\gamma\int_0^\infty e^{\gamma\lambda}e^{-c\gamma/\epsilon}\omega\{\mathbf{Q} \in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\}d\lambda + \int_{S_T}\left[\exp\left(\frac{\gamma\mathcal{M}^2 u}{\epsilon}\right) - 1\right]d\omega \equiv I + II.$$

Observe that if $\epsilon < c/\ell$ with $\ell > 1$ large

$$
\begin{aligned}
I &= \left[\int_0^1 + \int_1^\infty\right] e^{\gamma\lambda} e^{-c\gamma/\epsilon}\omega\{\mathbf{Q}\in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\} d\lambda \\
&\lesssim \gamma \int_0^1 \omega\{\mathbf{Q}\in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\} d\lambda + e^{-\ell\gamma}\gamma \int_1^\infty e^{\gamma\lambda}\omega\{\mathbf{Q}\in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\} d\lambda.
\end{aligned}
$$

If ℓ is chosen large enough we can hide the second term in the left side of (4.6), thus obtaining

$$
A \lesssim \gamma \int_0^\infty \omega\{\mathbf{Q}\in S_T : N\log u(\mathbf{Q}) > \gamma\lambda\} d\lambda + \int_{S_T}\left[\exp\left(\frac{\gamma\mathcal{M}^2 u}{\epsilon}\right) - 1\right] d\omega.
$$

All in all

$$
\int_{S_T} (e^{\Re} - 1)\, d\omega \lesssim \int_{S_T} (N\log u)\, d\omega + \int_{S_T}\left[\exp\left(\frac{\gamma\mathcal{M}^2 u}{\epsilon}\right) - 1\right] d\omega.
$$

Now we know that an estimate as (4.4) implies as in [1, p. 583]

$$
\begin{aligned}
\int_{S_T} (N\log u)\, d\omega &\le \left(\int_\Delta (N_\alpha \log u)^{p_0} d\omega\right)^{1/p_0} \lesssim \\
&\lesssim \left(\int_\Delta |\mathcal{M}_\alpha u|^{p_0} d\omega + \omega(\Delta)|u(\overline{\mathcal{A}}_r(Q_0, s_0))|^{p_0}\right)^{1/p_0} < \infty
\end{aligned}
$$

with the $p_0 > 1$ in the hypothesis of Theorem 1.3. The conclusion of the theorem follows. □

5 Distributional inequalities for multiplicative square function: Proof of Theorem 4.5

We retain notations from Section 4. Without loss of generality we assume $t = 1$ and define

$$
F_u = \{(Q,s)\in S : [\mathcal{M}_\beta u(Q,s)]^2 \le 1,\ M\chi_G \le 1/2\}, \qquad W = \Gamma_{\beta_0}(F_u), \qquad \beta_0 = (\alpha+\beta)/2,
$$

$$
\mathcal{N}u(Q,s) = \sup\{|\log u(X,t)| : (X,t)\in\Gamma_\alpha(Q,s)\cap W\},\ (Q,s)\in S.
$$

Note that $\mathcal{N}u(Q,s) \le N_\alpha \log u(Q,s)$ for $(Q,s)\in S$, and that $\mathcal{N}u(Q,s) = N_\alpha \log u(Q,s)$ for $(Q,s)\in F_u$. Therefore, if we prove

$$
\mathcal{N}u \in BMO(S, d\omega); \tag{5.1}
$$
$$
\mathcal{N}u \in L^q_{loc}(S, d\omega) \text{ for certain } q > 0, \tag{5.2}
$$

then Lemma 4.4 would imply the theorem.

In order to prove (5.2) we later in this section prove a version of Theorem 4.5, which is local, and where there is no exponential decay in the right-hand-side of the inequality.

Theorem 5.1. *Let $\Delta \equiv \Delta_r(Q_0,s_0) \subset S$, and for $\alpha > 0$ large enough, let $\beta > \alpha$. Suppose that there exists $\mathbf{P}^* \in S$, with* $\mathrm{dist}(\mathbf{P}^*;(Q_0,s_0)) \lesssim r$ *such that $N_\alpha \log u(\mathbf{P}^*) \le t$. Then, given λ and $\gamma > 1$ there exists $\delta = \delta(\alpha,\beta,\lambda,\gamma)$ and $\theta > 0$ such that*

$$\lambda\omega\Big\{(Q,s)\in\Delta : N^r_\alpha \log u(Q,s) > \gamma t,\ [\mathcal{M}^r_\beta u(Q,s)]^2 \le \delta t,\ M_\omega \chi_{G_{\delta t}}(Q,s) \le \theta\Big\} \le \omega(\Delta). \quad (5.3)$$

where as before $G_t = \Big\{(Q,s)\in S : \mathcal{M}_\beta u(Q,s) > t\Big\}$.

This theorem also implies as in [1, p. 583]

$$\int_\Delta |N_\alpha \log u|^{p_0} d\omega \lesssim \int_\Delta |\mathcal{M}_\alpha u|^{p_0} d\omega + \omega(\Delta)|u(\overline{\mathcal{A}}_r(Q_0,s_0))|^{p_0} < \infty$$

for the same $p_0 > 0$ for which, by the hypothesis of Theorem 1.3, $\mathcal{M}_\alpha u \in L^{p_0}_{loc}(S,d\omega)$. Hence (5.2) holds with $q = p_0$.

To prove (5.1) assuming Theorem 5.1, we use the idea from [9, p. 260], as developed in [4]. Let $r > 0$ so small that $0 < 20r < r_0$, and let $\Delta = \Delta_r(Q_0,s_0)$, with $(Q_0,s_0)\in S$. Define now the regions

$$\mathcal{G}^{\tau r}(Q,s) = \{(X,t)\in\Gamma_\alpha(Q,s)\cap W : \delta(X,t) \ge \tau r\},$$
$$\mathcal{G}_{\tau r}(Q,s) = \{(X,t)\in\Gamma_\alpha(Q,s)\cap W : \delta(X,t) < \tau r\},$$

where the constant $\tau > 0$ is chosen so that the region $\Phi(\Delta,F_u,\alpha)\setminus(B_{\rho_0}(X_0)\times\mathbb{R})$ contains $\mathcal{G}_{\tau r}(Q,s)$ for every $(Q,s)\in E$. Define also the maximal functions

$$\widetilde{\mathcal{N}}^r u(Q,s) = \sup\{|\log u(X,t)| : (X,t)\in\mathcal{G}^{\tau r}(Q,s)\},$$
$$\widetilde{\mathcal{N}}_r u(Q,s) = \sup\{|\log u(X,t)| : (X,t)\in\mathcal{G}_{\tau r}(Q,s)\}.$$

Consider now the following estimate of the interior local oscillation of $\log u$.

Lemma 5.2. *Let u be a positive solution to $Lu = 0$ on Ω, and given $(Q_0,s_0)\in\partial\Omega\cap\{t : -1 < t < T+1,\}$ let $(X_0,t_0)\in\Gamma_\alpha(Q_0,s_0)$. Choose $\rho > 0$ such that $0 < 10\rho < r_0$, $\rho \approx d(X_0,s_0)$ and $C_{2\rho}(X_0,t_0)\subset\Gamma(Q_0,s_0)$. Then for every $(X,t)\in C_\rho(X_0,t_0)$*

$$|\log u(X,t) - \log u(X_0,t_0)| \lesssim \int_{C_{2\rho}(X_0,t_0)} \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\delta(Y,s)^{-n} dY ds.$$

Assuming this lemma we explain how to finish the proof of (5.1) following an idea from [4]. Define

$$\widetilde{\mathcal{N}} u(Q,s) = \sup\{|\log u(X,t) - \log u(\xi_\Delta,s_0)| : (X,t)\in\mathcal{G}_{\tau r}(Q,s)\} \qquad \text{for } (Q,s)\in S_T.$$

Claim 1. There exists a constant $C > 0$ such that for $(Q,s)\in\Delta$

$$|\widetilde{\mathcal{N}}^r u(Q,s) - \widetilde{\mathcal{N}}^r u(Q_0,s_0)| \le C.$$

Claim 2. Let $\kappa > 0$ be the constant in Proposition 4.3. Then

$$\omega\{(Q,s)\in\Delta : \widetilde{\mathcal{N}} u(Q,s) > \lambda\} \lesssim \lambda^{-2\kappa}\omega(\Delta).$$

Since the Claim 2 is the heart of the matter in the proof of Theorem 5.1, we defer the proof of this claim until that theorem is proved.

Claim 1 is proved with an argument similar to that in [4, p. 288-289] (which is essentially in [2, p. 104]) once Lemma 5.2 is established. For completeness, and since we refer to this construction later on, we sketch the main idea to prove Claim 1.

Let $\mathbf{Q} \in \Delta$ and $\mathbf{X} \in \Gamma_\alpha^r(\mathbf{Q})$. Take $\mathbf{P} \in F_u$ such that $\mathbf{X} \in \Gamma_{\beta_0}(\mathbf{P})$ and set $\rho = \delta(\mathbf{X})$.

- If $\delta(\mathbf{X}) \leq r_0/20$ then there is $\overline{\mathbf{X}} \in \Gamma_\alpha^r(Q_0, s_0)$ with $\delta(\overline{\mathbf{X}}) \approx \delta(\mathbf{X})$ and such that $\overline{\mathbf{X}} \in \Gamma_{\beta_0}(\mathbf{P})$. We now can construct a sequence of points $\{\mathbf{X}_j : j = 1, \dots, \ell\}$ such that, setting $\tilde{\rho} \equiv \delta_0 \rho$ for certain δ_0 depending on the constant A_1 of Ω, one has

$$\mathbf{X}_1 = \mathbf{X}, \qquad \mathbf{X}_\ell = \overline{\mathbf{X}}, \qquad C_{\tilde{\rho}}(\mathbf{X}_j) \subset \Gamma_{\beta_0}(\mathbf{P}),$$
$$\delta(\mathbf{X}_j) \approx \rho, \qquad \left\|\mathbf{X}_j - \mathbf{X}_{j+1}\right\| \leq \tilde{\rho}/2, \qquad j = 1, \dots, \ell.$$

  Now by Lemma 5.2, for any j

$$\left|\log u(\mathbf{X}_j) - \log u(\mathbf{X}_{j+1})\right| \lesssim \int_{C_{\tilde{\rho}}(\mathbf{X}_j)} \frac{|\nabla u(X,t)|^2}{|u(X,t)|^2} \delta(X,t)^{-n} dX dt \leq [\mathcal{M}_\beta u(\mathbf{P})]^2 \leq 1$$

  since $\mathbf{P} \in F_u$. This implies that

$$|\log u(\mathbf{X})| \leq |\log u(\overline{\mathbf{X}})| + c \tag{5.4}$$

  with a constant $C > 0$ that depends on A_1, α and β.

- If $\delta(\mathbf{X}) > r_0/20$ then we can still construct a sequence of points $\{\mathbf{X}_j : j = 1, \dots, \ell\}$, this time satisfying

$$\mathbf{X}_1 = \mathbf{X}, \qquad \mathbf{X}_\ell = \Xi_T, \qquad C_{\tilde{\rho}}(\mathbf{X}_j) \subset \Gamma_{\beta_0}(\mathbf{P}), \qquad \|\mathbf{X}_j - \mathbf{X}_{j+1}\| \leq \tilde{\rho}/2 \qquad j = 1, \dots, \ell.$$

  Again by Lemma 5.2 we obtain

$$|\log u(\mathbf{X})| \leq |\log u(\Xi_T)| + c \tag{5.5}$$

  again with a constant $C > 0$ that depends on A_1, α and β.

Estimates (5.4) and (5.5) imply

$$|\log u(\mathbf{X})| \leq \widetilde{\mathcal{N}}^r u(Q_0, s_0) + c$$

for every $\mathbf{X} \in \Gamma_\alpha^r(\mathbf{Q})$ and $\mathbf{Q} \in \Delta$.

A similar argument allows us to prove that actually

$$|\log u(\mathbf{X})| \leq \widetilde{\mathcal{N}}^r u(\mathbf{Q}) + c$$

for every $\mathbf{X} \in \Gamma_\alpha^r(\mathbf{Q})$. Therefore

$$|\widetilde{\mathcal{N}}^r u(\mathbf{Q}) - \widetilde{\mathcal{N}}^r u(Q_0, s_0)| \lesssim 1 \tag{5.6}$$

for every $\mathbf{Q} \in \Delta$, and the Claim 1 is proved.

Assuming the two Claims, we can follow the argument in [4, p. 289] (see also [9, Lemma 4.4]) to obtain

$$\inf_{a\in\mathbb{R}} \int_{\Delta} |\mathcal{N}u - a|^{\kappa} d\omega \leq C\omega(\Delta)$$

which already implies that $\mathcal{N}u \in BMO(S,\omega)$, which is precisely (5.1). The proof of Theorem 4.5 is now finished, except for Lemma 5.2, Claim 2 and Theorem 5.1, which we prove next.

Proof of Lemma 5.2. To shorten notation, let $C_i = C_{i\rho}(X_0,t_0)$, $i = 1,2$. Apply Green's identity to $v(X,t) = \log u(X,t) - \log u(X_0,t_0)$ at $(X,t) = (X_0,t_0)$ to obtain

$$\int_{\partial_p C_2} [\log u(Q,s) - \log u(X_0,t_0)] d\omega_2^{(X_0,t_0)}(Q,s) = \int_{C_2} g_2(X_0,t_0;Y,s) L[\log u(Y,s)] dY ds, \quad (5.7)$$

where ω_2 and g_2 denote the parabolic measure and Green's function for L on C_2. In fact, for $(X,t) \in C_2$

$$\begin{aligned}\log u(X,t) - \log u(X_0,t_0) = &\int_{\partial_p C_2} [\log u(Q,s) - \log u(X_0,t_0)] d\omega_2^{(X,t)}(Q,s) - \\ &- \int_{C_2} g_2(X,t;Y,s) L[\log u(Y,s)] dY ds. \qquad (5.8)\end{aligned}$$

Applying Harnack's inequality to g_2 in (5.8), and using the fact that $|L\log u| \lesssim |\nabla u|^2/|u|^2$, and that the Radon-Nikodým derivative $d\omega_2^{(X_0,t_0)}/d\omega_2^{(X,t)}$ is essentially bounded by 1 we obtain

$$\begin{aligned}\left|\log u(X,t) - \log u(X_0,t_0)\right| &\lesssim \int_{C_2} g_2(X_0,t_0;Y,s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2} dY ds \\ &\lesssim \int_{C_2} G(X_0,t_0;Y,s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2} dY ds\end{aligned}$$

by (5.7), and where in the last inequality we have used the maximum principle. Here G denotes the Green's function of L on $\Omega \cap \{(X,t) : -1 < t < T+1\}$. Applying Harnack inequality and the comparison of parabolic measure and Green's function [10, Lemma 2.8] we conclude

$$\begin{aligned}|\log u(X,t) - \log u(X_0,t_0)| &\lesssim \int_{C_2} \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2} \delta(Y,s)^{-n} \omega(\Delta_{\alpha\delta(Y,s)}(Y,s)) dY ds \\ &\leq \int_{C_2} \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2} \delta(Y,s)^{-n} dY ds\end{aligned}$$

as desired. □

Proof of Theorem 5.1. Let E be the set in the left side of (5.3) and define

$$F = \{(Q,s) \in \Delta : [\mathcal{M}_\beta^r u(Q,s)]^2 \leq \delta t,\ M_\omega \chi_{G_{\delta t}}(Q,s) \leq \theta\}$$

Now we repeat the definition of W, $\mathcal{N}u$, $\mathcal{G}^{\tau r}$, etc. in the previous proof with F replacing F_u. This way, the argument leading to (5.6), along with the fact that for $(Q,s) \in E$ one has $[\mathcal{M}^r_\beta u(Q,s)]^2 \le \delta t$ imply that

$$E \subset \left\{(Q,s) \in \Delta : \widetilde{\mathcal{N}}u(Q,s) \ge \gamma t\right\}.$$

So it suffices to prove that for $\kappa > 0$ as in Proposition 4.3 and any $\zeta > 0$

$$\omega\{(Q,s) \in \Delta : \widetilde{\mathcal{N}}u(Q,s) \ge \zeta\} \le \zeta^{-\kappa}\omega(\Delta). \tag{5.9}$$

The following argument that establishes (5.9) also proves the Claim 2 in the proof of Theorem 4.5.

Let $H_\zeta = \{(Q,s) \in \Delta : \widetilde{\mathcal{N}}u(Q,s) > \zeta\}$. By Proposition 4.3 it suffices to prove

$$\tilde{\nu}(H_\zeta) \lesssim \zeta^\eta \tag{5.10}$$

for certain $\eta > 0$ and where $\tilde{\nu}$ is the measure defined in (4.1). Now note that by Chebyshev's inequality, and using the notations in and around the definition of $\tilde{\nu}$, we obtain

$$\tilde{\nu}(H_\zeta) \lesssim \frac{1}{\zeta}\int_E [\widetilde{\mathcal{N}}u]\,d\nu + \sum_j \frac{\omega(\Delta_j \cap H_\zeta)}{\omega(\Delta_j)}\nu(\widetilde{\Delta}). \tag{5.11}$$

Now the proof continues in three steps:

Step 1. We first prove that $\omega(H_\zeta \cap \Delta_j) \approx \omega(\Delta_j)$.

Proof. For $\mathbf{Q} \in H_\zeta \cap \Delta_j$ one has $d(\mathbf{Q};E) \approx \operatorname{diam}\Delta_j$. Also if $\mathbf{P} \in E$ satisfies $d(\mathbf{P};\mathbf{Q}) \approx \operatorname{diam}\Delta_j$ then there exists $\mathbf{X} \in \Gamma_\alpha(\mathbf{Q}) \cap \Gamma_{(\alpha+\beta)/2}(\mathbf{P})$ with $d(\mathbf{X};S_\Phi) \approx \operatorname{diam}\Delta_j$ and such that $|\log u(\mathbf{X}) - \log u(\xi_\Phi, s_0)| > \zeta$. So we choose β as a large multiple of α and we obtain that for certain $\rho_1 > 0$ one has

$$\widetilde{\mathcal{N}}u(\mathbf{Z}) > \zeta \qquad \text{for every} \quad \mathbf{Z} \in \Delta_{\rho_1 \operatorname{diam}\Delta_j}(\mathbf{Q}).$$

The doubling property of ω implies the claim of this first step. □

Step 2. Now we prove that

$$\tilde{\nu}(H_\zeta)\zeta \lesssim \int_D g(\Xi;Y,s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\,dY ds,$$

where g is the Green's function for L on D.

Proof. Note that by Step 1, (5.11) is transformed into

$$\tilde{\nu}(H_\zeta) \lesssim \frac{1}{\zeta}\int_E [\widetilde{\mathcal{N}}u]\,d\nu + \sum_j \nu(\widetilde{\Delta}). \tag{5.12}$$

Accordingly, there is two terms to be estimated. For $\mathbf{Q} \in E$ and $\mathbf{X} \in \mathcal{G}_{\tau r}(\mathbf{Q})$, by Green's identity

$$\begin{aligned}|\log u(\mathbf{X}) - \log u(\Xi)| &\le \int_{\partial_p D} |\log u(\mathbf{Z}) - \log u(\Xi)|\,d\nu^{\mathbf{X}}(\mathbf{Z}) + \int_D g(\mathbf{X};Y,s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\,dY ds\\ &\lesssim \int_{\partial_p D} |\log u(\mathbf{Z}) - \log u(\Xi)|\,d\nu(\mathbf{Z}) + \int_D g(\Xi;Y,s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\,dY ds\end{aligned}$$

by Harnack's principle, and since the Radon-Nikodým derivative $d\nu^{\mathbf{X}}/d\nu \leq 1$.

This already implies

$$\widetilde{\mathcal{N}}u(\mathbf{Q}) \lesssim \int_{\partial_p D} |\log u(\mathbf{Z}) - \log u(\Xi)|\, d\nu(\mathbf{Z}) + \int_D g(\Xi; Y, s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dY ds. \tag{5.13}$$

uniformly in $\mathbf{Q} \in E$.

On the other hand, for $\mathbf{Z} \in \widetilde{\Delta}_j$ we can choose as in the proof of Step 1, $\mathbf{X} \in \Gamma_\alpha(\mathbf{Z}) \cap \Gamma_{(\alpha+\beta)/2}(\mathbf{P})$ such that $|\log u(\mathbf{X}) - \log u(\Xi)| > \zeta$, where $\mathbf{P} \in E$ satisfies $d(\mathbf{P}; \mathbf{Z}) \approx \operatorname{diam}\Delta_j$.

Then, since $\mathbf{P} \in E$ we can assume that $[\mathcal{M}^r_\beta(\mathbf{P})]^2 < \epsilon\zeta$, with ϵ to be chosen. Also, since $|\log u(\mathbf{Z}) - \log u(\Xi)| \leq [\mathcal{M}^r_\beta(\mathbf{P})]^2$ we can conclude that for suitable small ϵ one has $|\log u(\mathbf{Z}) - \log u(\Xi)| > \zeta$. Applying Chebyshev's inequality

$$\nu(\widetilde{\Delta}_j) \leq \frac{1}{\zeta} \int_{\widetilde{\Delta}_j} |\log u(\mathbf{Z}) - \log u(\Xi)| d\nu(\mathbf{Z}).$$

Since the $\widetilde{\Delta}_j$ have finite overlapping we conclude from (5.13)

$$\tilde{\nu}(H_\zeta)\zeta \lesssim \int_D g(\Xi; Y, s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dY ds + \int_{\partial_p D} |\log u(\mathbf{Z}) - \log u(\Xi)| d\nu(\mathbf{Z}). \tag{5.14}$$

Finally notice that Green's identity applied to $v(X,t) = \log u(X,t) - \log u(\Xi)$ implies

$$\int_{\partial_p D} [\log u(\mathbf{Z}) - \log u(\Xi)] d\nu(\mathbf{Z}) = \int_D g(\Xi; Y, s) L \log u(Y,s)\, dY ds.$$

Hence

$$\int_{\partial_p D} [\log u(\mathbf{Z}) - \log u(\Xi)] d\nu(\mathbf{Z}) \lesssim \int_D g(\Xi; Y, s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dY ds. \tag{5.15}$$

Plugging (5.15) in (5.13) and (5.14) the claim is established. □

Step 3. We finally prove

$$\int_D g(\Xi; Y, s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dY ds \lesssim 1. \tag{5.16}$$

Proof. Set $D^r = \{\mathbf{X} \in D : d(\mathbf{X}; D_S) \leq \tau r\}$ where $\tau > 0$ is chosen small enough. Now we handle the part far from the boundary. By ellipticity

$$\begin{aligned}\int_{D\setminus D^r} g(\Xi; Y, s) \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dY ds &\lesssim \frac{1}{\inf\limits_{D\setminus D^r} |u|^2} \int_{D\setminus D^r} g(\Xi; Y, s) \langle A(Y,s)\nabla u(Y,s), \nabla u(Y,s)\rangle dY ds \\ &\leq \frac{1}{\inf\limits_{D\setminus D^r} |u|^2} \int_{D\setminus D^r} g(\widetilde{\Xi}; Y, s) \langle A(Y,s)\nabla u(Y,s), \nabla u(Y,s)\rangle dY ds\end{aligned}$$

by elliptic type Harnack principle, and where $\widetilde{\Xi} = (\xi_\Phi, s_0 - br^2)$ (see Proposition 4.2). Note that Harnack's inequality implies $u(\widetilde{\Xi}) \lesssim u(\mathbf{X})$ for every $\mathbf{X} \in D \setminus D^r$ and so we conclude

$$\begin{aligned}\int_{D\setminus D^r} g(\Xi; Y, s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dYds &\lesssim \frac{1}{\inf\limits_{D\setminus D^r} |u|^2}\int_{D\setminus D^r} g(\widetilde{\Xi}; Y, s) Lu^2(Y,s) dYds \\ &\le \frac{u(\widetilde{\Xi})}{\inf\limits_{D\setminus D^r} |u|^2} \lesssim 1 \end{aligned}\tag{5.17}$$

To handle the part close to the boundary, observe that

$$C \ge \int_{S\setminus G_1} [\mathcal{M}^r_\beta u]^2 d\omega^\Xi \ge \int_D \frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\widetilde{\Psi}(Y,s) dYds \tag{5.18}$$

where

$$\widetilde{\Psi}(Y,s) = \omega^\Xi\{\mathbf{Z} \in S \setminus G_1 : (Y,s) \in \Gamma^r_\beta(\mathbf{Z})\}.$$

Note that for $\mathbf{Y} \in D^r$ there exists $\widetilde{\mathbf{Y}} \in E \subset S \setminus G_1$ such that $\|\mathbf{Y} - \widetilde{\mathbf{Y}}\| \approx \delta(\mathbf{Y})$ and $\mathbf{Y} \in \Gamma_\beta(\widetilde{\mathbf{Y}})$. In other words, denoting $\widetilde{\Delta} = \Delta_{\gamma\delta(\widetilde{\mathbf{Y}})}(\widetilde{\mathbf{Y}})$ then

$$\widetilde{\Delta} \cap S \setminus G_1 \subset \{\mathbf{Z} \in S \setminus G_1 : \mathbf{Y} \in \Gamma^r_\beta(\mathbf{Z})\}.$$

Therefore by the Comparison principle in [10, Lemma 3.4]

$$\widetilde{\Psi}(\mathbf{Y}) \ge \omega^\Xi(\widetilde{\Delta} \cap S \setminus G_1) \approx \frac{\omega(\widetilde{\Delta} \cap S \setminus G_1)}{\omega(\widetilde{\Delta})}\omega^\Xi(\widetilde{\Delta}) \gtrsim \frac{\omega(\widetilde{\Delta} \cap S \setminus G_1)}{\omega(\widetilde{\Delta})} g(\Xi; \mathbf{Y})\delta^n(\mathbf{Y})$$

where in the last inequality we have used the comparison between Green's function and parabolic measure [10, Lemma 2.8]. Since $\widetilde{\mathbf{Y}} \in E$, then by definition $\omega(\widetilde{\Delta} \cap S \setminus G_1)/\omega(\widetilde{\Delta}) \gtrsim 1$ thus

$$\widetilde{\Psi}(\mathbf{Y}) \gtrsim g(\Xi; \mathbf{Y})\delta^n(\mathbf{Y}). \tag{5.19}$$

Since clearly

$$\int_{D^r} g_D(\Xi; Y, s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dYds \le \int_D g(\Xi; Y, s)\frac{|\nabla u(Y,s)|^2}{|u(Y,s)|^2}\, dYds$$

from (5.18) and (5.19) we obtain the claim of this step. □

As we mention before, (5.10) is essentially Claim 2. Also, we noted that the three steps described above imply (5.10), which in turn suffices to finish the proof of Theorem 4.5. □

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